1. Determinare la proiezione del vettore $u = (1,1,1)$ nel piano contenente i vettori $a = (2,1,0)$ e $b = (1,0,1)$.

$$v = a \times b = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = i - 2j - k \qquad \|v\|^2 = 1+4+1 = 6$$

$$W = u - \frac{\langle v, u \rangle v}{\|v\|^2}$$

$$= (1,1,1) - \frac{1}{6}(1-2-1)(1,-2,-1)$$

$$= \left(1+\frac{1}{3},\ 1-\frac{2}{3},\ 1-\frac{1}{3}\right) = \left(\frac{4}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

2) $A = \left[\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right]$ $\qquad B = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right]$

Per determinare dim A, si calcola il rango di

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \qquad r(M) \leq 3$$

$$\left| \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \right| = 2 \cdot (2) \neq 0 \qquad r(M) = 3 \Rightarrow \dim A = 3$$

Analogamente si determina dim B, calcolando il rango di

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad r(N) \leq 2 \qquad r(N) = 2 \Rightarrow \dim B = 2$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \neq 0$$

$$A + B = \left[\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] \subseteq \mathbb{R}^4$$

$\Rightarrow$ non può avere dimensione maggiore di 4

$$\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$$

Si determina $\dim(A+B)$

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} = -1 \begin{vmatrix} 0 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix} = -1\,(2) = -2 \neq 0$$

$\Rightarrow \dim(A+B) = 4$

Pertanto

$$\dim(A\cap B) = \dim A + \dim(B) - \dim(A+B)$$
$$= 3 + 2 - 4 = 1$$

$\Rightarrow A+B$ non è somma diretta perché $A\cap B \neq \{0\}$

3) Risoluzione (se possibile) del sistema $A\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = b$

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 2 & 1 & 1 & -1 \\ 1 & 2 & -1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$r(A) \leq 3$

$$\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$-3 - 2(-3) - 3 = -3 + 6 - 3 = 0$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3 \neq 0$$

$\Rightarrow r(A) = 3$

Si determina $r(A|b)$

$$(A|b) = \begin{pmatrix} 1 & -1 & 2 & 0 & 1 \\ 2 & 1 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 & 1 \end{pmatrix}$$

Non può essere maggiore di 3

$\Rightarrow r(A) = r(A|b) = 3$

Ci sono $\infty^{4-3} = \infty^1$ soluzioni

$$\begin{cases} x - y = 1 - 2z \\ 2x + y - u = 0 - z \\ x + 2y = 1 + z \end{cases}$$

$$x = \frac{\begin{vmatrix} 1-2z & -1 & 0 \\ -z & 1 & -1 \\ 1+z & 2 & 0 \end{vmatrix}}{3} = \frac{\begin{vmatrix} 1-2z & -1 \\ 1+z & 2 \end{vmatrix}}{3} = \frac{2 - 4z + 1 + z}{3} = -z + 1$$

$$y=\frac{\begin{vmatrix}1 & 1-2z & 0\\ 2 & -z & -1\\ 1 & 1+z & 0\end{vmatrix}}{3}=\frac{\begin{vmatrix}1 & 1-2z\\ 1 & 1+z\end{vmatrix}}{3}=\frac{\cancel{1}+z-\cancel{1}+2z}{3}=z$$

$$u=\frac{\begin{vmatrix}1 & -1 & 1-2z\\ 2 & 1 & -z\\ 1 & 2 & 1+z\end{vmatrix}}{3}=\frac{1}{3}\left[\begin{vmatrix}1 & -z\\ 2 & 1+z\end{vmatrix}-2\begin{vmatrix}-1 & 1-2z\\ 2 & 1+z\end{vmatrix}+\begin{vmatrix}-1 & 1-2z\\ 1 & -z\end{vmatrix}\right]=$$

$$=\frac{1}{3}\left(1+z+2z-2(\overbrace{-1-z-2+4z}^{3z-3})+z-1+2z\right)$$

$$=\frac{1}{3}\left(6z-6z+6\right)=2$$

$$\begin{pmatrix}x\\ y\\ z\\ u\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\\ 2\end{pmatrix}+z\begin{pmatrix}-1\\ 1\\ 1\\ 0\end{pmatrix}$$

4) $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad F(x,y) = (x-y, 0, 2x-2y)$

$G: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad G(x,y,z) = (x, 0, x+z)$

Si determina $\mathrm{Imm}(F)$.

$$\mathrm{Imm}\,F = \left\{ \begin{pmatrix} x-y \\ 0 \\ 2x-2y \end{pmatrix}, \ x, y \in \mathbb{R} \right\}$$

Data la matrice associata a F rispetto alle basi canoniche, $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix}$, la matrice ha rango 1. Dunque $\mathrm{Imm}\,F = \left[\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right]$ e $\dim \mathrm{Imm}\,F = 1$

Si determina $\mathrm{Ker}(F)$.

$$\mathrm{Ker}\,F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{matrix} x-y=0 \\ 0=0 \\ 2x-2y=0 \end{matrix} \right\} = \left[\begin{pmatrix} x \\ x \end{pmatrix} \right] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

$\dim \mathrm{Ker}\,F = 1$ infatti $\dim \mathbb{R}^2 - \dim \mathrm{Imm}\,F = 2 - 1 = 1$

$$G \circ F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x-y \\ 0 \\ 2x-2y \end{pmatrix} \longrightarrow \begin{pmatrix} x-y \\ 0 \\ x-y-2x+2y \end{pmatrix} = \begin{pmatrix} x-y \\ 0 \\ -x+y \end{pmatrix}$$

$G \circ F$ ha matrice associata data da

$$M(G) \cdot M(F) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}$$ che è esattamente la matrice associata a $G \circ F : \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x-y \\ 0 \\ y-x \end{pmatrix}$$

5. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$B = \{(1,1), (-1,0)\} \quad A = M^B_B(f) = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$$

$$M^C_C(f) = ?$$

$$M^C_C(f) = M^B_C(i)\, M^B_B(f)\, M^C_B(i)$$

$$M^B_C(i) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$M^C_B(i) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$M^C_C(f) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & 1 \end{pmatrix}$$

6). $A=\begin{pmatrix}2&-2&0\\-3&3&0\\0&0&-1\end{pmatrix}$

Determinare autovalori e autospazi

$$|\lambda I - A| = \begin{vmatrix}\lambda-2 & 2 & 0\\ 3 & \lambda-3 & 0\\ 0 & 0 & \lambda+1\end{vmatrix} = (\lambda+1)\left[(\lambda-2)(\lambda-3)-6\right]$$

$$=(\lambda+1)(\lambda^2+6-5\lambda-6)=$$

$$=(\lambda+1)\,\lambda\,(\lambda-5)$$

$\lambda=-1$ m.alg.$=1$

$\lambda=0$ m.alg.$=1$

$\lambda=5$ m.alg.$=1$

La matrice è diagonalizzabile (I criterio)

$$V_{-1}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{matrix}-3x+2y=0\\3x-4y=0\end{matrix}\right\}=\left\{\begin{matrix}x=y=0\\z\end{matrix}\right\}=\left[\begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

$$V_0=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{matrix}-2x+2y=0\\3x-3y=0\\z=0\end{matrix}\right\}=\left\{\begin{matrix}z=0\\x=y\end{matrix}\right\}=\left[\begin{pmatrix}1\\1\\0\end{pmatrix}\right]$$

$$V_5=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{matrix}3x+2y=0\\3x+2y=0\\6z=0\end{matrix}\right\}=\left\{\begin{matrix}z=0\\x=-\frac{2}{3}y\end{matrix}\right\}=\left[\begin{pmatrix}-2\\3\\0\end{pmatrix}\right]$$

$AM=MD$ $\quad M=\begin{pmatrix}0&1&-2\\0&1&3\\1&0&0\end{pmatrix}$ $\quad D=\begin{pmatrix}-1&0&0\\0&0&0\\0&0&5\end{pmatrix}$

7. Determinare il segno della forma quadratica

$$q(x,y,z) = x^2 + 2y^2 + 2xz + z^2$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-2 & 0 \\ -1 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\left((\lambda-2)(\lambda-1)\right) - (\lambda-2)$$

$$= (\lambda-2)\left[(\lambda-1)^2 - 1\right] =$$

$$= (\lambda-2)(\lambda^2 + 1 - 2\lambda - 1)$$

$$= (\lambda-2)\lambda(\lambda-2)$$

$\lambda = 0$ m.alg. 1

$\lambda = 2$ m.alg 2

q è semidefinito positivo

$$V_0 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{matrix} -x-z=0 \\ -2y=0 \\ -x-z=0 \end{matrix} \right\} = \left\{ \begin{matrix} y=0 \\ x=-z \end{matrix} \right\} = \left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right]$$

$$V_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - z = 0 \right\} = \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right]$$

Base ortonormale: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

8) Sia r con parametri direttori $v = (4, -1, 0)$ passante per $P = (0, 3, 0)$

Equazioni parametriche

$$\begin{cases} x = 4t \\ y = -t + 3 \\ z = 0 \end{cases}$$

Equazioni cartesiane

$$\begin{cases} \frac{x}{4} = 3 - y \\ z = 0 \end{cases} \Rightarrow \begin{cases} x = 12 - 4y \\ z = 0 \end{cases}$$

$Q \equiv (1, 4, 0)$ non appartiene alla retta

Sia s la retta di equazione

$$s \begin{cases} x = t - 1 \\ y = 3t \\ z = 0 \end{cases}$$

Le due rette non sono parallele perché v e $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ non sono paralleli.

Il punto di intersezione è $(0, 3, 0)$