1. Dati i punti $A=(1,2,3)$, $B=(2,2,3)$ $C=(2,3,4)$, determinare l'angolo tra i vettori $\vec{AB}$ e $\vec{AC}$.

$\vec{AB}=(1,0,0)$ $\qquad$ $\vec{AC}=(1,1,1)$

$$\cos\theta = \frac{\langle \vec{AB}, \vec{AC}\rangle}{\|\vec{AB}\|\,\|\vec{AC}\|} =$$

ove θ angolo tra $\vec{AB}$ e $\vec{AC}$

$$= \frac{1}{\sqrt{3}} \qquad \theta = \operatorname{acos}\left(\frac{1}{\sqrt{3}}\right)$$

Calcolare la proiezione ortogonale di $\vec{AC}$ su $\vec{AB}$

$$v = \left\langle \vec{AC}, \frac{\vec{AB}}{\|\vec{AB}\|} \right\rangle \frac{\vec{AB}}{\|\vec{AB}\|} = \frac{1}{1}(1,0,0) = (1,0,0)$$

2) Siano

$v_1 = (-1, 2, 3)$

$v_2 = (0, -1, 0)$

$v_3 = (-1, 1, 3)$

Sia $V = [v_1, v_2, v_3]$. Per determinare la dim V, si determina il rango di A

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ 3 & 0 & 3 \end{pmatrix} \qquad |A| = -1 \begin{vmatrix} -1 & -1 \\ 3 & 3 \end{vmatrix} = 0$$

$r(A) = 2$

$\Rightarrow$ dim $V = 2$ $\qquad$ base $= \{v_1, v_2\}$

Sia $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; y = 0, x = z \right\} = \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$

esso ha dimensione 1 e base $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Sottospazio somma

$$V + U = \left[\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$$

Si calcola la dimensione

$$\begin{vmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = -1(-1-3) = 4 \neq 0$$

dim $V + U = 3$

$\Rightarrow$ $U + V$ è somma diretta

Poiché $3 = \dim V + U = \dim V + \dim U + \dim (U \cap V)$
$= 2 + 1 + \dim (U \cap V)$

3).

$$\begin{cases} x+2y+3z=k+1 \\ 4x+5y+6z=k \\ 7x+8y+9z=k+1 \end{cases}$$

$$|A|=\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 45-48-2(36-42)+3(32-35)$$

$$= -3+12-9=0$$

$r(A)=2$

$$(A|b) = \begin{bmatrix} 1 & 2 & 3 & k+1 \\ 4 & 5 & 6 & k \\ 7 & 8 & 9 & k+1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & k+1 \\ 4 & 5 & k \\ 7 & 8 & k+1 \end{vmatrix} = \begin{vmatrix} 5 & k \\ 8 & k+1 \end{vmatrix} - 2\begin{vmatrix} 4 & k \\ 7 & k+1 \end{vmatrix} + (k+1)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 5k+5-8k-2(4k+4-7k)+(k+1)(-3)$$

$$= \cancel{-3k}+5-8+\cancel{6k}\cancel{-3k}-3$$

$$= -6 \neq 0$$

$r(A|b)=3$ per ogni valore di k

Il sistema è incompatibile per ogni valore di k.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 9z = 0 \end{cases}$$

$r(A) = 2$ - Il sistema omogeneo ha ∞ soluzioni

$\dim(\mathrm{Ker}\, A) = 1$

$$\begin{cases} x + 2y = -3z \\ 4x + 5y = -6z \end{cases}$$

$$x = \frac{\begin{vmatrix} -3z & 2 \\ -6z & 5 \end{vmatrix}}{-3} = \frac{-15z + 12z}{-3} = z$$

$$y = \frac{\begin{vmatrix} 1 & -3z \\ 4 & -6z \end{vmatrix}}{-3} = \frac{-6z + 12z}{-3} = -2z$$

Soluzioni : $z\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

4. $f: R^4 \longrightarrow R^3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 - x_2 \\ 2x_1 - x_2 + x_4 \\ -x_1 + x_2 + x_4 \end{pmatrix}$$

Si determina

Ker f, Imm f

La $A = M^C_C(f) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix}$ è la matrice

associata.

$r(A) \leq 3$

$\begin{vmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -2 + 3 = 1 \neq 0 \quad \Rightarrow r(A) = \dim \mathrm{Imm} f = 3$

base di Imm $f = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\dim \mathrm{Ker} f = 4 - 3 = 1$

$\mathrm{Ker} f = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \begin{matrix} x_1 - x_2 = 0 \\ 2x_1 - x_2 + x_4 = 0 \\ -x_1 + x_2 + x_4 = 0 \end{matrix} \right\} = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right]$

L'applicazione è suriettiva ma non iniettiva.

La controimmagine di $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ è data da

$$\begin{cases} x_1 - x_2 = 0 \\ 2x_1 - x_2 + x_4 = 2 \\ -x_1 + x_2 + x_4 = 1 \end{cases} \Rightarrow$$

$$x_1 = \frac{\begin{vmatrix} 0 & -1 & 0 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{1} = 1$$

$$x_2 = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix}}{1} = 1$$

$$x_4 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 2 & -1 & 2 \\ -1 & 1 & 1 \end{vmatrix}}{1} = \frac{-3+4}{1} = 1$$

x_3 qualunque

$$f^{-1}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 1 \\ x_3 \\ 1 \end{pmatrix} \right\}$$

5)

$f: R^4 \to R^2$ definito da

$$A = M^{B_1}_{B_2}(f) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$$

$$B_1 = \{(1,1,0,1), (1,0,0,0), (2,0,1,0), (0,1,0,0)\}$$

$$B_2 = \{(1,-1), (1,0)\}$$

$$M^C_C(f) = M^{B_2}_C(i) \cdot M^{B_1}_{B_2}(f)\, M^C_{B_1}(i)$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \overset{t}{a} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \overset{x-t-2z}{b} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \overset{z}{c} \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \overset{y-t}{d} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x = a + b + 2c \\ y = a + d \\ z = c \\ t = a \end{cases} \Rightarrow \begin{cases} x - t = b + 2z \\ y - t = d \\ c = z \\ a = t \end{cases} \quad \begin{cases} a = t \\ b = x - t - 2z \\ c = z \\ d = y - t \end{cases}$$

$$M^C_{B_1}(i) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$M^{B_2}_C(i) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow M^C_C(f) = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$f: \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \longrightarrow \begin{pmatrix} x + y - 2t \\ -y \end{pmatrix}$$

6)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

autovalori:

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 0 & \lambda-2 & 1 \\ -1 & 2 & \lambda-1 \end{vmatrix} = (\lambda-2)\left[(\lambda-2)(\lambda-1)-2\right]$$

$$= (\lambda-2)\left(\lambda^2 - 3\lambda + \cancel{2} - \cancel{2}\right)$$

$$= (\lambda-2)\,\lambda\,(\lambda-3)$$

$\lambda_1 = 2$ m. alg $=1$

$\lambda_2 = 0$ " " $=1$

$\lambda_3 = 3$ " " $=1$

autovalori distinti $\Rightarrow$ la matrice è diagonalizzabile

$$V_{\lambda_1} = \left\{ \begin{array}{l} z = 0 \\ -x+2y+z=0 \end{array} \right\} = \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right]$$

$$V_{\lambda_2} = \left\{ \begin{array}{l} -2x = 0 \\ -2y+z=0 \\ -x+2y-z=0 \end{array} \right\} = \left[\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right]$$

$$V_{\lambda_3} = \left\{ \begin{array}{l} x = 0 \\ y+z=0 \\ -x+2y+2z=0 \end{array} \right\} = \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right]$$

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \quad \text{t.c.} \quad AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

7. Sia $V = \mathcal{L}\left[\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]$.

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$

Occorre trovare una base ortonormale

$v_1' = v_1$ $\qquad\qquad \|v_1'\| = \sqrt{4} = 2$

$$v_2' = v_2 - \frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle} v_1' =$$

$$= \begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{\cancel{2}\,1}{\cancel{4}\,2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\1/2\\-1/2\end{pmatrix} \qquad \|v_2'\| = \sqrt{1} = 1$$

$$v_3' = v_3 - \frac{\langle v_3, v_1'\rangle}{\langle v_1', v_1'\rangle} v_1' - \frac{\langle v_3, v_2'\rangle}{\langle v_2', v_2'\rangle} v_2'$$

$$= \begin{pmatrix}0\\0\\0\\1\end{pmatrix} - \frac{1}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{+1/2}{1}\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\1/2\\-\frac{1}{2}\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{4}+\frac{1}{4}\\-\frac{1}{4}-\frac{1}{4}\\-\frac{1}{4}+\frac{1}{4}\\1-\frac{1}{4}-\frac{1}{4}\end{pmatrix} = \begin{pmatrix}0\\-\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix} \qquad \|v_3'\| = \sqrt{\frac{1}{2}}$$

Base ortogonale

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \begin{pmatrix}1/2\\-\frac{1}{2}\\1/2\\-1/2\end{pmatrix} \begin{pmatrix}0\\-\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}$$

Base ortonormale

$$\underset{v_1''}{\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}} \underset{v_2''}{\begin{pmatrix}1/2\\-\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}} \underset{v_3''}{\begin{pmatrix}0\\-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}}$$

Per completare la base, basta aggiungere

$$v_4'' = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Infatti $\langle v_3'', v_4'' \rangle = 0$

$\langle v_1'', v_4'' \rangle = 0$

$\langle v_2'', v_4'' \rangle = 0$

e $\| v_4'' \| = 1$

8. ~~D~~eterminare il piano π ortogonale alla retta $r: x+y-4=2x-z-4=0$ e passante per $(0,1,1)$

$r: \begin{cases} y=-x+4 \\ z=2x-4 \end{cases}$ $\quad$ $\begin{cases} x=t \\ y=-t+4 \\ z=2t-4 \end{cases}$ $\quad$ $(1,-1,2)$ parametri direttori $\perp$ al piano

$$\underset{\substack{\parallel \\ 1}}{a}(x-0)+\underset{\substack{\parallel \\ -1}}{b}(y-1)+\underset{\substack{\parallel \\ 2}}{c}(z-1)=0$$

$$x-y+1+2z-2=0$$

$$x-y+2z-1=0$$

Equazioni della retta s passante per $(1,0,1)$ e $(1,2,2)$

La retta ha parametri $(1-1,2-0,2-1)$
$=(0,2,1)$

$s\begin{cases} x=1. \\ y=+2t \\ z=1+t \end{cases}$ $\quad$ eq. parametriche

Eq. cartesiane

$\dfrac{y}{2}=z-1 \qquad x=1 \quad \Rightarrow \quad \begin{matrix} y-2z+2=0 \\ x-1=0 \end{matrix}$

Il piano è ortogonale a $(1,-1,2)$

La retta s ha parametri $(0,2,1)$

Siccome $\langle(1,-1,2)(0,2,1)\rangle = 0$, s e π sono paralleli.

Se la retta appartiene al piano, $(1,0,1)$ [e $(1,2,2)$] appartengono al piano

$1+1+2-2 \neq 0$

$1-2+1+4-2 \neq 0$

La retta non appartiene al piano.