3 Quadrics

3.1 Quadratic forms

The projective geometry of quadrics is the geometrical version of the part of linear algebra which deals with symmetric bilinear forms – the generalization of the dot product $\mathbf{a} \cdot \mathbf{b}$ of vectors in $\mathbf{R}^3$. We recall:

Definition 8 *A symmetric bilinear form on a vector space V is a map $B : V \times V \to F$ such that*

- $B(v, w) = B(w, v)$
- $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w)$

The form is said to be nondegenerate if $B(v, w) = 0$ for all $w \in V$ implies $v = 0$.

If we take a basis $v_1, \ldots, v_n$ of V, then $v = \sum_i x_i v_i$ and $w = \sum_i y_i v_i$ so that

$$B(v, w) = \sum_{i,j} B(v_i, v_j) x_i y_j$$

and so is uniquely determined by the symmetric matrix $\beta_{ij} = B(v_i, v_j)$. The bilinear form is nondegenerate if and only if β_{ij} is nonsingular.

We can add symmetric bilinear forms: $(B+C)(v, w) = B(v, w) + C(v, w)$ and multiply by a scalar $(\lambda B)(v, w) = \lambda B(v, w)$ so they form a vector space isomorphic to the space of symmetric $n \times n$ matrices which has dimension $n(n + 1)/2$. If we take a different basis

$$w_i = \sum_j P_{ji} v_j$$

then

$$B(w_i, w_j) = B(\sum_k P_{ki} v_k, \sum_\ell P_{\ell j} v_\ell) = \sum_{k,\ell} P_{ki} B(v_k, v_\ell) P_{\ell j}$$

so that the matrix $\beta_{ij} = B(v_i, v_j)$ changes under a change of basis to

$$\beta' = P^T \beta P.$$

Most of the time we shall be working over the real or complex numbers where we can divide by 2 and then we often speak of the quadratic form $B(v,v)$ which determines the bilinear form since

$$B(v+w,v+w)=B(v,v)+B(w,w)+2B(v,w)$$

Here we have the basic result:

Theorem 10 *Let B be a quadratic form on a vector space V of dimension n over a field F. Then*

- *if $F=\mathbf{C}$, there is a basis such that if $v=\sum_i z_i v_i$*

$$B(v,v)=\sum_{i=1}^{m} z_i^2$$

- *if $F=\mathbf{R}$, there is a basis such that*

$$B(v,v)=\sum_{i=1}^{p} z_i^2-\sum_{i=j}^{q} z_j^2.$$

If B is nondegenerate then $m=n=p+q$.

Proof: The proof is elementary – just *completing the square*. We note that changing the basis is equivalent to changing the coefficients x_i of v by an invertible linear transformation.

First we write down the form in one basis, so that

$$B(v,v)=\sum_{i,j}\beta_{ij}x_ix_j$$

and ask: *is there a term $\beta_{ii}\neq 0$?*. If not, then we create one. If the coefficient of x_ix_j is non-zero, then putting $y_i=(x_i+x_j)/2, y_j=(x_i-x_j)/2$ we have

$$x_ix_j=y_i^2-y_j^2$$

and so we get a term $\beta'_{ii}\neq 0$.

If there is a term $\beta_{ii}\neq 0$, then we note that

$$\frac{1}{\beta_{ii}}(\beta_{i1}x_1+\ldots+\beta_{in}x_n)^2=\beta_{ii}x_i^2+2\sum_{k\neq i}\beta_{ik}x_kx_i+R$$

where R involves the x_k with $k \neq i$. So if

$$y_i = \beta_{i1}x_1 + \ldots + \beta_{in}x_n$$

then

$$B(v,v) = \frac{1}{\beta_{ii}}y_i^2 + B_1$$

where B_1 is a quadratic form in the $n-1$ variables x_k, $k \neq i$.

We now repeat the procedure to find a basis such that if v has coefficients $y_1, \ldots, y_n$, then

$$B(v,v) = \sum_{i=1}^{m} c_i y_i^2.$$

Over **C** we can write $z_i = \sqrt{c_i}y_i$ and get a sum of squares and over **R** we put $z_i = \sqrt{|c_i|}y_i$ to get the required expression. □

Example: Consider the quadratic form in $\mathbf{R}^3$:

$$B(v,v) = x_1x_2 + x_2x_3 + x_3x_1.$$

We put

$$y_1 = (x_1 + x_2)/2, \qquad y_2 = (x_1 - x_2)/2$$

to get

$$B(v,v) = y_1^2 - y_2^2 + x_3(2y_1).$$

Now complete the square:

$$B(v,v) = (y_1 + x_3)^2 - y_2^2 - x_3^2$$

so that with $z_1 = y_1 + x_3, z_2 = y_2, y_3 = x_3$ we have $p = 1, q = 2$.

3.2 Quadrics and conics

Definition 9 *A quadric in a projective space $P(V)$ is the set of points whose representative vectors satisfy $B(v,v) = 0$ where B is a symmetric bilinear form on V. The quadric is said to be nonsingular if B is nondegenerate. The dimension of the quadric is* $\dim P(V) - 1$.

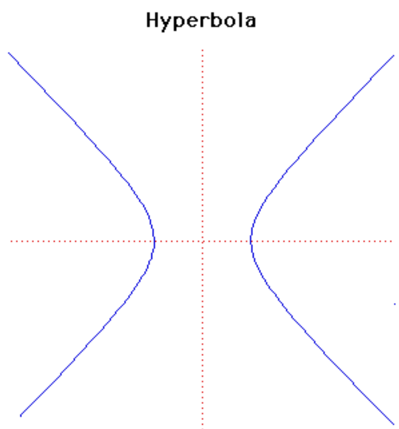
A quadric in a projective line is either empty or a pair of points. A quadric in a projective plane, a one-dimensional quadric, is called a conic.

Note that bilinearity of B means that $B(\lambda v, \lambda v) = \lambda^2 B(v, v)$ so that the set of points $[v] \in P(V)$ such that $B(v, v) = 0$ is well-defined. Also, clearly B and λB define the same quadric. The converse is not true in general, because if $F = \mathbf{R}$ and B is positive definite, then $B(v, v) = 0$ implies $v = 0$ so the quadric defined by B is the empty set. A little later we shall work over the complex numbers in general, as it makes life easier. But for the moment, to get some intuition, let us consider conics in $P^2(\mathbf{R})$ which are non-empty, and consider the intersection with $\mathbf{R}^2 \subset P^2(\mathbf{R})$ defined by the points $[x_0, x_1, x_2]$ such that $x_0 \neq 0$. Using coordinates $x = x_1/x_0, y = x_2/x_0$ this has the equation

$$\beta_{11}x^2 + 2\beta_{12}xy + \beta_{22}y^2 + 2\beta_{01}x + 2\beta_{02}y + \beta_{00} = 0.$$

Examples:

1. Consider the hyperbola $xy = 1$:



In $P^2(\mathbf{R})$ it is defined by the equation

$$x_1x_2 - x_0^2 = 0$$

and the line at infinity $x_0 = 0$ meets it where $x_1x_2 = 0$ i.e. at the two points $[0, 1, 0], [0, 0, 1]$.

Now look at it a different way: as in Theorem 10 we rewrite the equation as

$$(\frac{1}{2}(x_1 + x_2))^2 - (\frac{1}{2}(x_1 - x_2))^2 - x_0^2 = 0$$

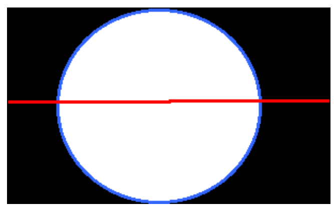
and then if $x_1 + x_2 \neq 0$, we put

$$y_1 = \frac{x_1 - x_2}{x_1 + x_2}, \quad y_2 = \frac{2x_0}{x_1 + x_2}$$

and the conic intersects the copy of $\mathbf{R}^2 \subset P^2(\mathbf{R})$ (the complement of the line $x_1 + x_2 = 0$) in the circle

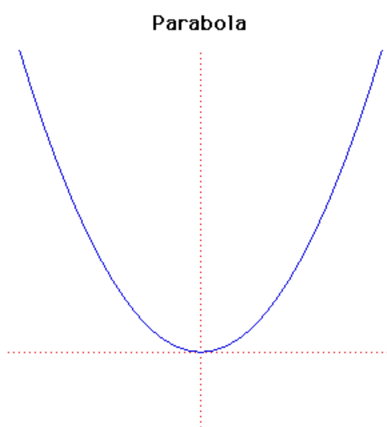
$$y_1^2 + y_2^2 = 1.$$

The original line at infinity $x_0 = 0$ meets this in $y_2 = 0$:



So a projective transformation allows us to view the two branches of the hyperbola as the two semicircles on each side of the line at infinity.

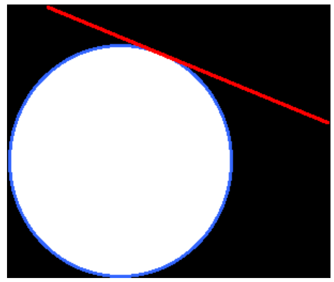
2. Now look at the parabola $y = x^2$.



The equation in homogeneous coordinates is

$$x_2 x_0 = x_1^2$$

and the line $x_0 = 0$ meets it in one point $[0, 0, 1]$. In projective space it still looks like a circle, but the single branch of the parabola is the complement of the point where the line at infinity meets the circle tangentially:



Thus the three different types of conics in $\mathbf{R}^2$ – ellipses, hyperbolas and parabolas – all become circles when we add in the points at infinity to make them sit in $P^2(\mathbf{R})$.

3.3 Rational parametrization of the conic

Topologically, we have just seen that the projective line $P^1(\mathbf{R})$ and a conic in $P^2(\mathbf{R})$ are both homeomorphic to a circle. In fact a much stronger result holds over any field.

Theorem 11 *Let C be a nonsingular conic in a projective plane $P(V)$ over the field F, and let A be a point on C. Let $P(U) \subset P(V)$ be a projective line not containing A. Then there is a bijection*

$$\alpha : P(U) \to C$$

such that, for $X \in P(U)$,the points $A, X, \alpha(X)$ are collinear.

Proof: Suppose the conic is defined by the nondegenerate symmetric bilinear form B. Let $a \in V$ be a representative vector for A, then $B(a, a) = 0$ since A lies on the conic. Let $x \in P(U)$ be a representative vector for $X \in P(U)$. Then a and x are linearly independent since X does not lie on the line $P(U)$. Extend a, x to a basis a, x, y of V.

Now B restricted to the space spanned by a, x is not identically zero, because if it were, the matrix of B with respect to this basis would be of the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}$$

which is singular. So at least one of $B(x, x)$ and $B(a, x)$ is non-zero.

Any point on the line AX is represented by a vector of the form $\lambda a + \mu x$ and this lies on the conic C if

$$0 = B(\lambda a + \mu x, \lambda a + \mu x) = 2\lambda\mu B(a, x) + \mu^2 B(x, x).$$

When $\mu = 0$ we get the point X. The other solution is $2\lambda B(a, x) + \mu B(x, x) = 0$ i.e. the point with representative vector

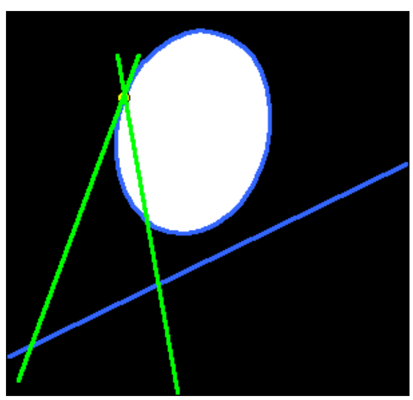
$$w = B(x, x)a - 2B(a, x)x \tag{4}$$

which is non-zero since the coefficients are not both zero.

We define the map $\alpha : P(U) \to C$ by

$$\alpha(X) = [w]$$

which has the collinearity property of the statement of the Theorem. If $Y \in C$ is distinct from A, then the line AY meets the line $P(U)$ in a unique point, so α^{-1} is well-defined on this subset. By the definition of α in (4), $\alpha(X) = A$ if and only if $B(a, x) = 0$. Since B is nonsingular $f(x) = B(a, x)$ is a non-zero linear map from V to F and so defines a line, which meets $P(U)$ in one point. Thus α has a well-defined inverse and is therefore a bijection. □



Remark: There is a more invariant way of seeing this map by using duality. The line A^o in $P(V')$ is dual to the point A. Each point $Y \in A^0$ defines a line Y^o in $P(V)$ through A which intersects the conic C in a second point $\alpha(Y)$. What we do in the more concrete approach of the theorem is to compose this natural bijection with the projective transformation $A^0 \to P(U)$ defined by $Y \mapsto Y^0 \cap P(U)$.

Example: Consider the case of the conic

$$x_0^2 + x_1^2 - x_2^2 = 0.$$

Take $A = [1, 0, 1]$ and the line $P(U)$ defined by $x_0 = 0$. Note that this conic and the point and line are defined over any field since the coefficients are 0 or 1.

A point $X \in P(U)$ is of the form $X = [0, 1, t]$ or $[0, 0, 1]$ and the map α is

$$\begin{aligned} \alpha([0,1,t]) &= [B((0,1,t),(0,1,t))(1,0,1) - 2B((1,0,1),(0,1,t))(0,1,t)] \\ &= [1-t^2, 2t, 1+t^2] \end{aligned}$$

or $\alpha([0, 0, 1]) = [-1, 0, 1]$.

This has an interesting application if we use the field of rational numbers $F = \mathbf{Q}$. Suppose we want to find all right-angled triangles whose sides are of integer length. By Pythagoras, we want to find positive integer solutions to

$$x^2 + y^2 = z^2.$$

But then $[x, y, z]$ is a point on the conic. Conversely, if $[x_0, x_1, x_2]$ lies on the conic, then multiplying by the least common multiple of the denominators of the rational numbers x_0, x_1, x_2 gives integers such that $[x, y, z]$ is on the conic.

But what we have seen is that *any* point on the conic is either $[-1, 0, 1]$ or of the form

$$[x, y, z] = [1 - t^2, 2t, 1 + t^2]$$

for some rational number $t = p/q$, so we get all integer solutions by putting

$$x = q^2 - p^2, \quad y = 2pq, \quad z = q^2 + p^2.$$

For example, $p = 1, q = 2$ gives $3^2 + 4^2 = 5^2$ and $p = 2, q = 3$ gives $5^2 + 12^2 = 13^2$.

One other consequence of Theorem 11 is that we can express a point (x, y) on the general conic

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

in the form

$$x = \frac{p(t)}{r(t)}, \qquad y = \frac{q(t)}{r(t)}$$

where p, q and r are quadratic polynomials in t. Writing x, y as rational functions of t is why the process we have described is sometimes called the *rational parametrization of the conic.* It has its uses in integration. We can see, for example, that

$$\int \frac{dx}{x + \sqrt{ax^2 + bx + c}}$$

can be solved by elementary functions because if $y = x + \sqrt{ax^2 + bx + c}$ then

$$(y - x)^2 - ax^2 - bx - c = 0$$

and this is the equation of a conic. We can solve it by $x = p(t)/r(t), y = q(t)/r(t)$ and with this substitution, the integral becomes

$$\int \frac{r'(t)p(t) - p'(t)r(t)}{q(t)r(t)} dt$$

and expanding the rational integrand into partial fractions we get rational and logarithmic terms after integration.

3.4 Polars

We used a nondegenerate symmetric bilinear form B on a vector space V to define a quadric in $P(V)$ by the equation $B(v, v) = 0$. Such forms also define the notion of orthogonality $B(v, w) = 0$ and we shall see next what geometrically this corresponds to. First the linear algebra: given a subspace $U \subseteq V$ we can define its *orthogonal subspace* $U^\perp$ by

$$U^\perp = \{v \in V : B(u, v) = 0 \text{ for all } u \in U\}.$$

Note that unlike the Euclidean inner product, U and $U^\perp$ can intersect non-trivially – indeed a point with representative vector v lies on the quadric if it is orthogonal to itself. Note also that $U^\perp$ is the same if we change B to λB.

Orthogonal subspaces have a number of properties:

- $U = (U^\perp)^\perp$
- if $U_1 \subseteq U_2$, then $U_2^\perp \subseteq U_1^\perp$
- $\dim U^\perp + \dim U = \dim V$

These can be read off from the properties of the annihilator $U^0 \subseteq V'$, once we realize that a nondegenerate bilinear form on V defines an isomorphism between V and its dual V'. This is the map $\beta(v) = f_v$ where

$$f_v(w) = B(v, w).$$

The map $\beta : V \to V'$ defined this way is obviously linear in v and has zero kernel since $\beta(v) = 0$ implies $B(v, w) = 0$ for all w which means that $v = 0$ by nondegeneracy. Since $\dim V = \dim V'$, β is therefore an isomorphism, and one easily checks that

$$\beta(U^\perp) = U^o.$$

Definition 10 *If $X \in P(V)$ is represented by the one-dimensional subspace $U \subset V$, then the polar of X is the hyperplane $P(U^\perp) \subset P(V)$.*

At this stage, life becomes much easier if we work with the field of complex numbers $F = \mathbf{C}$. We should retain our intuition of conics, for example, as circles but realize that these really are just pictures for guidance. It was Jean-Victor Poncelet (1788-1867) who first systematically started to do geometry over $\mathbf{C}$ (he was also the one to introduce duality) and the simplifications it affords are really worthwhile. Poncelet's work on projective geometry began in Russia. As an officer in Napoleon's army, he was left for dead after the battle of Krasnoe, but was then found and spent several years as a prisoner of war, during which time he developed his mathematical ideas.

We consider then a complex projective plane $P(V)$ with a conic $C \subset P(V)$ defined by a non-degenerate symmetric bilinear form B. A *tangent* to C is a line which meets C at one point.

Proposition 12 *Let C be a nonsingular conic in a complex projective plane, then*

- *each line in the plane meets the conic in one or two points*
- *if $P \in C$, its polar line is the unique tangent to C passing through P*
- *if $P \notin C$, the polar line of P meets C in two points, and the tangents to C at these points intersect at P.*

Proof: Let $U \subset V$ be a 2-dimensional subspace defining the projective line $P(U)$ and let u, v be a basis for U. Then the point $[\lambda u + \mu v]$ lies on the conic if

$$0 = B(\lambda u + \mu v, \lambda u + \mu v) = \lambda^2 B(u, u) + 2\lambda\mu B(u, v) + \mu^2 B(v, v) \tag{5}$$

Over the complex numbers this can be factorized as

$$0 = (a\lambda - b\mu)(a'\lambda - b'\mu)$$

giving the two (possibly coincident) points of intersection of the line and the conic

$$[bu + av], \quad [b'u + a'v].$$

Suppose the point P lies on C, and let u be a representative vector for P, so that $B(u, u) = 0$. Then any line through P is $P(U)$ where U is spanned by u and v. Then from (5) the points of intersection are given by

$$2\lambda\mu B(u, v) + \mu^2 B(v, v) = 0.$$

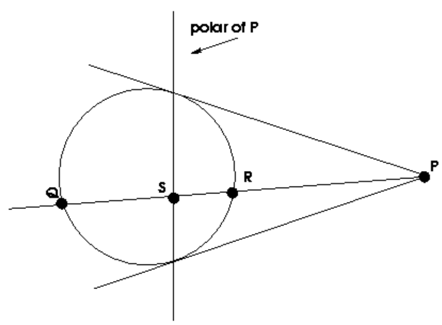
If the only point of intersection is $[u]$ then $\mu = 0$ is the only solution to this equation which means that $B(u, v) = 0$. Since any vector $w \in U$ is a linear combination of u and v and $B(u, u) = B(u, v) = 0$ this means $B(u, w) = 0$ and $P(U)$ is the polar line of X.

From the above, if P does not lie on C, its polar must meet C in two distinct points with representative vectors v_1, v_2. We then have

$$B(u, v_1) = 0 = B(v_1, v_1) \tag{6}$$

Since $B(u, u) \neq 0$ and $B(v_1, v_1) = 0$, u and v_1 are linearly independent and span a 2-dimensional space U_1. From (6) $P(U_1)$ is the polar of $[v_1] \in C$ and hence is the tangent to C at $[v_1]$. Similarly for $[v_2]$. $\square$

The picture to bear in mind is the following real one, but even that does not tell the full story, since if P is inside the circle, its polar line intersects it in two complex conjugate points, so although we can draw the point and its polar, we can't see the two tangents.



Quadrics are nonlinear subsets of $P(V)$ but they nevertheless contain many linear subspaces. For example if $Q \subset P(V)$ is a nonsingular quadric, then $P(U) \subset Q$ if and only if $B(u,u) = 0$ for all $u \in U$. This implies

$$2B(u_1, u_2) = B(u_1 + u_2, u_1 + u_2) - B(u_1, u_1) - B(u_2, u_2) = 0$$

and hence

$$U \subset U^{\perp}.$$

Since

$$\dim U + \dim U^{\perp} = \dim V$$

this means that

$$2 \dim U \leq \dim U + \dim U^{\perp} = \dim V.$$

In fact, over $\mathbf{C}$ the maximum value always occurs. If $\dim V = 2m$, then from Theorem 10 there is a basis in which $B(v,v)$ can be written as

$$x_1^2 + x_2^2 + \ldots + x_{2m}^2 = (x_1 + ix_2)(x_1 - ix_2) + \ldots + (x_{2m-1} + ix_{2m})(x_{2m-1} - ix_{2m})$$

and so if U is defined by

$$x_1 - ix_2 = x_3 - ix_4 = \ldots = x_{2m-1} - ix_{2m} = 0,$$

then $\dim U = \dim V/2$ and $U \subseteq U^{\perp}$ so $U = U^{\perp}$. Over $\mathbf{R}$ this occurs when $p = q = m$ and the form can be reduced to

$$x_1^2 + \ldots + x_m^2 - x_{m+1}^2 \ldots - x_{2m}^2.$$

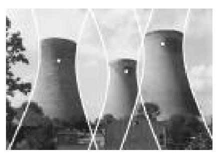
We can see this in more detail for the quadric surface

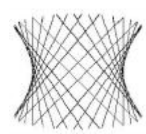
$$x_1^2 - x_2^2 = x_3^2 - x_4^2$$

in $P^3(\mathbf{R})$. This intersects the copy of $\mathbf{R}^3 \subset P^3(\mathbf{R})$ defined by $x_4 \neq 0$ in the hyperboloid of revolution

$$x^2 + y^2 - z^2 = 1$$

where $x = x_2/x_4, y = x_3/x_4, z = x_1/x_4$. This is the usual "cooling tower" shape





There are two one-parameter families of lines in the quadric given by:

$$\begin{aligned} \lambda(x_1 - x_2) &= \mu(x_3 - x_4) \\ \mu(x_1 + x_2) &= \lambda(x_3 + x_4) \end{aligned}$$

and

$$\begin{aligned} \lambda(x_1 - x_2) &= \mu(x_3 + x_4) \\ \mu(x_1 + x_2) &= \lambda(x_3 - x_4). \end{aligned}$$

In fact these two families of lines provide "coordinates" for the projective quadric: the map

$$F : P^1(\mathbf{R}) \times P^1(\mathbf{R}) \to Q$$

defined by

$$F([u_0, u_1], [v_0, v_1]) = [u_0v_0 + u_1v_1, u_1v_1 - u_0v_0, u_0v_1 + u_1v_0, u_1v_0 - u_0v_1]$$

is a bijection.

3.5 Pencils of quadrics

The previous sections have dealt with the geometrical interpretation of a symmetric bilinear form. Now we look at the theory behind a *pair* of bilinear forms and we shall see how the geometry helps us to determine algebraic properties of these.

We saw in Theorem 10 that over $\mathbf{C}$, any quadratic form B can be expressed in some basis as

$$B(v,v) = x_1^2 + \ldots + x_n^2.$$

In particular the matrix of B is diagonal (actually the identity matrix). If we have a pair A, B of symmetric bilinear forms we ask whether we can simultaneously diagonalize them both. The answer is:

Proposition 13 *Let α, β be symmetric $n \times n$ matrices with complex entries, and suppose α is non-singular. Then if the equation $\det(\lambda\alpha - \beta) = 0$ has n distinct solutions $\lambda_1, \ldots, \lambda_n$ there is an invertible matrix P such that*

$$P^T \alpha P = I, \qquad P^T \beta P = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

Proof: From Theorem 10 we can find an invertible matrix Q such that $Q^T \alpha Q = I$. Write $\beta' = Q^T \beta Q$, so that β' is also symmetric. Then

$$\det Q^T \det(\lambda\alpha - \beta) \det Q = \det(Q^T(\lambda\alpha - \beta)Q) = \det(\lambda I - \beta')$$

and so the roots of $\det(\lambda\alpha - \beta) = 0$ are the eigenvalues of β'. By assumption these are distinct, so we have a basis of eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.

If $v_\ell = (x_1, \ldots, x_n)$ and $v_k = (y_1, \ldots, y_n)$ then

$$\lambda_\ell \sum_i x_i y_i = \sum_{i,j} \beta'_{ij} x_j y_i = \sum_{i,j} \beta'_{ji} x_j y_i = \lambda_k \sum_i x_i y_i$$

and since $\lambda_\ell \neq \lambda_k$, we have

$$\sum_i x_i y_i = 0.$$

Thus v_k and v_ℓ are orthogonal if $k \neq \ell$. We also must have $(v_i, v_i) \neq 0$ since otherwise v_i is orthogonal to each element of the basis $v_1, \ldots, v_n$, so we can write

$$w_i = \frac{1}{\sqrt{(v_i, v_i)}} v_i$$

and obtain an orthonormal basis. With respect to this basis, β' is diagonal so if R is the invertible matrix defining the change of basis, $R^T \beta' R = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $R^T R = I$. Putting $P = QR$ we get the result. $\square$

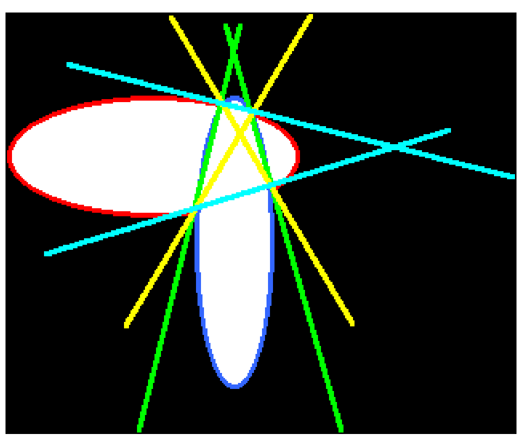
Now let us try and set this in a geometric context. Let A, B be symmetric bilinear forms on a complex vector space V which define different quadrics Q and Q'. Then

Definition 11 *The pencil of quadrics in $P(V)$ generated by Q and Q' is the set of quadrics defined by $\lambda A + \mu B$ where $(\lambda, \mu) \neq (0, 0)$.*

Another way of saying this is to let SV denote the vector space of symmetric bilinear forms on V, in which case a pencil of quadrics is a line in $P(SV)$ – a family of quadrics parametrized by a projective line. The singular quadrics in this pencil are given by the equation $\det(\lambda\alpha + \mu\beta) = 0$. If α is nonsingular then for a solution to this equation $\mu \neq 0$, and so under the hypotheses of the proposition, we can think of the points $[\lambda_i, -1] \in P^1(\mathbf{C})$ as defining n singular quadrics in the pencil. The geometry of this becomes directly visible in the case that $P(V)$ is a plane. In this case a singular quadric has normal form x_1^2 or $x_1^2 + x_2^2 = (x_1 - ix_2)(x_1 + ix_2)$ and so is either a double line or a pair of lines.

Theorem 14 *Let C and C' be nonsingular conics in a complex projective plane and assume that the pencil generated by C and C' contains three singular conics. Then*

- *the pencil consists of all conics passing through four points in general position*
- *the singular conics of the pencil consist of the three pairs of lines obtained by joining disjoint pairs of these four points*
- *each such pair of lines meets in a point with representative vector v_i where $\{v_1, v_2, v_3\}$ is a basis for V relative to which the matrices of C and C' are simultaneously diagonalizable.*



Proof: The proof consists of reducing to normal form and calculating. Since by hypothesis there are three singular conics in the pencil, Proposition 13 tells us that there is a basis in which the two conics are defined by bilinear forms A, B with equations:

$$x_1^2 + x_2^2 + x_3^2 = 0, \qquad \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 0$$

and we directly find the points of intersection are the four points

$$[\sqrt{\lambda_2 - \lambda_3}, \pm\sqrt{\lambda_3 - \lambda_1}, \pm\sqrt{\lambda_1 - \lambda_2}]$$

where we fix one choice of square root of $\lambda_2 - \lambda_3$. To show that these are in general position we need to show that any three are linearly independent, but, for example

$$\det \begin{pmatrix} \sqrt{\lambda_2 - \lambda_3} & \sqrt{\lambda_3 - \lambda_1} & \sqrt{\lambda_1 - \lambda_2} \\ \sqrt{\lambda_2 - \lambda_3} & -\sqrt{\lambda_3 - \lambda_1} & \sqrt{\lambda_1 - \lambda_2} \\ \sqrt{\lambda_2 - \lambda_3} & -\sqrt{\lambda_3 - \lambda_1} & -\sqrt{\lambda_1 - \lambda_2} \end{pmatrix} = 4\sqrt{\lambda_1 - \lambda_2}\sqrt{\lambda_2 - \lambda_3}\sqrt{\lambda_3 - \lambda_1}$$

and since the λ_i are distinct this is non-zero.

Now clearly if $[u]$ is one of these four points $A(u, u) = 0, B(u, u) = 0$ and so $(\lambda A + \mu B)(u, u) = 0$, and every conic in the pencil passes through them. Conversely by Theorem 3 we can take the points to be

$$[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1].$$

The conics which pass through these points have matrices β_{ij} where

$$\beta_{11} = \beta_{22} = \beta_{33} = \beta_{12} + \beta_{23} + \beta_{31} = 0 \tag{7}$$

The vector space of symmetric 3×3 matrices is of dimension 6 spanned by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and the four equations (7) are linearly independent, so define a 2-dimensional space of bilinear forms spanned by A and B – this is the pencil.

We need to understand the singular quadrics in the pencil, for example

$$\lambda_1(x_1^2 + x_2^2 + x_3^2) - (\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2) = (\lambda_1 - \lambda_2)x_2^2 - (\lambda_3 - \lambda_1)x_3^2 = 0$$

But this factorizes as

$$(\sqrt{\lambda_1 - \lambda_2}x_2 - \sqrt{\lambda_3 - \lambda_1}x_3)(\sqrt{\lambda_1 - \lambda_2}x_2 + \sqrt{\lambda_3 - \lambda_1}x_3) = 0.$$

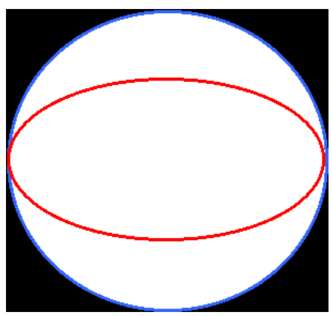
These two lines, by belonging to the pencil, pass through the points of intersection, but since those points are in general position, each line passes through only two of them, so these are the required pairs of lines.

The intersection of $\sqrt{\lambda_1 - \lambda_2}x_2 - \sqrt{\lambda_3 - \lambda_1}x_3 = 0$ and $\sqrt{\lambda_1 - \lambda_2}x_2 + \sqrt{\lambda_3 - \lambda_1}x_3 = 0$ is $[1, 0, 0]$, which is the first basis vector in which the two conics are diagonalized. Similarly the other two pairs of lines give the remaining basis vectors. $\Box$

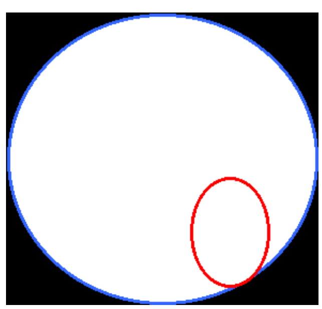
This geometrical approach tells us when two symmetric bilinear forms in three variables can and can not be simultaneously diagonalized. If the values λ_i are not all distinct, and the two forms *can* be simultaneously diagonalized, then they are either multiples of each other or are of the form

$$x_1^2 + x_2^2 + x_3^2 = 0, \qquad \lambda x_1^2 + x_2^2 + x_3^2 = 0.$$

In this case the two conics intersect where $x_1 = 0, x_2 = \pm i x_3$, i.e. at two points:



The non-diagonalizable case is where the intersection is at an odd number of points:



3.6 Exercises

1. Which of the following quadratic forms defines a non-singular conic?

- $x_0^2 - 2x_0x_1 + 4x_0x_2 - 8x_1^2 + 2x_1x_2 + 3x_2^2$
- $x_0^2 - 2x_0x_1 + x_1^2 - 2x_0x_2$.

2. Take five points in a projective plane such that no three are collinear. Show that there exists a unique non-singular conic passing through all five points.

3. Let C be a non-singular conic in a projective plane $P(V)$. Show that if $X \in P(V)$ moves on a fixed line ℓ, then its polar passes though a fixed point Y. What is the relationship between the point Y and the line ℓ?

4. Let $\tau : P^1(\mathbf{R}) \to P^1(\mathbf{R})$ be a projective transformation and consider its graph

$$\Gamma_\tau \subset P^1(\mathbf{R}) \times P^1(\mathbf{R})$$

i.e. $\Gamma_\tau = \{(X, Y) : Y = \tau(X)\}$. Using the one-to-one correspondence between $P^1(\mathbf{R}) \times P^1(\mathbf{R})$ and a quadric surface in $P^3(\mathbf{R})$, show that Γ_τ is the intersection of the quadric surface with a plane.

5. Prove that if $L \subseteq V$ and $M \subseteq V$ are vector subspaces of the same dimension then

$$\dim(L \cap M^\perp) = \dim(L^\perp \cap M).$$

6. Show that the two quadratic forms

$$x^2 + y^2 - z^2, \qquad x^2 + y^2 - yz$$

cannot be simultaneously diagonalized.

7. Let $P^5(\mathbf{R}) = P(\mathbf{R}^6)$ be the space of all conics in $P^2(\mathbf{R})$. Show that the conics which pass through three non-collinear points form a projective plane $P(V) \subset P^5(\mathbf{R})$. Show further that the conics parametrized by this plane and which are tangent to a given line form a conic in $P(V)$.

8. Prove that the set of tangent lines to a nonsingular conic in $P(V)$ is a conic in the dual space $P(V')$.