CHAPTER 3

Exam

The exam is a discussion of talks and exercises. Choose from the list either one talk or one starred exercise or two non starred exercises.

TALKS

- Prove [**Be**][Theorems IV.13, IV.16] and the necessary Lemmata.
- Prove [**Be**][Theorem III.4] (need to know cohomology).
- Section 2 of [**Re2**].
- Section 8.3.1 (without Theorem 8.3.6) of [**Do1**].

EXERCISES

. Show that any irreducible rational curve of degree d with a a point of multiplicity $d-1$ can be mapped to a line via a Cremona transformation of $\mathbb{P}^2$

. Let $X \subset \mathbb{P}^n$ be a variety of degree d and dimension k. Show that $d \geq n-k+1$.

* Show that any surface of degree d in P^{d+1} is rational and it is either the Veronese surface in $\mathbb{P}^5$ or $S(1,a)$ for some a.

. Let $S \subset \mathbb{P}^3$ be a quartic surface with 3 double lines meeting in a triple point. Prove that S is a projection of the Veronese surface $V \subset \mathbb{P}^5$.

* Let $S \subset \mathbb{P}^3$ be a cubic surface. Show that there is a set of 12 lines $\{l_1,\ldots,l_6,r_1,\ldots,r_6\}$ such that

$$l_i \cap l_j = r_i \cap r_j = l_i \cap r_j = \emptyset \text{ for } i \neq j,$$

and $l_i \cap r_i$ is a point. Determine how many set of such lines exists on S.

. Let $S \subset \mathbb{P}^3$ be a quartic with a double conic. Show that S is the projection of a del Pezzo surface of degree 4 in $\mathbb{P}^4$.

* Show that any surface with infinitely many (-1)-curves is rational and give an example of such a surface.

. Show that there are irreducible curves $C \subset \mathbb{P}^2$ of degree d with $\frac{(d-1)(d-2)}{2}$ double points.

. Let $S_d \subset \mathbb{P}^n$ be a smooth rational surface of degree d. Prove that $d \geq n-1$. Prove that any surface $S_3 \subset \mathbb{P}^4$ is the blow up of $\mathbb{P}^2$ in a point.

* Prove Castelnuovo Theorem stating that any $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be factored by De Jonquieres transformations and linear automorphisms. A De Jonquieres transformation is associated to the linear system of curves of degree d with a point of multiplicity $d-1$ and $2(d-1)$ simple points.

. Let $\mathcal{L}$ be an homaloidal system. Prove the so called Noether equations.

$$\sum m_i = 3(d-3) \sum m_i^2 = d^2-1,$$

where m_i are the multiplicities of points in $\operatorname{Bs}\mathcal{L}$. Show that for any rational curve $C_d \subset \mathbb{P}^2$ of degree $d \leq 5$ there exists a birational modification $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\omega(C_d)$ is a line.

* Prove that for any degree $d \geq 6$ there are rational curves of degree d that cannot be mapped onto a line by Cremona modifications.

. Let $\mathcal{L}$ be the linear system of quartics through 10 points in $\mathbb{P}^2$. Show that $\varphi_{\mathcal{L}}(\mathbb{P}^2)$ is a sextic surface (Bordiga surface) that contains 10 lines and 10 disjoint plane cubics such that each line meets a single cubic (this is called a double ten).

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