Soluzioni parziale I° turno

1. Esercizio 1
   - $\varphi = \frac{3}{4}\pi$.
   - $w' = \frac{4}{5}\overrightarrow{i} + \frac{2}{5}\overrightarrow{j}$.
   - $w'' = \frac{14}{9}\overrightarrow{i} - \frac{10}{9}\overrightarrow{j} - \frac{17}{9}\overrightarrow{k}$.
2. Esercizio 2
   - $(0,0) \notin W_1$.
   - W_2 non è chiuso rispetto al prodotto.
   - $W_3 \sqsubseteq \mathbb{R}^3$.
3. Esercizio 3
   - $\beta_V = \{(1,0,-1),(0,1,-1)\}$, $\beta_W = \{(1,1,0),(0,0,1)\}$.
   - $V + W = \mathbb{R}^3$.
   - $V + W \neq V \oplus W$.
4. Esercizio 4
   - AB non è invertibile, AC lo è.
   - $(AC)^{-1} = \begin{pmatrix} -1 & -3 & 5 \\ 1 & 4 & -6 \\ 0 & -1 & 2 \end{pmatrix}$.
5. Esercizio 5
   - Per $\alpha = 0$: $\dim W = 2$.
   - Per $\alpha \neq 0$: $\dim W = 3$.
6. Esercizio 6
   - Per $k \neq 0$, $k \neq 1$: $\mathrm{Sol}(S) = \{(\frac{1}{k-1}, -\frac{3}{k-1}, \frac{2}{k-1})\}$.
   - Per $k = 0$: $\mathrm{Sol}(S) = \{(x, 2-x, 2x) : x \in \mathbb{R}\}$.
   - Per $k = 1$: $\mathrm{Sol}(S) = \emptyset$.

CORREZIONE PARZIALE I° TURNO

$u = \hat{j} + \hat{k}$

$v = 2\hat{i} + \hat{j}$

$w = 2\hat{i} - 2\hat{j} - \hat{k}$

1) $\langle u, w \rangle = -2 - 1 = -3$

$|u| = \sqrt{2}$

$|w| = \sqrt{4+4+1} = 3$

$$\cos\varphi = \frac{-\cancel{3}}{\sqrt{2}\ \cancel{3}} = -\frac{1}{\sqrt{2}} \qquad \varphi = \frac{3}{4}\pi$$

2) $|v| = \sqrt{5}$ $\quad \langle v, w \rangle = 4 - 2 = 2$

$$w' = \left\langle w, \frac{v}{|v|} \right\rangle \frac{v}{|v|} = \frac{1}{|v|^2} \langle w, v \rangle v$$

$$= \frac{2}{5}(2\hat{i} + \hat{j}) = \frac{4}{5}\hat{i} + \frac{2}{5}\hat{j}$$

3) $$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -\hat{i} + 2\hat{j} - 2\hat{k}$$

Proiezione di w su $u \times v$

$$w' = \left\langle w, \frac{u \times v}{|u \times v|} \right\rangle \frac{u \times v}{|u \times v|} = \frac{1}{|u \times v|^2} \langle w, u \times v \rangle\, u \times v$$

$|u \times v|^2 = 1 + 4 + 4 = 9$

$$\langle w, u \times v \rangle = \begin{vmatrix} 2 & -2 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} \underset{C_2 \leftarrow C_2 - C_3}{=} \begin{vmatrix} 2 & -1 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} =$$

$$= -\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = -(2+2) = -4$$

(2)

$$w' = -\frac{4}{9}(-\hat{\imath}+2\hat{\jmath}-2\hat{k}) = \cancel{\ldots}$$
$$= \frac{4}{9}\hat{\imath} - \frac{8}{9}\hat{\jmath} + \frac{8}{9}\hat{k}$$

Proiezione di w sul piano generato da u e v

$$w'' = w - w' = (2\hat{\imath} - 2\hat{\jmath} - \hat{k}) - \left(\frac{4}{9}\hat{\imath} - \frac{8}{9}\hat{\jmath} + \frac{8}{9}\hat{k}\right)$$
$$= \frac{14}{9}\hat{\imath} - \frac{10}{9}\hat{\jmath} - \frac{17}{9}\hat{k}$$

ES 2

$0 \notin W_1$

W_2 non è chiuso rispetto al prodotto scalare-vettore

$W_3 \subseteq \mathbb{R}^3$

ES 3

$V = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$

$W = \{(x,y,z) \in \mathbb{R}^3 : x=y\}$

$$V = \{(x,y,-x-y) : x,y \in \mathbb{R}\} =$$
$$= \{x(1,0,-1) + y(0,1,-1) : x,y \in \mathbb{R}\} =$$
$$= [(1,0,-1),(0,1,-1)]$$

$$W = \{(x,x,z) : x,z \in \mathbb{R}\} = [(1,1,0),(0,0,1)]$$

$\mathcal{G} = \{(1,0,-1),(0,1,-1),(1,1,0),(0,0,1)\}$ è un insieme di generatori per $V+W$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1-(-1) =$$
$$= 2 \neq 0$$

$B = \{(1,0,-1),(0,1,-1),(1,1,0)\}$ è una base per W_1+W_2 ③

$\Rightarrow W_1+W_2 = \mathbb{R}^3$

$$\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
$$= 4-3 = 1 \neq 0$$

$\Rightarrow W_1+W_2 \neq W_1 \oplus W_2$

ES 4

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det B = 0$

$\det(AB) = \det A \det B = 0$

$$AC = \begin{pmatrix} -2 & -1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\det AC = \begin{vmatrix} -2 & -1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} \underset{\substack{C_1 \leftarrow C_1 - 2C_2 \\ C_3 \leftarrow C_3 + 2C_2}}{=} \begin{vmatrix} 0 & -1 & 0 \\ -2 & 2 & 5 \\ -1 & 1 & 3 \end{vmatrix} =$$

$$= \begin{vmatrix} -2 & 5 \\ -1 & 3 \end{vmatrix} = -6+5 = -1 \neq 0$$

$\Rightarrow A \cdot C$ è invertibile

$$AC_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$AC_{12} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$AC_{13} = \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0$$

$$AC_{21} = -\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = 3$$

$$AC_{22} = \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = -4$$

$$AC_{23} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} = -(-2+1) = +1$$

$$AC_{31} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$$

$$AC_{32} = -\begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -(-2-4) = 6$$

$$AC_{33} = \begin{vmatrix} -2 & -1 \\ 2 & 2 \end{vmatrix} = -4+2 = -2$$

$$\mathrm{adj}_J(AC) = \begin{pmatrix} 1 & -1 & 0 \\ 3 & -4 & 1 \\ -5 & 6 & -2 \end{pmatrix} \qquad \mathrm{adj}_J(AC)^T = \begin{pmatrix} 1 & 3 & -5 \\ -1 & -4 & 6 \\ 0 & 1 & -2 \end{pmatrix}$$

$$(AC)^{-1} = \frac{1}{\det(AC)}\, \mathrm{adj}_J(AC)^T =$$

$$= \begin{pmatrix} -1 & -3 & 5 \\ 1 & 4 & -6 \\ 0 & -1 & 2 \end{pmatrix}$$

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$v_1 = (0, 1, \alpha, -3)$

$v_2 = (2, 1, -\alpha, 1)$

$v_3 = (\alpha - 1, 2\alpha, 0, -2)$

$W = [v_1, v_2, v_3]$

$$B = \begin{pmatrix} 0 & 2 & \alpha - 1 \\ 1 & 1 & 2\alpha \\ \alpha & -\alpha & 0 \\ -3 & 1 & -2 \end{pmatrix}$$

$$\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2 \neq 0 \qquad r(B) \geq 2$$

$$\begin{vmatrix} 0 & 2 & \alpha - 1 \\ 1 & 1 & 2\alpha \\ \alpha & -\alpha & 0 \end{vmatrix} \underset{C_2 \leftarrow C_2 + C_1}{=\!=} \begin{vmatrix} 0 & 2 & \alpha - 1 \\ 1 & 2 & 2\alpha \\ \alpha & 0 & 0 \end{vmatrix} =$$

$$= \alpha \begin{vmatrix} 2 & \alpha - 1 \\ 2 & 2\alpha \end{vmatrix} = 2\alpha \begin{vmatrix} 1 & \alpha - 1 \\ 1 & 2\alpha \end{vmatrix}$$

$$= 2\alpha(2\alpha - \alpha + 1) = 2\alpha(\alpha + 1)$$

$$\begin{vmatrix} 0 & 2 & \alpha - 1 \\ 1 & 1 & 2\alpha \\ -3 & 1 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2\alpha \\ -3 & -2 \end{vmatrix} + (\alpha - 1) \begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} =$$

$$= -2(-2 + 6\alpha) + 4(\alpha - 1)$$

$$= \cancel{4} - 12\alpha + 4\alpha - \cancel{4} = -8\alpha$$

Per $a=0$: $\tau(B)=2 \Rightarrow \dim W = 2$.

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Per $a \neq 0$: $\tau(B)=3 \Rightarrow \dim W = 3$

ES 6

$$\begin{pmatrix} 1 & 1 & k \\ k^2 & k & k \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ k \\ 0 \end{pmatrix}$$

$$\det A_k = \begin{vmatrix} 1 & 1 & k \\ k^2 & k & k \\ -2 & 0 & 1 \end{vmatrix} \xlongequal[C_1 \leftarrow C_1 + 2C_3]{} \begin{vmatrix} 1+2k & 1 & k \\ k^2+2k & k & k \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+2k & 1 \\ k^2+2k & k \end{vmatrix} = k + 2k^2 - k^2 - 2k =$$

$$= k^2 - k = k(k-1)$$

Per $k \neq 0$, $k \neq 1$: il sistema è di Cramer

$$x = \frac{\begin{vmatrix} 2 & 1 & k \\ k & k & k \\ 0 & 0 & 1 \end{vmatrix}}{k(k-1)} = \frac{\cancel{k}\begin{vmatrix} 2 & 1 & k \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}}{\cancel{k}(k-1)} = \frac{\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}}{k-1} = \frac{1}{k-1}$$

$$y = \frac{\begin{vmatrix} 1 & 2 & k \\ k^2 & k & k \\ -2 & 0 & 1 \end{vmatrix}}{k(k-1)} = \frac{\cancel{k}\begin{vmatrix} 1 & 2 & k \\ k & 1 & 1 \\ -2 & 0 & 1 \end{vmatrix}}{\cancel{k}(k-1)} \underset{C_1 \leftarrow C_1 + 2C_3}{=} \frac{\begin{vmatrix} 1+2k & 2 & k \\ k+2 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}}{k-1}$$

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$$= \frac{\begin{vmatrix} 1+2k & 2 \\ k+2 & 1 \end{vmatrix}}{k-1} = \frac{1+\cancel{2k}-\cancel{2k}-4}{k-1} = -\frac{3}{k-1}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 2 \\ k^2 & k & k \\ -2 & 0 & 0 \end{vmatrix}}{k(k-1)} = \frac{-2\begin{vmatrix} 1 & 2 \\ k & k \end{vmatrix}}{k(k-1)} = \frac{-2\cancel{k}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}{\cancel{k}(k-1)} =$$

$$= \frac{-2(1-2)}{k-1} = \frac{2}{k-1}$$

$$Sol(S) = \left\{ \left(\frac{1}{k-1}, -\frac{3}{k-1}, \frac{2}{k-1} \right) \right\}$$

<u>Per $k = 0$</u>

$$A_0 = \begin{pmatrix} 1 & \boxed{1 & 0} \\ 0 & 0 & 0 \\ -2 & \boxed{0 & 1} \end{pmatrix} \qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$$r(A_0) = 2$$

$$B_0 = (A_0 \ \ b) = \begin{pmatrix} 1 & \boxed{1 & 0} & 2 \\ 0 & 0 & 0 & 0 \\ -2 & \boxed{0 & 1} & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

Tutti gli orlati di $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ sono nulli

$\Rightarrow r(B_0) = r(A_0) = 2 \Rightarrow$ il sistema è possibile

$$\begin{cases} x+y=2 \\ -2x+z=0 \end{cases} \qquad \begin{cases} y=2-x \\ z=2x \end{cases}$$

(8)

$Sol(S)=\{(x,2-x,2x) : x\in\mathbb{R}\}$ ∞^1 soluzioni

Per $k=1$

$$A_1=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{pmatrix} \qquad \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}=1 \qquad r(A_1)=2$$

$$B_1=\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$$

$r(B_1)=3\neq r(A_1) \Rightarrow$ il sistema è impossibile

Riassumendo:

Per $k\neq 0, k\neq 1$: il sistema ammette una ed una sola soluzione e
$Sol(S)=\left\{\left(\frac{1}{k-1}, -\frac{3}{k-1}, \frac{2}{k-1}\right)\right\}$

Per $k=0$: ∞^1 soluzioni
$Sol(S)=\{(x,2-x,2x) : x\in\mathbb{R}\}$

Per $k=1$: il sistema è impossibile $Sol(S)=\emptyset$