CHAPTER 3

Gauss map

In this chapter we will study the Gauss map of surfaces in $\mathbb{R}^3$.

3.1. Surfaces in $\mathbb{R}^3$

Let $S \subset \mathbb{R}^3$ be a submanifold of dimension 2. Let $\{U_i, \varphi_i\}$ be a DS on S. For any $p \in U_i$ we have a well defined tangent space T_pS and its orthogonal complement $T_pS^\perp$. The map φ_i^{-1} induces local coordinates (u, v) and a base $(X_u(p), X_v(p))$ for any T_pS. Where X_u, X_v are the vector fields associated to ∂_u and ∂_v respectively, recall Lemma 1.7.3. To uniformize our notation with those classically used for surfaces in $\mathbb{R}^3$ we define $\mathbf{x}(u, v) := \varphi_i^{-1}$, $\mathbf{x}_u := X_u$, $\mathbf{x}_v := X_v$, hence

$$T_{(u,v)}S = \langle \mathbf{x}_u, \mathbf{x}_v \rangle.$$

Define

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{||\mathbf{x}_u \wedge \mathbf{x}_v||}$$

the **normal versor**. This produces a smooth morphism **the Gauss map**

$$N : U_i \to S^2.$$

Remark 3.1.1. Recall that T_pS is independent on the local chart chosen. On the other hand the choice of N is not canonical. We could have chosen

$$\frac{\mathbf{x}_v \wedge \mathbf{x}_u}{||\mathbf{x}_u \wedge \mathbf{x}_v||}$$

instead. That is the normal versor is defined only up to a sign. This is related to the problem of orientability of S. One says that a surface is **orientable** if the Gauss map is defined on all of S. That is it is possible to glue the Gauss maps on local charts to produce a well defined map $N : S \to S^2$. Think at the Möbius band, maybe some Escher picture, to understand the geometric meaning of this notion. Altrenatively this is equivalent to have a trivial Normal bundle.

From now on to simplify the treatment we will assume that $S \subset \mathbb{R}^3$ is an orientable surface, that is there is a well defined Gauss map $N : S \to S^2$. The local description shows that N is differentiable and the differential is $DN : TS \to TS^2$, with

$$DN_p : T_pS \to T_{N(p)}S^2.$$

Note that for any point of S^2 the tangent space is $T_pS = \langle p \rangle^\perp$, recall exercise 1.8.10. Therefore we may interpret DN_p as a linear self map on T_pS. As such we may consider it is a way to measure the way the tangent space is varying in a neighborhood of a point.

For curves the variation of tangent direction is measured by the curvature, a number or a 1x1 matrix. Here we have a two dimensional vector space to control therefore we need a 2x2 matrix. This is what DN_p is devoted to.

From now on we will always consider

$$DN_p : T_pS \to T_pS$$

as a linear endomorphism of T_pS. Our first computation is the following.

LEMMA 3.1.2. *Let $DN_p : T_pS \to T_pS$, then $DN_p(X_u) = N_u(p)$ and $DN_p(X_v) = N_v(p)$.*

PROOF. Let $\alpha(t)$ be an integral curve of X_u with $\alpha(0) = p$. Then

$$DN_p(\mathbf{x}_u) = DN_p(\alpha'(0)) = \frac{d}{dt}N(\alpha(t))_{|t=0} = N_u(p),$$

and similarly for $\mathbf{x}_v$. □

Since a linear map is determined by the image of a basis Lemma 3.1.2 is the local way to determine the differential of the Gauss map. Let us start computing it in some special cases

3.1.0.5. *DN_p of a sphere.* Let $S \subset \mathbb{R}^3$ be a sphere centered at the origin of radius r. We already know that for any point $p \in S$ $T_pS = \langle p \rangle^\perp$ that is

$$N(p) = \frac{p}{||p||} = \frac{1}{r}p,$$

and

$$DN_p = \frac{1}{r}Id.$$

3.1.0.6. *DN_p of a cylinder.* Let $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = r^2\}$, then we may parametrize it via $\varphi^{-1} : \mathbb{R}^2 \to S$ with $\varphi^{-1}(u, v) = (r \cos u, r \sin u, v)$

$$N(x, y, z) = \frac{(x, y, 0)}{r},$$

and

$$DN_p = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & 0 \end{bmatrix}.$$

See more examples in the exercises at the end of the chapter.

DEFINITION 3.1.3. A linear endomorphism $f : \mathbb{R}^s \to \mathbb{R}^s$ is called **self-adjoint** if for any pair of vectors $v, w \in \mathbb{R}^s$

$$\langle f(v) \cdot w \rangle = \langle v \cdot f(w) \rangle,$$

where $\langle \cdot \rangle$ is the euclidean scalar product.

LEMMA 3.1.4. *DN_p is self-adjoint*

PROOF. To prove the claim it is enough to prove that

$$\langle DN_p(\mathbf{x}_u) \cdot \mathbf{x}_v \rangle = \langle \mathbf{x}_v \cdot DN_p(\mathbf{x}_u) \rangle.$$

Since N is orthogonal to X_v and X_u we have

$$0 = \partial_u \langle N \cdot \mathbf{x}_v \rangle = \langle N_u \cdot \mathbf{x}_v \rangle + \langle N \cdot \mathbf{x}_{uv} \rangle$$

and

$$0 = \partial_v \langle N \cdot \mathbf{x}_u \rangle = \langle N_v \cdot \mathbf{x}_u \rangle + \langle N \cdot \mathbf{x}_{vu} \rangle.$$

Therefore we conclude by Lemma 3.1.2 and equality of mixed partials. □

REMARK 3.1.5. Let us recall some important facts of self adjoint operators from linear algebra. Let $A : \mathbb{R}^s \to \mathbb{R}^s$ be a self-adjoint linear operator

a) on any hortonormal base A is represented by a symmetric matrix
b) if a versor v maximize the quantity $\langle A(v)\cdot v\rangle$ then v is an eigenvector of A
c) A can be diagonalized by a hortonormal basis.

Let us see what this means for DN_p. First consider the bilinear symmetric form

$$B : T_pS \times T_pS \to \mathbb{R}$$

given by

$$B(v, w) = \langle v\cdot DN_p(w)\rangle,$$

and the associated quadratic form

$$Q(w) = B(w, w).$$

DEFINITION 3.1.6. In the above notation the **second fundamental form** of the surface S at the point p is

$$II_p(v) = -Q(v) = -\langle v\cdot DN_p(v)\rangle,$$

for $v \in T_pS$.

This lemma explains the minus sign and sheds some light on the matter, I hope.

LEMMA 3.1.7. *Let $\alpha(t)$ be a regular curve with $\alpha(0) = p$ and $\alpha'(0) = v$. Then*

$$II_p(v) = k_\alpha(0)\langle n_\alpha(0)\cdot N(p)\rangle,$$

where k_α and n_α are, respectively, curvature and normal versor of α.

PROOF. We may assume that α is parametrized by arc length. Since

$$\langle \alpha'(t)\cdot N(\alpha(t))\rangle = 0,$$

derivation with respect to t yields

$$0 = \langle \alpha(t)''\cdot N(\alpha(t)\rangle + \langle \alpha'(t)\cdot DN_{\alpha(t)}(\alpha'(t))\rangle,$$

to conclude apply Frenet formulas and our definition

$$k_\alpha(0)\langle n_\alpha(0)\cdot N(p)\rangle - II_p(v) = 0.$$

□

Let α be, as in Lemma, parametrized by arc length, then α' and N are hortogonal versors at any point of $\alpha(t)$. Therefore we may associate a hortonormal moving frame

$$(\alpha', N, \alpha' \wedge N).$$

Since $|\alpha'| = 1$, then α' is hortogonal to α'' and

$$k_\alpha n_\alpha = \alpha'' = k_n N + k_g \alpha' \wedge N.$$

Since the basis is hortonormal we have

$$k_\alpha^2 = k_n^2 + k_g^2,$$

further note that $k_n(0) = \langle \alpha''(0)\cdot N(p)\rangle = II_p(\alpha'(0))$.

DEFINITION 3.1.8. k_n is called the **normal curvature**, k_g is called the **geodesic curvature**.

REMARK 3.1.9. A **geodesic** is a curve with zero geodesic curvature, that is a curve whose normal is parallel to the normal of the surface at any point. We are not going to explore it, but geodesics are local minimum of distance, that is the curve of minimal distance between two points in a neighborhood of a surface.

In this way it is easy to derive Meusnier Theorem

COROLLARY 3.1.10 (Meusnier Theorem). *Let $\alpha(t)$ be a regular curve on a surface S, with $\alpha(0) = p$ and $\alpha'(o) = v$. Then the normal curvature $k_n(0)$depends only on v.*

PROOF. In our notation we have $k_n(0) = k_\alpha(0)\langle n_\alpha(0) \cdot N(p)\rangle = II_p(v)$ □

Let us go a bit further.

LEMMA 3.1.11. *Let $p \in S$ be a point, T_pS the tangent space and $H \ni p$ a plane. If $T_pH \neq T_pS$, then $C := H \cap S$ is a submanifold in a neighborhood of p and $T_pC = T_pH \cap T_pS$. In particular if H is parallel to $N(p)$ the resulting manifold $C := H \cap S$ is called a* **normal section**.

PROOF. We may assume that $H = (z = 0) \ni p = (0, 0, 0)$. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be the projection on the z coordinate. By hypothesis we have $T_xS \not\subset H$. Let $i : S \to \mathbb{R}^3$ be the inclusion map, then $h \circ i : S \to \mathbb{R}$ is a differentiable map of constant rank 1 in a neighborhood of p. Therefore $(h \circ i)^{-1}(0) = H \cap S$ is a 1-manifold in a neighborhood of p and T_pC is the kernel of $D(h \circ i)_p = (z = 0 \cap T_0S)$. □

Let $C = H \cap S$ be a normal section at p. Then $T_pC = T_pH \cap T_pS$ and we may choose a local parametrization by arc length, $\alpha(t)$, with $\alpha(0) = p$ and $n_\alpha(0) = N(p)$. This yields

$$k_n = k_\alpha = II_p(\alpha'(0)).$$

In particular all normal curvature, i.e. the second fundamental form, are encoded in normal sections.

Let $S^1 \subset T_pS$ be the set of versors, and $k_n : S^1 \to \mathbb{R}^1$ the map given by

$$k_n(v) = II_p(v).$$

Since S^1 is compact there is a maximum, say k_1, for $k_n(S^1)$. Let v_1 be such that $k_n(v_1) = k_1$, and v_2 an orthogonal versor. Then by Remark 3.1.5 v_1 is an eigenvector and we may diagonalize DN_p on the basis (v_1, v_2). On the orthonormal basis (v_1, v_2) the matrix of DN_p is given by

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix},$$

that is for $w = av_1 + bv_2 \in T_pS$ we have

$$II_p(w) = a^2k_1 + b^2k_2.$$

When we restrict to versors $v \in S^1$ there is a θ such that $v = \cos\theta v_1 + \sin\theta v_2$. Therefore we have the **Euler formula**

$$k_n(v) = \cos\theta^2 k_1 + \sin\theta^2 k_2, \tag{12}$$

and k_1, k_2 are maximum and minimum of normal curvatures.

DEFINITION 3.1.12. When $k_1 \neq k_2$ the eigenvalues k_1 and k_2 of DN_p are called **principal curvatures** and the eigenversors v_1, v_2 are called **principal directions**. The leaf of a principal direction distribution is called **line of curvature**.

A 2x2 matrix has not many invariants.

DEFINITION 3.1.13. The **Gaussian** curvature at the point p is $K(p) := k_1 k_2 = \det DN_p$, the **mean curvature** is $H(p) = 1/2(k_1 + k_2) = 1/2Trace(DN_p)$.

REMARK 3.1.14. By the examples we already worked out we have:
3.1.0.7. *Sphere.* A radius r sphere has $k_1 = k_2 = 1/r$, therefore $K(p) = 1/r^2$ and $H(p) = 1/r$,
3.1.0.8. *Cylinder.* A radius r cylinder has $k_1 = 1/r$, $k_2 = 0$, therefore $K(p) = 0$ and $H(p) = 1/2r$,
3.1.0.9. *Plane.* A plane has $k_1 = k_2 = 0$ therefore $K(p) = H(p) = 0$.

As we will see in a while the sign and vanishing of $K(p)$ has a geometric meaning.

DEFINITION 3.1.15. Let $p \in S$ be a point. We say that p is
elliptic if $K(p) > 0$
hyperbolic if $K(p) < 0$
parabolic if $K(p) = 0$
umbilical if $k_1 = k_2$
planar if $DN_p \equiv 0$.

The Gaussian curvature encodes both global and local geometric properties of the surface.

PROPOSITION 3.1.16. *Let $p \in l \subset S$ be a smooth point on a line l, then $K(p) \leq 0$.*

PROOF. Let H be a plane containing l and normal to S at p, then

$$0 = k_l = k_n(v),$$

where $v = T_p l$. Therefore p cannot be elliptic. □

Umbilical points can be easily found as follows

LEMMA 3.1.17. *Let $S \subset \mathbb{R}^3$ be a surface then the set of umbilical points is given by the equation $H^2 - K = 0$, for H and K the mean and Gaussian curvature.*

PROOF. The equation $H^2 - K = 0$ translates, in terms of principal curvatures, as

$$(k_1 + k_2)^2 - 4k_1 k_2 = (k_1 - k_2)^2 = 0.$$

Hence it defines the set of points where $k_1 = k_2$. □

Note that by Equation (12) at hyperbolic points there are exactly two directions $v_1, v_2 \in T_p S$ such that $k_n(v_i) = 0$. Moreover $K(p) < 0$ then there is a neighborhood U_p of hyperbolic points. This allows to define the asymptotic curves.

DEFINITION 3.1.18. Let $p \in S$ be a hyperbolic point and v_1, v_2 such that $k_n(v_i) = 0$. Then v_i are called **asymptotic directions**. The leaf of the distribution of an asymptotic direction is called **asymptotic line**.

REMARK 3.1.19. Note that thanks to Frobenius Theorem we know that both lines of curvature (for non umbilical points) and asymptotic lines (for hyperbolic points) exist. These pairs of vector fields are always linearly independents and may be used to define a local parametrization of the surface S. Note that lines of curvature are mutually orthogonal, this is in general not the case for asymptotic lines.

The first global result is the following.

PROPOSITION 3.1.20. *Let $S \subset \mathbb{R}^3$ be a connected surface all of whose points are umbilical then S is contained in either a sphere or a plane.*

PROOF. Since connected manifolds are also path-connected it is enough to prove the statement on a neighborhood of any point. Let $U_p \subset S$ be a local chart with coordinates $\mathbf{x}(u, v)$. Then for any $v = a_1\mathbf{x}_u + a_2\mathbf{x}_v \in T_qS$ we have

$$DN_q(v) = \lambda(q)v,$$

for some smooth map $\lambda : U_p \to \mathbb{R}$, that is

$$N_u a_1 + N_v a_2 = \lambda(a_1\mathbf{x}_u + a_2\mathbf{x}_v).$$

Hence $N_u = \lambda\mathbf{x}_u$ and $N_v = \lambda\mathbf{x}_v$ and differentiating with mixed derivatives we get

$$\lambda_u\mathbf{x}_v - \lambda_v\mathbf{x}_u = 0.$$

The latter forces $\lambda_u = \lambda_v = 0$ and λ is therefore constant. If $\lambda \equiv 0$ then N is constant and it is easy to see, by derivation, that

$$\langle\mathbf{x}(u, v){\cdot}N\rangle = \text{constant},$$

hence U_p is contained in the plane $p + N(p)^\perp$.

To conclude let $\lambda \neq 0$ then, again by derivation,

$$\mathbf{x}(u, v) - \frac{1}{\lambda}N = \text{constant}.$$

Then U_p is contained in the sphere of radius $1/\lambda$ centered in $p - 1/\lambda N$. □

PROPOSITION 3.1.21. *Let $S \subset \mathbb{R}^3$ be a compact connected surface with $K(p) \neq 0$ for any $p \in S$. Then S is diffeomorphic to the sphere. In particular there are not smooth compact surfaces of negative curvature at any point.*

PROOF. By hypothesis the Gauss map $N : S \to S^2$ is well defined and since $K(p) \neq 0$ it is a local diffeomorphism. To conclude we need to prove that it is bijective. We already observed that N is an open map therefore $N(S)$ is open in S^2 and since S is compact $N(S)$ is also closed. This shows that N is surjective. Then N is a covering and since S^2 is simply connected $\sharp N^{-1}(x) = 1$ for any $x \in S^2$. □

Next we show that a compact surfaces always possesses elliptic points.

PROPOSITION 3.1.22. *Let $S \subset \mathbb{R}^3$ be a compact surface. Then there is $p \in S$ with $K(p) > 0$. In particular in Proposition 3.1.21 we have $K(p) > 0$ for any $p \in S$.*

PROOF. Let $x \in S$ such that $||x|| \geq ||p||$ for any $p \in S$. Then the norm function $f(x) = ||x||$ has a maximum at x, therefore $T_xS = \langle x\rangle^\perp$. Then any normal section is a plane curve C with x maximum for the norm function. This shows that k_n has a fixed sign and therefore p is elliptic. □

REMARK 3.1.23. In Proposition 3.1.22 the compactness assumption is needed, think for instance to a plane. It is far more complicate, but possible, to produce examples of non smooth compact surfaces for which all smooth points have negative curvature. A less sophisticated example is given by smooth non compact surfaces of constant negative curvature, see Exercise 3.3.8.

3.1.1. Local equations of DN_p. Let now $\mathbf{x} := \varphi_i^{-1} : \mathbb{R}^2 \to U_p \subset S$ be a coordinate chart, with $\langle \mathbf{x}_u, \mathbf{x}_v \rangle = T_{\mathbf{x}(u,v)}S$, then

$$N(u,v) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{||\mathbf{x}_u \wedge \mathbf{x}_v||}.$$

Recall that by Lemma 3.1.2 $N_u = DN(\mathbf{x}_u)$ and $N_v = DN(\mathbf{x}_v)$. Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^2$ be a curve with $\alpha(0) = p$, then we may consider $\beta = \mathbf{x} \circ \alpha$ to get $\beta(t) = \mathbf{x}(u(t), v(t))$. In this notations

$$II(\beta') = -\langle DN(\beta') \cdot \beta' \rangle = -(u')^2 \langle N_u \cdot \mathbf{x}_u \rangle - 2u'v' \langle N_u \cdot \mathbf{x}_v \rangle - (v')^2 \langle N_v \cdot \mathbf{x}_v \rangle \tag{13}$$

Note that deriving $\langle N(u,v) \cdot \mathbf{x}_u(u,v) \rangle = 0$ and $\langle N(u,v) \cdot \mathbf{x}_v(u,v) \rangle = 0$ we get

$$\langle N_u \cdot \mathbf{x}_u \rangle = -\langle N \cdot \mathbf{x}_{uu} \rangle, \ \langle N_u \cdot \mathbf{x}_v \rangle = \langle N_v \cdot \mathbf{x}_u \rangle = -\langle N \cdot \mathbf{x}_{uv} \rangle, \ \langle N_v \cdot \mathbf{x}_v \rangle = -\langle N \cdot \mathbf{x}_{uu} \rangle.$$

Therefore Equation (13) takes the form

$$II(\beta') = (u')^2 \langle N \cdot \mathbf{x}_{uu} \rangle + 2u'v' \langle N \cdot \mathbf{x}_{uv} \rangle + (v')^2 \langle N \cdot \mathbf{x}_{vv} \rangle \tag{14}$$

Recalling the **first fundamental form** of S we also have

$$I(\beta') = (u')^2 \langle \mathbf{x}_u \cdot \mathbf{x}_u \rangle + 2u'v' \langle \mathbf{x}_u \cdot \mathbf{x}_v \rangle + (v')^2 \langle \mathbf{x}_v \cdot \mathbf{x}_v \rangle,$$

where $I(v)$ is the first fundamental form of S. Classically all these have the following names.

DEFINITION 3.1.24.
$E = \langle \mathbf{x}_u \cdot \mathbf{x}_u \rangle$, $F = \langle \mathbf{x}_u \cdot \mathbf{x}_v \rangle$, $G = \langle \mathbf{x}_v \cdot \mathbf{x}_v \rangle$
$e = -\langle N_u \cdot \mathbf{x}_u \rangle$, $f = -\langle N_u \cdot \mathbf{x}_v \rangle$, $g = -\langle N_v \cdot \mathbf{x}_v \rangle$.

A direct computation furnishes the so called **Weingarten equations**. We have

$$\begin{bmatrix} N_u \\ N_v \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^t \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_v \end{bmatrix},$$

for (a_{ij}) the matrix representing DN with respect to the basis $(\mathbf{x}_u, \mathbf{x}_v)$. Taking the scalar product yields

$$-\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^t \begin{bmatrix} E & F \\ F & G \end{bmatrix},$$

that is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^t = -\begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}.$$

Finally we derive the equations

$$a_{11} = \frac{fF - eG}{EG - F^2}, \ a_{12} = \frac{gF - fG}{EG - F^2}$$
$$a_{21} = \frac{eF - fE}{EG - F^2}, \ a_{22} = \frac{fF - gE}{EG - F^2}$$

and also the expressions of Gaussian and mean curvature

$$K = \frac{eg - f^2}{EG - F^2}, \ H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.$$

Recall that $H^2 - K = \frac{(k_1 - k_2)^2}{4}$ therefore we also have the expression for the principal curvatures

$$k_{1,2} = H \pm \sqrt{H^- K^2}.$$

Local expressions are useful do study the local behaviour of surfaces.

Proposition 3.1.25. *Let $p \in S$ be an elliptic point. Then there is a neighborhood $U_p \subset S$ such that $U_p \cap (p + T_pS) = \{p\}$, that is the surface is, locally, on one side of the tangent space.*

Let $p \in S$ be a hyperbolic point then for any neighborhood U_p the surface is on both side of the plane $p + T_pS$.

Proof. Let $\mathbf{x}(u, v)$ be a parametrization with $p = (0, 0, 0) = \mathbf{x}(0, 0)$ and $T_pS = (z = 0) \subset \mathbb{R}^3$ with coordinate (x, y, z). Fix $N(0) = (0, 0, 1)$, then the behaviour of the point, with respect to the tangent space, is dictated by the z coordinate.

By Taylor's formula we have

$$\mathbf{x}(u, v) = \mathbf{x}_u(0, 0)u + \mathbf{x}_v(0, 0)v + \frac{1}{2}(\mathbf{x}_{uu}(0, 0)u^2 + 2\mathbf{x}_{uv}(0, 0)uv + \mathbf{x}_{vv}(0, 0)v^2) + o(2),$$

thus using Equation (14) we find

$$\langle \mathbf{x}(u, v) \cdot N(0, 0)\rangle = \frac{1}{2} II_{(0,0)}(\mathbf{x}_u(0, 0)u + \mathbf{x}_v(0, 0)v) + o(2).$$

In particular the sign of the z coordinate depends only on the sign of $II_{(0,0)}$. So for an elliptic point the sign is constant and never vanishes in a neighborhood, while for a hyperbolic point it changes. □

Remark 3.1.26. Note that for neither parabolic nor planar point there is anything like this. The cylinder has all points on one side of the tangent space. Plane has all points on the tangent space. While for "monkey saddle"

$$(u, v) \mapsto (u, v, u^3 - 3v^2u),$$

$(0, 0)$ is a planar point and points are on both sides. Similar examples for parabolic points can be described with revolution surfaces.

3.2. Ruled surfaces

In this section we are interested in surfaces covered by lines.

Definition 3.2.1. A one parameter family of lines is a pair of smooth maps $\alpha : I \to \mathbb{R}^3$ and $\tau : I \to \mathbb{R}^3$ together with the map

$$\mathbf{x} : I \times J \to \mathbb{R}^3$$

given by

$$(u, v) \mapsto \alpha(u) + v\tau(u).$$

Assume that $0 \in J$, $|\tau(u)| = 1$, for any $u \in I$ then the image $S := \mathbf{x}(I \times J)$ is called a **ruled surface**. The (portion of) lines $\mathbf{x}_{|\{v\}\times J}$ are called the **rulings** of S while $\mathbf{x}_{|I\times\{0\}}$ is called a **directrix** of S. The surface S is said to be ruled by the map $\mathbf{x}$.

Remark 3.2.2. The simplest examples of ruled surfaces are: planes, cylinders, and cones. Note that we do not ask S to be smooth and in general it is not. On the other hand our assumption $|\tau(u)| \equiv 1$ forces $\langle \tau(u) \cdot \tau'(u)\rangle = 0$. In particular $\mathbf{x}_u = \alpha' + v\tau'$ and $\mathbf{x}_v = \tau$ therefore $\mathbf{x}$ is almost everywhere a smooth parametrization of S and

$$T_{\mathbf{x}(u,v)}S = \langle \alpha' + v\tau', \tau\rangle.$$

The directrix of S is clearly non unique. In our hypothesis, the so called non-cylindrical case when $\tau'(u) \neq 0$ for any $u \in I$, it is possible to introduce a "special" directrix called the **line of striction**. For this define

$$\beta(u) = \alpha(u) - \frac{\langle \alpha'(u) \cdot \tau'(u) \rangle}{\langle \tau'(u) \cdot \tau'(u) \rangle} \tau(u). \tag{15}$$

Then

$$\langle \beta'(u) \cdot \tau'(u) \rangle = 0,$$

for any $u \in I$, see the exercises at the end of the chapter for further details.

Let $p \in S$ be a smooth point of a ruled surface. We already know by Proposition 3.1.16 that $K_p \leq 0$. Moreover by Exercise 3.3.3 the Gaussian curvature vanishes only if the tangent plane is constant along the line.

DEFINITION 3.2.3. A **developable surface** is a ruled surface with fixed tangent plane along the ruling, away from the directrix.

In particular developable surfaces have zero Gaussian curvature.

EXAMPLE 3.2.4. Keeping in mind Remark 3.2.2 we may easily write down two examples of developable surfaces

3.2.0.1. *Cylinders.* $\tau(u) = v$ constant

3.2.0.2. *Cones.* $\alpha(u) = p$ constant

There is a third one which is a bit less immediate.

3.2.0.3. *Tangent developable.* Let $\alpha : I \to \mathbb{R}^3$ be a smooth curve parametrized by arc length with non vanishing curvature. Then the developable surface S associated to α is the ruled surface given by the parametrization $h : I \times \mathbb{R} \to \mathbb{R}^3$, with

$$h(u, v) = \alpha(u) + v\alpha'(u).$$

In particular S is smooth away from the directrix $\alpha(I)$. The points of $\alpha(I)$ can be either smooth or non smooth points of S. In particular along a ruling, away from the directrix, we have

$$T_{\mathbf{x}(\overline{u},v)}S = \langle \alpha'(\overline{u}), \alpha'(\overline{u}) + v\alpha''(\overline{u}) \rangle = \langle \alpha'(\overline{u}), \alpha''(\overline{u}) \rangle.$$

Hence the tangent space is constant along the ruling and $K \equiv 0$ away from the directrix.

We aim to prove the converse of this statement that is

THEOREM 3.2.5. *A surface S with zero Gaussian curvature and no planar points is a developable surface.*

REMARK 3.2.6. Developable surfaces are isometric to a plane, that is can be developed on a plane.

Even if it is not strictly necessary we take this theorem as an excuse to introduce a global point of view on the Gauss mapping using projective geometry. Our next task is therefore to develop the theory of projective spaces to give a proof of Theorem 3.2.5.

3.3. Exercises

Exercise 3.3.1. Compute the image of the Gauss map for the following surfaces:

- $S = \{x^2 + y^2 + z^2 = r^2\}$ (sphere)
- $S = \{x^2 + y^2 = r^2\}$ (cylinder)
- $S = \{z = x^2 - y^2\}$ (hyperbolic paraboloid)

Exercise 3.3.2. Compute the differential of the Gauss map for the following surfaces:

- $S = \{z = x^2 - y^2\}$
- $S = \{z = ax^2 + by^2\}$
- $S = \{z = 0\}$

Exercise 3.3.3. Let $l \subset S$ be a line. Show that the points of l are parabolic if and only if T_pS is constant in the direction of l.

Exercise 3.3.4. Let S be given by the following parametrization

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, av),$$

for some $a \neq 0$. Determine the differential of the Gauss map at any point $p \in S$.

Exercise 3.3.5. Let $S = \{z = xy^2\} \subset \mathbb{R}^3$ show that S is a submanifold and $(0, 0, 0)$ is a planar point.

Exercise 3.3.6. Let $S = \{xyz = 1\} \subset \mathbb{R}^3$ show that S is a submanifold and compute $K(p)$ for any $p \in S$.

Exercise 3.3.7. Let $q \in k[x, y, z]$ be a polynomial of degree 2 and $S = \{q(x, y, z) = 0\} \subset \mathbb{R}^3$. Prove that if S is a submanifold $K(p)K(q) > 0$ for any pair of points $p, q \in S$.

Exercise 3.3.8. Lets now investigate a very interesting surface, called the pseudosphere. It is the surface of revolution obtained by rotating the tractrix about the x-axis, and so it is parametrized by

$$\mathbf{x}(u, v) = (u - tanh(u), sech(u) \cos v, sech(u) \sin v),$$

for $u > 0, v \in [0, 2\pi)$. Note that the circles (of revolution) are lines of curvature and the various tractrices are lines of curvature. In the plane of one tractrix, say t the surface normal and the curve normal agree. Prove that the curvature of the tractrix is $\frac{1}{sinh(u)}$ and $N(p) = -n_t$ therefore $k_1 = -\frac{1}{sinh(u)}$ Prove that the normal curvature of the circle is $sinhu$ (hint: to do this observe that $k_n = k \cos \theta = cosh(u)tanh(u) = sinh(u)$)

Exercise 3.3.9. Give examples of smooth developable surfaces and of singular developable surfaces.

Exercise 3.3.10. Show that the line of striction defined in Equation (15) has the required properties and is unique. Show that a ruled surface is smooth outside the line of striction.