CHAPTER 2

Vector bundles

In this chapter we prove Frobenius Theorem about integrability of vector fields and use this construction to motivate and study vector bundles on manifolds.

2.1. One parameter vector fields

Let M be a manifold. We already encountered vector fields, they can be either seen as derivations on $C^\infty(M)$ or sections of the tangent bundle, recall Proposition 1.7.24. In this section we want to focus on slightly different perspective. Let $C \subset M$ be a submanifold of dimension 1, that is a curve on M. Then through any point $p \in C$ we have a tangent space $T_pC \subset T_pM$ and also the tangent vector induced by a parametrization of C. Let now X be a vector field. Then at any point $p \in M$, $X(p) \in T_pM$ is a tangent vector. Restrict to a local chart (U_p, φ) then $X = \sum a_i(x_1, \dots, x_n)\partial_i$, and consider a differentiable function $f : J \to \varphi(U_p)$, with $f(t) = (x_1(t), \dots, x_m(t))$. When we think of f as a curve in U_p, the function $f'(t) := \frac{df}{dt}(t) = Df_t(1)$ describes its tangent vectors. It is natural to ask whether there is such a f with $f'(t) = X(f(t))$. In other words if there is a curve whose tangent vectors are described by the vector field. Such a curve is called an integral curve of the vector field X.

Definition 2.1.1. Let X be a vector field on M. A curve $f : J \to M$ is an **integral curve** of X if for any $t \in J$, $f'(t) = Df_t(1) = X_{f(t)}$.

By definition integral curves are solutions of the following equation

$$Df_t(1) = X(f(t)). \tag{6}$$

In a chart (U_p, φ), with $\varphi(p) = (0, \dots, 0)$ we have $\varphi \circ f(t) = (x_1(t), \dots, x_m(t))$, $D\varphi(X) = \sum a_i(x_1, \dots, x_m)\partial_i$, therefore Equation (6) translates in the following system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = a_i(x_1, \dots, x_m),$$

together with initial condition $(\varphi \circ f)(0) = (0, \dots, 0)$. Therefore, thanks to Cauchy existence result, integral curves always exists locally.

Remark 2.1.2. Note that even if integral curves always exists they do not need to be submanifolds. For example, consider $M = S^1 \times S^1 \subset \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}^2_{(y_1,y_2)}$. Fix any irrational number a, the integral manifold of the non-vanishing vector field $X_a = (x_2\partial_{x_1} - x_1\partial_{x_2}) + a(y_2\partial_{y_1} - y_1\partial_{y_2})$ is dense in M.

This allows to give a geometric point of view on differential equations. Moreover we may also consider vector fields depending on a parameter t.

DEFINITION 2.1.3. Let M be a m-manifold and $J \subset \mathbb{R}$ an interval, with $0 \in J$. A **one parameter vector field** is a map

$$X : M \times J \to TM,$$

such that for any $\bar{t} \in J$, the assignment $X(p, \bar{t})$ is a vector field on M. A curve $f(t)$ is **integral** for $X(p, t)$ if

$$\frac{df(t)}{dt} = X(f(t), t),$$

for any $t \in J$. Let us indicate with $\mathcal{X}(M)_J$ the set of one parameter vector fields on M defined on $M \times J$.

EXAMPLE 2.1.4. This is an evolution of Example 1.7.25. Consider $M = \mathbb{R}^3$. Fix coordinates (x, y, z) on $\mathbb{R}^3$ and the canonical basis $(\partial_i(p))$ for each $T_pM \simeq \mathbb{R}^3$. Let $f : \mathbb{R} \to \mathbb{R}^3$, with $f(t) = (x(t), y(t), z(t))$ be a smooth function with $f(0) = (0, 0, 0)$. Let $X : M \setminus f(\mathbb{R}) \times \mathbb{R} \to T(M \setminus f(\mathbb{R}))$ be the vector field defined as

$$X((x, y, z), t) = -G((x - x(t))/r(t)^3, (y - y(t))/r(t)^3, (z - z(t))/r(t)^3),$$

with $r(t) = \sqrt{(x - x(t))^2 + (y - y(t))^2 + (z - z(t))^2}$. This is the gravitational field of an object of unit mass that moves along the curve $f(t)$. We had to exclude $f(\mathbb{R})$ to ensure that X is well defined on the manifold.

PROPOSITION 2.1.5. *Let $X(p, t) \in \mathcal{X}(M)_J$ be a one parameter vector field. Then there is an open subset $W_{M \times \{0\}} \subset M \times J$ and a smooth function $G : W \to M$, such that*

- *for any $\overline{x} \in M$ the curve $f(t) := G(\overline{x}, t)$ is integral for X*
- *$f(0) = \overline{x}$, $G(x, 0) = x$*

PROOF. Fix $p \in M$ and let (U_p, φ) be a local chart, after maybe shrinking U_p, we may assume, by Cauchy theorem, that the solution exists and is unique in $U_p \times (-\delta_p, \delta_p)$, for some $\delta_p > 0$. That is there exists a function $G_p : U_p \times (-\delta_p, \delta_p) \to M$ with the required properties. We have now to glue these local solutions. This can be done thanks to the uniqueness part of Cauchy theorem. To conclude observe that $\cup_{p \in M} U_p \times (-\delta_p, \delta_p)$ is an open neighborhood of $M \times \{0\}$. □

DEFINITION 2.1.6. The function G produced in proposition 2.1.5 is called the **flow of the one parameter vector field**.

REMARK 2.1.7. Note that the flow is such that $G(\overline{x}, t)$ is an integral curve of the vector field for any t, and $G(x, 0) = x$. Hence we may rewrite the differential equation of the field via the following equations of the flow:

$$DG_{(x,0)}(\partial_i) = \partial_i, \quad DG_{(x,t)}(\frac{d}{dt}) = X(x, t) \tag{7}$$

If $W \supset M \times (-\delta, \delta)$ and $s \in (-\delta, \delta)$, then by uniqueness we have

$$G(p, s) = G(G(p, s), 0). \tag{8}$$

Let $\theta_s : M \to M$ be defined as

$$\theta_s(p) := G(p, s).$$

Then by Equations (7), (8) we have that θ_s is a local diffeomorphism. Moreover θ_s is a bijection by uniqueness of solutions. Therefore θ_s is a diffeomorphism and it is homotopy equivalent to the identity. In other words the flow describes the manifold M has a dynamical system whose points are moved according to the one parameter vector field. This opens our arguments to dynamics on the manifolds. We refer the interested reader to [**1**].

If M is compact we only need a finite number of local charts, then there is a positive δ such that the flow is defined on $M \times (-\delta, \delta)$, but something even better is at hand.

PROPOSITION 2.1.8. *Let M be a compact manifold and $X(p,t) \in \mathcal{X}(M)_{\mathbb{R}}$ a one parameter vector field then there is a flow $G : M \times \mathbb{R} \to M$. Moreover*

$$G(p, s_1 + s_2) = G(G(p, s_1), s_2),$$

therefore there is a morphism of groups

$$\mathbb{R} \to Diff(M, M),$$

all diffeomorphisms built in this way are homotopy to the identity. Finally this produces a map

$$\mathcal{X}(M)_{\mathbb{R}} \to \mathrm{Hom}(\mathbb{R}, Diff(M, M)).$$

PROOF. Let $f : (a, b) \to M$ be an integral curve and assume that $f(0) = p$. For the first statement it is enough to show that we may prolong f to an integral curve $\tilde{f} : (a - \delta, b + \delta) \to M$, such that $f(t) = \tilde{f}(t)$ for $t \in (a, b)$. M is compact therefore the flow $G : M \times (-\delta, \delta) \to M$ is well defined, for some $\delta > 0$. Choose $\bar{t} \in (a, a + \delta)$ and define the integral curve $G(f(\bar{t}), t)$ on $(t - \delta, t + \delta)$. By Equation (8) we have

$$G(p, t + \bar{t}) = G(f(\bar{t}), t),$$

hence by uniqueness of solution it has to agree with f and it prolongs it.

Since the flow $G : M \times \mathbb{R} \to M$ is well defined then, again by uniqueness of solutions we have that

$$G(p, s_1 + s_2) = G(G(p, s_1), s_2).$$

□

On non compatc manifold it is in general not true that a vector field $X \in \mathcal{X}(M)_{\mathbb{R}}$ defines a flow on $M \times \mathbb{R}$. This naturally leads to the following definition

DEFINITION 2.1.9. Let $X \in \mathcal{X}(M)_{\mathbb{R}}$ be a one paramenter vector field. Then X is called complete if there exists a flow $G : M \times \mathbb{R} \to TM$ associated to X.

As observed before to a complete vector field is associated a one paramenter group of diffeomorphism homotopically equivalent to the identity.

REMARK 2.1.10. Note that the theory of one paramenter vector fields contains that of vector fields, simply defining $X(p,t) = X(p)$, for any $p \in M$.

2.2. Frobenius Theorem

It is quite natural in our set up to ask for integral submanifolds of higher dimension. That is we talked about integral curves associated to a vector field on a manifold M, but what happens if we choose two or more vector fields? Is it possible to "integrate" them? In other words is it possible to describe submanifolds $N \subset M$ such that at any point T_pN is spanned by the chosen vector fields?

Let us start with a simple example. Let $W \subset \mathbb{R}^3$ be open and consider a system of partial differential equations

$$\partial z/\partial x = g(x, y, z), \quad \partial z/\partial y = h(x, y, z).$$

Given $(a, b, c) \in W$, a solution, if any, will be a function $z = f(x, y)$ such that

$$c = f(a, b),\ f_x(x, y) = g(x, y, f(x, y)),\ f_y(x, y) = h(x, y, f(x, y)).$$

From a geometric point of view if we let let $F(x, y, z) = z - f(x, y)$ then $V := \{F = 0\}$ is a surface in $W \subset \mathbb{R}^3$. Recall that $(\partial_x, \partial_y, \partial_z)$ is a basis for derivations on W, hence, by Lemma 1.7.3, we have

$$T_{(x,y,z)}V = (\partial_x F, \partial_y F, \partial_z F)^\perp = (-f_x, -f_y, 1)^\perp = \langle \partial_x + g(x, y, z)\partial_z, \partial_y + h(x, y, z)\partial_z \rangle.$$

In other words if we consider the two vector fields X and Y, given by

$$X = \partial_x + f_x \partial_z,\ Y = \partial_y + f_y \partial_z$$

then V is an integral submanifold for $\{X, Y\}$. Note further that for this particular choice of vector fields, since $f_{xy} = f_{yx}$ we have

$$[X, Y] = XY - YX = 0$$

This shows that, in this set up, our initial question has a necessary condition, namely $[X, Y] = 0$, and it reflects the independence on the order of partial derivatives. It is therefore easy to guess that some condition on integrability are needed in this more general framework. It is time to introduce some definitions.

Definition 2.2.1. Let M be a manifold, a **distribution** D **of rank** k is the assignment of a subspace $D_p \subset T_pM$ such that:

a) $\dim D_p = k$ for any $p \in M$,
b) for any $p \in M$ there is a chart (U_p, φ) and k vector fields $\{X_1, \ldots, X_k\} \subset \mathcal{X}(U_p)$ such that for any $q \in U_p$ $D_p = \langle X_1(q), \ldots, X_k(q) \rangle$. Such a set $\{X_1, \ldots, X_k\}$ is called a **local basis** at q

We say that a vector field $Y \in \mathcal{X}(M)$ belongs to the distribution D,

$$Y \in D,$$

if for any $p \in M$ $Y(p) \in D_p$. A distribution is said **involutive** if for any pair of vector fields $X, Y \in D$ we have $[X, Y] \in D$. A distribution is **integrable** at p if there exists an open $W_p \subset M$ and a submanifold $F \ni p$ such that for any $q \in W \cap F$

$$T_qF = D_q,$$

such a F is called a **leaf** of the distribution. Integrable distributions are also called **foliations**.

Example 2.2.2. Let $M = \mathbb{R}^{n+k}$ and $D_p := \{\partial_i(p)\}_{i=1,\ldots n} \subset T_pM$. Let D be the distribution defined by the D_p. Then D is clearly involutive and the leaves of D are the fibers of the projection onto the last k coordinates $\pi : M \to \mathbb{R}^k$. Despite this may seem a very special case we will prove that any foliation is locally of this type.

Remark 2.2.3. The notion of integrable distribution extends that of integral curve. Note that a distribution of rank 1 is a single vector field, hence is always involutive since $[X, X] = XX - XX = 0$. The result on integral curves in the preceding section can be rephrased saying that a rank 1 distribution is always involutive and a foliation.

We aim to study foliations. The first step is to prove that Example 2.2.2 locally describes any rank 1 foliation. The following is just a rephrasing of the existence of integral curves with a local change of variables.

LEMMA 2.2.4. *Let $D = \{X\}$ be a rank 1 distribution, i.e. a non vanishing vector field, on M. Let $p \in M$ a point then there is a local chart (U_p, φ) such that for any $q \in \mathrm{U}_p$, $X(q) = \partial_1(q)$.*

PROOF. The statement is local therefore we may assume, after shrinking M, that $M \simeq B_\epsilon(0) \subset \mathbb{R}^m$, moreover we may assume that $X(0) = \partial_1(0)$. Let $X = \sum_1^m a_i \partial_i$, with a_i smooth functions, $a_1(0) \neq 0$, for all $q \in M$, and $a_i(0) = 0$ for $i \geq 2$. Consider the following system of ordinary differential equations

$$\frac{dx^i}{dx_1} = \frac{a_i(x_1, \dots, x_m)}{a_1(x_1, \dots, x_m)} \text{ for } i = 2, \dots, m. \tag{9}$$

Then for any $(z_2, \dots, z_m)$ the system has a unique solution

$$x^i = f^i(x_1, \dots, x_m),$$

with initial data

$$x^i(0, z_2, \dots, z_m) = z_i,$$

for $i = 2, \dots, m$. Moreover the x^i are smooth functions in the variables $(x_1, z_2, \dots, z_m)$. Consider the following sistem

$$x_1 = z_1, \; x_2 = x^2(z_1, \dots, z_m), \dots, \; x_m = x^m(z_1, \dots, z_m).$$

By construction the Jacobian $(\partial x^i / \partial z_j)$ evaluated in $(0, \dots, 0)$ is the identity therefore by the inverse function theorem we may change coordinates from $(x_1, \dots, x_m)$ to $(z_1, \dots, z_m)$. In these coordinates, by Equation (9), we may rewrite

$$X = \sum a_i \frac{\partial}{\partial x_i} = \sum (a_1 \frac{\partial x_i}{\partial z_1}) \frac{\partial}{\partial x_i} = a_1 \frac{\partial}{\partial z_1}.$$

To conclude it is then enough to normalize the first coordinate with

$$x^1(z_1, \dots, z_n) := \int_0^{z_1} \frac{dt}{a_1(t, x^2, \dots, x^m)}.$$

□

THEOREM 2.2.5 (Frobenius Theorem). *Let M be a m-manifold and D a distribution of rank k. Then D is integrable if and only if it is involutive.*

One direction of the Frobenius is clear. If D is integrable then the vector fields $X, Y \in D$ belong to $TF \subset TM$ therefore $[X, Y] \in TF = D$. To prove Frobenius Theorem we start with a local version of it.

PROPOSITION 2.2.6. *Let D be an involutive distribution of rank k on M. Let $p \in M$ be a point, then there is a local chart (U_p, φ) such that for all $q \in U_p$ we have*

$$D_q = \langle \partial_1(q), \dots, \partial_k(q) \rangle.$$

PROOF. Let $\{X_1, \dots, X_k\}$ be a local basis for the distribution, after eventually shrinking the open neighborhood of p. We prove the Proposition by induction on k. The first step is Lemma 2.2.4. Then we may assume the Proposition is true for distributions of rank $k-1$. By Lemma 2.2.4 we have $M \simeq B_\epsilon(0) \subset \mathbb{R}^m$ and $X_k = \partial_k$. Define, for $j \leq k-1$ the vector fields

$$Y_j = X_j - X_j(x_k) X_k,$$

then $Y_j(x_k) = 0$, for $j \leq k-1$ and $X_k(x_k) = 1$. Moreover by definition

$$D = \langle Y_1, \dots, Y_{k-1}, X_k \rangle,$$

and evaluating the bracket on x_k we see that

$$0 = [Y_i, Y_j](x_k) = (\sum b_{ijh} Y_h)(x_k) + a_{ij} X_k(x_k) = a_{ij}$$

hence

$$D_Y = \langle Y_1, \dots, Y_{k-1} \rangle$$

is involutive. By induction hypothesis we have a coordinate system, say $(y_1, \dots, y_m)$ such that

$$\{\frac{\partial}{\partial y_i}\}_{i=1,\dots,k-1} = D_Y.$$

Since $\frac{\partial}{\partial y_i}$, for $i = 1, \dots, k-1$, is a linear combination of Y_j, for $i = 1, \dots, k-1$ we still have

$$\frac{\partial}{\partial y_i}(x_k) = 0,$$

for $i = 1, \dots, k-1$. Let

$$[\frac{\partial}{\partial y_i}, X_k](x_k) = (\sum_1^{k-1} c_{ikh} \frac{\partial}{\partial y_h} + c_i X_k)(x_k), \tag{10}$$

then, as before, evaluating on the function x_k we get $c_i = 0$. That is

$$[\frac{\partial}{\partial y_i}, X_k] = \sum_1^{k-1} c_{ikh} \frac{\partial}{\partial y_h}. \tag{11}$$

Let $X_k = \sum_1^n b_j \frac{\partial}{\partial y_j}$, since $(\frac{\partial}{\partial y_j})$ is a local basis then plugging in Equation (11) we get

$$\frac{\partial b_j}{\partial y_i} = 0,$$

for $i \leq k-1$ and $k \leq j \leq n$. That is $b_j = b_j(y_k, \dots, y_m)$ for $j \geq k$. Let $Y_k = \sum_{j=k}^m b_j \frac{\partial}{\partial y_j}$, then

$$D = \{Y_1, \dots, Y_k\}.$$

Moreover by Lemma 2.2.4 there is a coordinate change, $(y_1, \dots, y_m)$ to $(z_1, \dots, z_m)$ such that

$$y_i = z_i, \text{ for } i = 1, \dots, k-1,$$

and

$$Y_k = \frac{\partial}{\partial z_k}.$$

Hence in this coordinate system $Y_i = \frac{\partial}{\partial z_i}$, For $i = 1, \dots k$. □

We are now in the condition to conclude Frobenius Theorem.

Proof of Frobenius Theorem. We need to produce the leaves of a rank k distribution D. Fix $p \in M$ a point. Then by Proposition 2.2.6 there is a local chart (U_p, φ) such that $D = \langle \partial_i \rangle$, for $i \leq k$. Let $\pi : \mathbb{R}^m \to \mathbb{R}^{m-k}$ the projection onto the last $m-k$ coordinates then $\pi \circ \varphi$ is a smooth function of constant rank k and

$$\ker(D(\pi \circ \varphi)_q) = D_q,$$

for any $q \in U_p$. Therefore for any $(z_{k+1}, \dots, z_m) \subset \mathrm{im}(\pi \circ \varphi)$, the subset $(\pi \circ \varphi)^{-1}(z_{k+1}, \dots, z_m)$ is a k-manifold and it is the required leaf. □

REMARK 2.2.7. With some more effort, but no new ideas, one can prove that the leaf passing through a point p is

$$F_p := \{q \in M | \text{there exists a piece-wise smooth integral curve of } D \text{ joining } p \text{ and } q\}.$$

We may use Frobenius Theorem to produce a new point of view on coordinates.

COROLLARY 2.2.8. *Let M be a manifold assume that $\{X_1, \ldots, X_m\}$ are vector fields such that $[X_i, X_j] = 0$ for any pair i, j and $\{X_1(p), \ldots, X_m(p)\}$ is a local basis for T_pM. Then the X_i define local coordinates in a neighborhood of p.*

PROOF. By hypothesis $\{X_1, \ldots, X_m\}$ is a distribution of rank m in a neighborhood of p and by Frobenius it is integrable. Moreover, following the proof of Proposition 2.2.6, this yields a change of coordinate change such that $X_i = \frac{\partial}{\partial z_i}$, for $i = 1, \ldots, m$. □

The above Corollary shifts the attention from coordinates to vector fields. This is sometimes useful when treating special structures, coming from theoretical descriptions, where it is difficult or even not possible to introduce explicit local coordinates.

2.3. Vector bundles

We already realized how useful could be the Tangent bundle of a manifold. Let M be a m-manifold and $\{U_i, \varphi_i\}$ a DS, then

$$TM_{|U_i} \simeq \mathbb{R}^m \times U_i.$$

In particular locally any manifold of dimension m has isomorphic tangent bundle and the geometry of M encoded in TM only depends on the way we glue together these pieces.

This suggests the possibility to define in an abstract way some gluing condition and attach to a manifold M various type of objects like TM. Before plunging in the abstract description let us work out a special example.

2.3.1. Cotangent bundle. Let M be a manifold of dimension m and $f \in C^\infty(M)$ a smooth function. Then we have $f : M \to \mathbb{R}$ and $Df : TM \to \mathbb{R}$. In particular for any $p \in M$ let

$$df(p) := Df_p : T_pM \to \mathbb{R}.$$

Then $df(p)$ is a linear map, that is a linear functional on T_pM. Therefore we may consider

$$df(p) \in T_pM^*.$$

As we did for the tangent bundle we define the set

$$TM^* = \cup_{p \in M} T_pM^*,$$

there is a natural projection $\pi : TM^* \to M$, and df is just a section of π. As we did for TM let us work out a DS to produce a manifold.

Let $\{U_i, \varphi_i\}$ be a DS on M, with local coordinates $(x_1(p), \ldots, x_m(p))$. Then, keep in mind Remark 1.7.16, define

$$(dx_i(p), \ldots, dx_m(p))$$

the dual basis of T_pM^*. It is worthwhile to spend a couple of lines on this dual basis.

Remark 2.3.1. We know that $TU_i = U_i \times \mathbb{R}^m$ and $\{\partial_1, \dots, \partial_m\}$ are vector fields such that for any $p \in U_i$, the set $\{\partial_1(p), \dots, \partial_m(p)\}$ is a basis of T_pM. Therefore we may define

$$dx_i : U_i \to U_i \times (\mathbb{R}^m)^*$$

as

$$dx_i(p)(\partial_j(p)) = \delta_{ij}.$$

The dx_i are sections of the map

$$\pi : \cup_{p \in U_i} T_pM^* \to U_i$$

and $\{dx_1(p), \dots, dx_m(p)\}$ is a basis of T_pM^* for any $p \in U_i$.

Since U_i is a local chart it is easy to see that dx_i are smooth morphism with the usual DS of the product. This also offers a closer look on the differential of a function $f \in C^\infty(U_i)$. For any vector $X(p) \in T_pM$ the element $dx_i(p)$ assigns a number $dx_i(p)(X(p))$ that is the i^{th} component of $X(p)$ in the base $\{\partial_i(p)\}$. For $f \in C^\infty(U_i)$ we have by definition

$$\frac{\partial f}{\partial x_i}(p) = Df_p(\partial_i(p)).$$

Hence we may rewrite df in the local base $\{dx_i\}$ as

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

Since $df(p)$ is a linear form on T_pM we may apply it to a vector field $X \in \mathcal{X}(M)$. We defined $df(p) = Df_p$ hence we have

$$df(p)(X_p) = X_p(f),$$

for $X_p \in D(M)_p$ a derivation. This shows that we may apply df to a vector field $X \in \mathcal{X}(U_i)$ to get a an element in $C^\infty(M)$

$$df(X)(p) := X(p)(f).$$

Thus df is a field of linear functions and $df(X(p))$ is a linear approximation of f in the direction of $X(p)$ In particular in the local expression we found we have

$$df(X)(p) = \sum_i \frac{\partial f}{\partial x_i}(p) dx_i(p)(X(p)).$$

Let us go back to the DS. Note that for any smooth function $F : M \to N$ we have the differential $DF : TM \to TN$. Since $DF_p : T_pM \to T_{f(p)}N$ is a linear map we have the transposed linear map $DF_p^* : T_{F(p)}N^* \to T_pM^*$ where

$$DF_p^*(h)(v) = h(DF_p(v)),$$

and, with the choice of canonical dual basis, DF_p^* is given by the transpose matrix of DF_p. Hence fix a local chart $\varphi_i : U_i \to V$ and dual basis $\{dx_i\}$. Then we define a local chart on TU_i^* by

$$(\varphi_i, (D(\varphi_i^{-1})^t)) : TU_i^* \to U_i \times \mathbb{R}^m.$$

In particular, recalling Equation (5) at page 20, the change of coordinates is given by

$$(\varphi_i \circ \varphi_j^{-1}, (D(\varphi_i^{-1} \circ \varphi_j)^t).$$

We proved the following

PROPOSITION 2.3.2. *Let M be a m-manifold. Then the* **cotangent bundle** TM^* *is a $2m$-manifold and $\pi : TM^* \to M$ is a smooth map.*

As in the Tangent bundle case, sections of cotangent bundle have a geometric meaning.

DEFINITION 2.3.3. A section of $\pi : TM^* \to M$ is called a **differential 1-form**. The space of differential 1-forms is called $\Omega^1(M)$.

REMARK 2.3.4. One forms are given, locally, by

$$\sum a_j(x_1, \dots, x_m)dx_j,$$

for $a_j \in C^\infty(U_i)$. In particular for any $f \in C^\infty(M)$ we may write

$$df = \sum \frac{\partial f}{\partial x_i} dx_i \in \Omega^1(M)$$

this defines the (external) differentiation

$$d : C^\infty(M) \to \Omega^1(M).$$

The image of this map is the set of **exact forms**.

Differential 1-forms, and their friends k-forms obtaining wedging the former, are related to integration on manifolds and Riemannian geometry, see [**2**] for an excellent introduction.

An important, and quite surprising, difference between vector fields and 1-forms is the behaviour with respect to morphisms. We observed that in general it is not possible to define a vector field through a morphism, recall Remark 1.7.26. On the other hand let $F : M \to N$ be a morphism and $\alpha \in \Omega^1(N)$ a 1-form. Then we may define

$$F^*\alpha(v) = \alpha(DF(v)),$$

for $v \in T_pM$, and it is a straightforward check, left to the reader, that $F^*\alpha$ is a 1-form. This produces the pull-back map for 1-forms

$$F^* : \Omega^1(N) \to \Omega^1(M).$$

Further note that this operation commutes with differentiation of functions, that is $d(F^*(f)) = F^*(df)$, where $F^*(f) := f \circ F$.

Via the pull-back it is possible to produce 1-forms on submanifolds of a manifold M. Indeed let $N \subset M$ be a submanifold and $\alpha \in \Omega^1(M)$. Then $i^*\alpha \in \Omega^1(N)$, where $i : N \to M$ is an embedding. In particular via the 1-forms of $\mathbb{R}^N$ we produce 1-forms on a submanifold $N \subset \mathbb{R}^N$. In general the behaviour of $i^*\alpha$ form may be different from that of α.

EXAMPLE 2.3.5. Let $M \subset \mathbb{R}^N$ be a submanifold, and $\alpha = dx_1 \in \Omega^1(\mathbb{R}^N)$. Then α is never zero, that is $\alpha(p)$ is not the zero form for any $p \in \mathbb{R}^N$. On the other hand if $q \in M$ is such that $T_qM \subset (1, 0, \dots, 0)^\perp$, then $i^*\alpha(p)$ is zero. This suggests that pull-back form may be used to study the geometry of submanifolds.

2.3.2. Vector bundles. It is time to provide an abstract description, and hence a generalization, of the bundles we introduced so far.

DEFINITION 2.3.6. Let M and F be manifolds. A (smooth) **fibration** on M with fiber F is

a) a manifold E
b) a morphism $\pi : E \to M$
c) an open covering $\{U_i\}$
d) diffeomorphisms $f_i : \pi^{-1}U_i \to U_i \times F$ that commutes with π

The diffeomorphisms f_i are called **trivializations**. We may, and will, assume that $\{U_i, \varphi_i\}$ are a DS on M.

Note that the diffeomorphism f_i forces $\pi^{-1}(x) \simeq F$ for any $x \in M$. Moreover we have the **transition function** $f_{ij} = f_i \circ f_j^{-1}$ that are diffeomorphisms on $U_{ij} \times F$, where $U_{ij} = U_i \cap U_j$. In particular for any $x \in U_{ij}$ the map $f_{ij|\{x\}\times F}$ is a diffeomorphism of F. The commutation in d) forces also the following **cocycle conditions**

$$f_{ij} = f_{ji}^{-1} \quad f_{ij}f_{jk} = f_{ik}.$$

A **section** of a fibration $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = id_M$.

We will not develop the theory of fibrations in full generality, for this the interested reader may refer to [**3**]. We restrict to vector bundles where both the fibers and the diffeomorphisms are particularly simple.

DEFINITION 2.3.7. A rank k (real) **vector bundle** is a fibration $\pi : E \to M$ with $F \simeq \mathbb{R}^k$ and diffeomorphism

$$f_{ij|\{x\}\times\mathbb{R}^k} \in GL(k, \mathbb{R}), \text{ for any } x \in U_{ij}.$$

REMARK 2.3.8. The manifold $M \times \mathbb{R}^k$ is naturally a vector bundle, called the trivial vector bundle. We may use the DS $\{(U_i, \varphi_i)\}$ to define a DS on $M \times \mathbb{R}^k$ via

$$(\varphi_i, id_{\mathbb{R}^k}) \circ f_i : \pi^{-1}(U_i) \to \mathbb{R}^m \times \mathbb{R}^k.$$

TM and TM^* are m-vector bundles with trivialization given, respectively, by $(\varphi_i, D\varphi_i)$ and $(\varphi_i, (D\varphi_i^{-1})^t)$.

Examples of fibrations, different from vector bundles, are:

- any diffeomorphism is a fibration with fiber a connected 0-manifold,
- the antipodal map $a : S^n \to \mathbb{P}^n_{\mathbb{R}}$ as a fibration with $F = \{p, -p\}$,
- the Hopf fibration

$$h : S^3 \to S^2 \quad (a, b, c, d) \mapsto (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)).$$

  an S^1 fibration over S^2. A way to see this is to consider it on complex numbers, there it can be defined as $h(z_0, z_1) = (|z_0|^2 - |z_1|^2, 2z_0\overline{z}_1)$, where we realize $S^3 := \{(z_0, z_1) \in \mathbb{C}^2 | |z_0|^2 + |z_1|^2 = 1\}$ and $S^2 := \{(x, w) \in \mathbb{R}\times\mathbb{C} | |w|^2 + x^2 = 1\}$. Then it is not difficult to see that $h(z_0, z_1) = h(z_2, z_3)$ if and only if there is a $\lambda \in \mathbb{C}$ with $|\lambda|^2 = 1$ such that $z_2 = \lambda z_0$ and $z_3 = \lambda z_1$. This shows that the fibers are S^1. With more effort one can prove that it is a fibration.

Note that the geometric information carried by a vector bundle are all encoded in its transition functions.

There are some operations we may perform on vector bundles.

2.3.2.1. *Restriction.* Let $\pi : E \to M$ be a k-vector bundle and $N \subset M$ a submanifold. Then $\pi_{|\pi^{-1}(N)} : \pi^{-1}(N) \to N$ is a k-vector bundle on N and the trivialization functions are exactly the same fiberwise.

2.3.2.2. *Product.* Let $E \to M$ and $G \to N$ be rank a and b vector bundles. Then $E \times G \to M \times N$ is naturally a $(a+b)$-vector bundle. When $M = N$ we may go a bit further. Let $\Delta \subset M \times M$ be the diagonal. Then it is easy to see that $\Delta \simeq M$, by projection on one of the factors, therefore we may define, by restriction, $E \times G$ as a rank $(a+b)$ vector bundle on M, this is usually called either the product vector bundle or the direct sum.

2.3.2.3. *Dual, tensor, wedge, sym.* All standard operations on vector spaces can be carried out on vector bundles. We already encountered the dual during the construction of the cotangent bundle. In a similar fashion we may define $E \otimes G$, $\bigwedge^r G$ and $Sym^s(E)$ using as transition functions the corresponding matrices.

2.3.2.4. *Morphisms.* Let $\pi_E : E \to M$ and $\pi_G : G \to N$ be two rank a and b vector bundles. Let $h : E \to G$, be a smooth map such that it induces a smooth function $\tilde{h} : M \to N$, that is $\tilde{h} \circ \pi_E = \pi_G \circ h$. Then $h_x := h_{|F_x} : F_x \simeq \mathbb{R}^a \to F_{\tilde{h}(x)} \simeq \mathbb{R}^b$ is a map for any $x \in M$.

DEFINITION 2.3.9. We say that h is a **vector bundle morphism** if h_x is a linear map for any $x \in M$ and we will say it is ***-jective** if h_x is *-jective. The map h is a vector bundle isomorphism if $\tilde{h}$ is a diffeomorphism and h_x is a linear isomorphism, for any $x \in M$.

REMARK 2.3.10. The differential of a smooth function $Df : TM \to TN$ is a vector bundle morphism. Given a manifold M and two vector bundles $E \to M$ and $G \to M$ of rank a and b. A vector bundle morphism that commutes with id_M is simply given by a smooth function $\psi : M \to M_{a,b}(\mathbb{R})$.

Note that for any vector bundle morphism $h(F_x) \subset F_{\tilde{h}(x)}$ is a vector subspace.

The sum and scalar multiplication on a vector bundle E are vector bundle morphisms.

DEFINITION 2.3.11. Let $h : E/M \to G/M$ be an injective vector bundle morphism inducing the identity, then $h(E) \subset G$ may be seen in a natural way as a subvector bundle A **vector subbundle** is the image of an injective vector bundle morphism, that induces the identity on the base.

Let $E \subset G$ be a vector subbundle of rank $a \leq b$. Then it is natural to consider its quotient Q. Fiberwise the associated vector space is just $Q_x = G_x/E_x$. To define it globally observe that $E \subset G$ is given by a smooth function $q : M \to M_{a,b}(\mathbb{R})$ and for any $x \in M$ the matrix $q(x)$ has a independent columns. Since we may work locally we assume that the first a columns are independent on $W \subset M$ and therefore we may identify Q_x with $\{x_1 = \ldots = x_a = 0\} \subset \mathbb{R}^b$, define locally $Q = \cup Q_x$ together with a map

$$W \times \mathbb{R}^{b-a} \to Q \quad (p, (x_1, \ldots, x_{b-a})) \mapsto (p, (0, \ldots, 0, x_1, \ldots, x_{b-a})).$$

This defines the **quotient bundle**. There is a quotient bundle that is particularly interesting for us. Let $X \subset M$ be a submanifold. Then we have the inclusion embedding $i : X \to M$ that gives as a bundle morphism $Di : TX \to TM$, it is easy to check that it is an injective morphism and moreover if we take the restriction $Di(TX)_{|X}$ we may look at it as a subbundle of $TM_{|X}$. Therefore we have a well defined quotient

$$NX := TM_{|X}/Di(TX),$$

the **normal bundle** of X in M. Note that NX is a vector bundle of rank $m - \dim X$.

REMARK 2.3.12. We can now reinterpret the notion of distribution. A rank k distribution D on a manifold M is a vector subbundle $E \subset TM$ of rank k. The integrability condition is just to say that for any point $p \in M$ there is k-submanifold $N_p \subset M$ such that $D_{|N_p} = TN$.

2.4. Exercises

EXERCISE 2.4.1. Let $X_1 = y^2\partial_x$ and $X_2 = x^2\partial_y$ be two vector field on $\mathbb{R}^2$. Prove that X_1 and X_2 are complete but $X_1 + X_2$ is not complete.

EXERCISE 2.4.2. Let $\{X_1, \dots, X_s\}$ be a local basis for a distribution D. Prove that D is involutive if $[X_i, X_j] \in D$.

EXERCISE 2.4.3. Determine which of the following local bases produce an integrable distribution on an open subset of $\mathbb{R}^3$:

- $\{\partial_x + \partial_y, \partial_z\}$
- $\{5\partial_x, 7\partial_z\}$
- $\{y\partial_x + cos(x)\partial_z - 77\sin(z + y^2)\partial_y\}$
- $\{\partial_x - \partial_y, \partial_z - \partial_x, \partial_y - \partial_x\}$
- $\{\partial_x + y\partial_z, \partial_y\}$
- $\{y\partial_x, x\partial_y\}$

EXERCISE 2.4.4. Let $\mathcal{D}$ be the distribution on $\mathbb{R}^3$ associated to the local basis $\{x_1\partial_2 - x_2\partial_1, \partial_3\}$. Prove that it is integrable and find the leaf of the foliation.

EXERCISE 2.4.5. Let $F : M \to N$ be a surjective map of constant rank. Show that for any $p \in N$ the sets $F^{-1}(p)$ are the leaves of a foliation.

EXERCISE 2.4.6. Show that on $\mathbb{R}$ any 1-form is exact. Produce a non exact one form on S^1.

EXERCISE 2.4.7. Let $\pi : E \to M$ be a rank a vector bundle. Let $f : N \to M$ be a smooth map. Let

$$f^*E := \{(x, v) \in N \times E | f(x) = \pi(v)\} \subset N \times E$$

be the **pull-back** vector bundle. Prove that the projection on the first factor has a natural structure of vector bundle and the projection on the second factor produces a commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array} .$$

Show that if E is trivial then f^*E is trivial.

EXERCISE 2.4.8. Let $G := \{([x], v) \in \mathbb{P}^1_{\mathbb{R}} \times \mathbb{R}^2 | v \in x\}$, prove that it is a manifold. Prove that the canonical projection on the first factor is a vector bundle. Prove that it is not the trivial vector bundle. (same for $\mathbb{P}^n_{\mathbb{R}}$)

EXERCISE 2.4.9. Determine the normal bundle of a plane $P \subset \mathbb{R}^3$ and of $S^2 \subset \mathbb{R}^3$

EXERCISE 2.4.10. Let $\pi : E \to M$ be a rank k-vector bundle. Assume that there are k sections $\{s_1, \dots s_k\}$ such that $\{s_1(x), \dots, s_k(x)\}$ are linearly independent for any $x \in M$. Prove that $E = M \times \mathbb{R}^k$.