Chapter 30

Fundamental Integration Formulas

IF $F(x)$ IS A FUNCTION whose derivative $F'(x) = f(x)$ on a certain interval of the x axis, then $F(x)$ is called an *antiderivative* or *indefinite integral* of $f(x)$. The indefinite integral of a given function is not unique; for example, x^2, $x^2 + 5$, and $x^2 - 4$ are all indefinite integrals of $f(x) = 2x$, since $\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2 - 4) = 2x$. All indefinite integrals of $f(x) = 2x$ are then included in $F(x) = x^2 + C$, where C, called the *constant of integration*, is an arbitrary constant.

The symbol $\int f(x)\, dx$ is used to indicate the indefinite integral of $f(x)$. Thus we write $\int 2x\, dx = x^2 + C$. In the expression $\int f(x)\, dx$, the function $f(x)$ is called the *integrand*.

FUNDAMENTAL INTEGRATION FORMULAS. A number of the formulas below follow immediately from the standard differentiation formulas of earlier chapters, while others may be checked by differentiation. Formula 25, for example, may be checked by showing that

$$\frac{d}{dx}\left(\frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C\right) = \sqrt{a^2 - x^2}$$

Absolute value signs appear in certain of the formulas. For example, for formula 5 we write $\int \frac{dx}{x} = \ln |x| + C$ instead of

$$\int \frac{dx}{x} = \ln x + C \text{ for } x > 0 \qquad \text{and} \qquad \int \frac{dx}{x} = \ln(-x) + C \text{ for } x < 0$$

and for formula 10 we have $\int \tan x\, dx = \ln |\sec x| + C$ instead of

$$\int \tan x\, dx = \ln \sec x + C \qquad \text{for all } x \text{ such that } \sec x \geq 1$$

and

$$\int \tan x\, dx = \ln(-\sec x) + C \qquad \text{for all } x \text{ such that } \sec x \leq -1$$

1. $\int \frac{d}{dx}[f(x)]\, dx = f(x) + C$
2. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$
3. $\int af(x)\, dx = a \int f(x)\, dx$, a any constant
4. $\int x^m\, dx = \frac{x^{m+1}}{m+1} + C, \quad m \neq -1$
5. $\int \frac{dx}{x} = \ln |x| + C$
6. $\int a^x\, dx = \frac{a^x}{\ln a} + C, \quad a > 0, a \neq 1$
7. $\int e^x\, dx = e^x + C$
8. $\int \sin x\, dx = -\cos x + C$

9. $\int \cos x\, dx = \sin x + C$

10. $\int \tan x\, dx = \ln|\sec x| + C$

11. $\int \cot x\, dx = \ln|\sin x| + C$

12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

13. $\int \csc x\, dx = \ln|\csc x - \cot x| + C$

14. $\int \sec^2 x\, dx = \tan x + C$

15. $\int \csc^2 x\, dx = -\cot x + C$

16. $\int \sec x \tan x\, dx = \sec x + C$

17. $\int \csc x \cot x\, dx = -\csc x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C$

21. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$

22. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$

23. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + C$

24. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln|x + \sqrt{x^2 - a^2}| + C$

25. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C$

26. $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \ln(x + \sqrt{x^2 + a^2}) + C$

27. $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln|x + \sqrt{x^2 - a^2}| + C$

THE METHOD OF SUBSTITUTION. To evaluate an antiderivative $\int f(x)\, dx$, it is often useful to replace x with a new variable u by means of a *substitution* $x = g(u)$, $dx = g'(u)\, du$. The equation

$$\int f(x)\, dx = \int f(g(u))g'(u)\, du \qquad (30.1)$$

is valid. After finding the right side of *(30.1)*, we replace u with $g^{-1}(x)$; that is, we obtain the result in terms of x. To verify *(30.1)*, observe that, if $F(x) = \int f(x)\, dx$, then $\frac{d}{du} F(x) = \frac{d}{dx} F(x) \frac{dx}{du} = f(x)g'(u) = f(g(u))g'(u)$. Hence, $F(x) = \int f(g(u))g'(u)\, du$, which is *(30.1)*.

EXAMPLE 1: To evaluate $\int (x + 3)^{11}\, dx$, replace $x + 3$ with u; that is, let $x = u - 3$. Then $dx = du$, and we obtain

$$\int (x + 3)^{11}\, dx = \int u^{11}\, du = \tfrac{1}{12} u^{12} + C = \tfrac{1}{12}(x + 3)^{12} + C$$

QUICK INTEGRATION BY INSPECTION. Two simple formulas enable us to find antiderivatives almost immediately. The first is

$$\int g'(x)[g(x)]^r\, dx = \frac{1}{r + 1} [g(x)]^{r+1} + C \quad r \neq -1 \qquad (30.2)$$

This formula is justified by noting that $\frac{d}{dx}\left\{\frac{1}{r+1}[g(x)]^{r+1}\right\} = g'(x)[g(x)]^r$.

EXAMPLE 2: (*a*) $\int \frac{(\ln x)^2}{x}\,dx = \int \frac{1}{x}(\ln x)^2\,dx = \frac{1}{3}(\ln x)^3 + C$

(*b*) $\int x\sqrt{x^2+3}\,dx = \frac{1}{2}\int (2x)(x^2+3)^{1/2}\,dx = \frac{1}{2}\left[\frac{1}{3/2}(x^2+3)^{3/2}\right] + C = \frac{1}{3}[\sqrt{x^2+3}]^3 + C$

The second quick integration formula is

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C \qquad (30.3)$$

This formula is justified by noting that $\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}$.

EXAMPLE 3: (*a*) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$

(*b*) $\int \frac{x^2}{x^3-5}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3-5}\,dx = \frac{1}{3}\ln|x^3-5| + C$

Solved Problems

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In Problems 1 to 8, evaluate the indefinite integral at the left.

1. $\int x^5\,dx = \frac{x^6}{6} + C$

2. $\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

3. $\int \sqrt[3]{z}\,dz = \int z^{1/3}\,dz = \frac{z^{4/3}}{4/3} + C = \frac{3}{4}z^{4/3} + C$

4. $\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-2/3}\,dx = \frac{x^{1/3}}{1/3} + C = 3x^{1/3} + C$

5. $\int (2x^2 - 5x + 3)\,dx = 2\int x^2\,dx - 5\int x\,dx + 3\int dx = \frac{2x^3}{3} - \frac{5x^2}{2} + 3x + C$

6. $\int (1-x)\sqrt{x}\,dx = \int (x^{1/2} - x^{3/2})\,dx = \int x^{1/2}\,dx - \int x^{3/2}\,dx = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$

7. $\int (3s+4)^2\,ds = \int (9s^2 + 24s + 16)\,ds = 9(\frac{1}{3}s^3) + 24(\frac{1}{2}s^2) + 16s + C = 3s^3 + 12s^2 + 16s + C$

8. $\int \frac{x^3 + 5x^2 - 4}{x^2}\,dx = \int (x + 5 - 4x^{-2})\,dx = \frac{1}{2}x^2 + 5x - \frac{4x^{-1}}{-1} + C = \frac{1}{2}x^2 + 5x + \frac{4}{x} + C$

9. Evaluate (*a*) $\int (x^3+2)^2(3x^2)\,dx$, (*b*) $\int (x^3+2)^{1/2}x^2\,dx$, (*c*) $\int \frac{8x^2\,dx}{(x^3+2)^3}$, and (*d*) $\int \frac{x^2\,dx}{\sqrt[4]{(x^3+2)}}$ by means of (*30.2*).

(a) $\int (x^3+2)^2(3x^2)\,dx = \frac{1}{3}(x^3+2)^3 + C$

(b) $\int (x^3+2)^{1/2}x^2\,dx = \frac{1}{3}\int (x^3+2)^{1/2}(3x^2)\,dx = \frac{1}{3}\frac{2}{3}(x^3+2)^{3/2} + C = \frac{2}{9}(x^3+2)^{3/2} + C$

(c) $\int \frac{8x^2}{(x^3+2)^3}\,dx = \frac{8}{3}\int (x^3+2)^{-3}(3x^2)\,dx = \frac{8}{3}\left(-\frac{1}{2}\right)(x^3+2)^{-2} + C = -\frac{4}{3}\frac{1}{(x^3+2)^2} + C$

(d) $\int \frac{x^2}{\sqrt[4]{x^3+2}}\,dx = \frac{1}{3}\int (x^3+2)^{-1/4}(3x^2)\,dx = \frac{1}{3}\frac{4}{3}(x^3+2)^{3/4} + C = \frac{4}{9}(x^3+2)^{3/4} + C$

All four integrals can also be evaluated by making the substitution $u = x^3 + 2$, $du = 3x^2\,dx$.

10. Evaluate $\int 3x\sqrt{1-2x^2}\,dx$.

Formula (*30.2*) yields

$$\int 3x\sqrt{1-2x^2}\,dx = 3(-\tfrac{1}{4})\int (1-2x^2)^{1/2}(-4x)\,dx = -\tfrac{3}{4}\tfrac{2}{3}(1-2x^2)^{3/2} + C$$
$$= -\tfrac{1}{2}(1-2x^2)^{3/2} + C$$

We could also use the substitution $u = 1 - 2x^2$, $du = -4x\,dx$.

11. Evaluate $\int \frac{(x+3)\,dx}{(x^2+6x)^{1/3}}$.

Formula (*30.2*) yields

$$\int \frac{(x+3)\,dx}{(x^2+6x)^{1/3}} = \frac{1}{2}\int (x^2+6x)^{-1/3}(2x+6)\,dx = \frac{1}{2}\frac{3}{2}(x^2+6x)^{2/3} + C$$
$$= \frac{3}{4}(x^2+6x)^{2/3} + C$$

We could also use the substitution $u = x^2 + 6x$, $du = (2x+6)\,dx$.

In Problems 12 to 15, evaluate the indefinite integral on the left.

12. $\int \sqrt[3]{1-x^2}\,x\,dx = -\frac{1}{2}\int (1-x^2)^{1/3}(-2x\,dx) = -\frac{1}{2}\frac{3}{4}(1-x^2)^{4/3} + C = -\frac{3}{8}(1-x^2)^{4/3} + C$

13. $\int \sqrt{x^2-2x^4}\,dx = \int (1-2x^2)^{1/2}x\,dx = -\frac{1}{4}\int (1-2x^2)^{1/2}(-4x\,dx) - \frac{1}{4}\frac{2}{3}(1-2x^2)^{3/2} + C$

$\quad = -\frac{1}{6}(1-2x^2)^{3/2} + C$

14. $\int \frac{(1+x)^2}{\sqrt{x}}\,dx = \int \frac{1+2x+x^2}{x^{1/2}}\,dx = \int (x^{-1/2} + 2x^{1/2} + x^{3/2})\,dx = 2x^{1/2} + \frac{4}{3}x^{3/2} + \frac{2}{5}x^{5/2} + C$

15. $\int \frac{x^2+2x}{(x+1)^2}\,dx = \int \left[1 - \frac{1}{(x+1)^2}\right]dx = x + \frac{1}{x+1} + C' = \frac{x^2}{x+1} + 1 + C' = \frac{x^2}{x+1} + C$

FORMULAS 5 TO 7

16. Evaluate $\int dx/x$.

Formula 5 gives $\int \frac{dx}{x} = \ln|x| + C$.

17. Evaluate $\int \frac{dx}{x+2}$, using (*30.3*).

$\int \frac{dx}{x+2} = \ln|x+2| + C$. We also could use formula 5 and the substitution $u = x + 2$, $du = dx$.

18. Evaluate $\int \frac{dx}{2x-3}$, using (*30.3*).

$\int \frac{dx}{2x-3} = \frac{1}{2}\int \frac{2\,dx}{2x-3} = \frac{1}{2}\ln|2x-3| + C$. Another method is to make the substitution $u = 2x - 3$, $du = 2\,dx$.

In Problems 19 to 27, evaluate the integral at the left.

19. $$\int \frac{x\,dx}{x^2-1} = \frac{1}{2}\int \frac{2x\,dx}{x^2-1} = \frac{1}{2}\ln|x^2-1| + C = \frac{1}{2}\ln|x^2-1| + \ln c = \ln\left(c\sqrt{|x^2-1|}\right),\ c > 0$$

20. $$\int \frac{x^2\,dx}{1-2x^3} = -\frac{1}{6}\int \frac{-6x^2\,dx}{1-2x^3} = -\frac{1}{6}\ln|1-2x^3| + C = \ln\frac{c}{\sqrt[6]{|1-2x^3|}},\ c > 0$$

21. $$\int \frac{x+2}{x+1}\,dx = \int\left(1 + \frac{1}{x+1}\right)dx = x + \ln|x+1| + C$$

22. $$\int e^{-x}\,dx = -\int e^{-x}(-dx) = -e^{-x} + C$$

23. $$\int a^{2x}\,dx = \frac{1}{2}\int a^{2x}(2\,dx) = \frac{1}{2}\frac{a^{2x}}{\ln a} + C$$

24. $$\int e^{3x}\,dx = \frac{1}{3}\int e^{3x}(3\,dx) = \frac{e^{3x}}{3} + C$$

25. $$\int \frac{e^{1/x}\,dx}{x^2} = -\int e^{1/x}\left(-\frac{dx}{x^2}\right) = -e^{1/x} + C$$

26. $$\int (e^x+1)^3 e^x\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(e^x+1)^4}{4} + C,$$ where $u = e^x + 1$ and $du = e^x\,dx$, or

$$\int (e^x+1)^3 e^x\,dx = \int (e^x+1)^3\,d(e^x+1) = \frac{(e^x+1)^4}{4} + C$$

27. $$\int \frac{dx}{e^x+1} = \int \frac{e^{-x}\,dx}{1+e^{-x}} = -\int \frac{-e^{-x}\,dx}{1+e^{-x}} = -\ln(1+e^{-x}) + C = \ln\frac{e^x}{1+e^x} + C$$
$$= x - \ln(1+e^x) + C$$

The absolute-value sign is not needed here because $1 + e^{-x} > 0$ for all values of x.

FORMULAS 8 TO 17

In Problems 28 to 47, evaluate the integral at the left.

28. $$\int \sin\tfrac{1}{2}x\,dx = 2\int(\sin\tfrac{1}{2}x)(\tfrac{1}{2}\,dx) = -2\cos\tfrac{1}{2}x + C$$

29. $\int \cos 3x\,dx = \frac{1}{3}\int (\cos 3x)(3\,dx) = \frac{1}{3}\sin 3x + C$

30. $\int \sin^2 x \cos x\,dx = \int \sin^2 x(\cos x\,dx) = \dfrac{\sin^3 x}{3} + C$

31. $\int \tan x\,dx = \int \dfrac{\sin x}{\cos x}\,dx = -\int \dfrac{-\sin x\,dx}{\cos x} = -\ln|\cos x| + C = \ln|\sec x| + C$

32. $\int \tan 2x\,dx = \frac{1}{2}\int (\tan 2x)(2\,dx) = \frac{1}{2}\ln|\sec 2x| + C$

33. $\int x \cot x^2\,dx = \frac{1}{2}\int (\cot x^2)(2x\,dx) = \frac{1}{2}\ln|\sin x^2| + C$

34. $\int \sec x\,dx = \int \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + C$

35. $\int \sec\sqrt{x}\,\dfrac{dx}{\sqrt{x}} = 2\int (\sec x^{1/2})(\frac{1}{2}x^{-1/2}\,dx) = 2\ln|\sec\sqrt{x} + \tan\sqrt{x}| + C$

36. $\int \sec^2 2ax\,dx = \dfrac{1}{2a}\int (\sec^2 2ax)(2a\,dx) = \dfrac{\tan 2ax}{2a} + C$

37. $\int \dfrac{\sin x + \cos x}{\cos x}\,dx = \int (\tan x + 1)\,dx = \ln|\sec x| + x + C$

38. $\int \dfrac{\sin y\,dy}{\cos^2 y} = \int \tan y \sec y\,dy = \sec y + C$

39. $\int (1 + \tan x)^2\,dx = \int (1 + 2\tan x + \tan^2 x)\,dx = \int (\sec^2 x + 2\tan x)\,dx$

$= \tan x + 2\ln|\sec x| + C$

40. $\int e^x \cos e^x\,dx = \int (\cos e^x)(e^x\,dx) = \sin e^x + C$

41. $\int e^{3\cos 2x}\sin 2x\,dx = -\dfrac{1}{6}\int e^{3\cos 2x}(-6\sin 2x\,dx) = -\dfrac{e^{3\cos 2x}}{6} + C$

42. $\int \dfrac{dx}{1 + \cos x} = \int \dfrac{1 - \cos x}{1 - \cos^2 x}\,dx = \int \dfrac{1 - \cos x}{\sin^2 x}\,dx = \int (\csc^2 x - \cot x \csc x)\,dx$

$= -\cot x + \csc x + C$

43. $\int (\tan 2x + \sec 2x)^2\,dx = \int (\tan^2 2x + 2\tan 2x \sec 2x + \sec^2 2x)\,dx$

$= \int (2\sec^2 2x + 2\tan 2x \sec 2x - 1)\,dx = \tan 2x + \sec 2x - x + C$

44. $\int \csc u\,du = \int \dfrac{du}{\sin u} = \int \dfrac{du}{2\sin\frac{1}{2}u\cos\frac{1}{2}u} = \int \dfrac{(\sec^2\frac{1}{2}u)(\frac{1}{2}\,du)}{\tan\frac{1}{2}u} = \ln|\tan\frac{1}{2}u| + C$

45. $\int (\sec 4x - 1)^2\, dx = \int (\sec^2 4x - 2\sec 4x + 1)\, dx = \frac{1}{4}\tan 4x - \frac{1}{2}\ln|\sec 4x + \tan 4x| + x + C$

46. $\int \frac{\sec x \tan x\, dx}{a + b\sec x} = \frac{1}{b}\int \frac{(\sec x \tan x)(b\, dx)}{a + b\sec x} = \frac{1}{b}\ln|a + b\sec x| + C$

47. $\int \frac{dx}{\csc 2x - \cot 2x} = \int \frac{\sin 2x\, dx}{1 - \cos 2x} = \frac{1}{2}\int \frac{(\sin 2x)(2\, dx)}{1 - \cos 2x} = \frac{1}{2}\ln(1 - \cos 2x) + C'$

$$= \frac{1}{2}\ln(2\sin^2 x) + C' = \frac{1}{2}(\ln 2 + 2\ln|\sin x|) + C' = \ln|\sin x| + C$$

FORMULAS 18 TO 20

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In Problems 48 to 72, evaluate the integral at the left.

48. $\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$

49. $\int \frac{dx}{1 + x^2} = \arctan x + C$

50. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \operatorname{arcsec} x + C$

51. $\int \frac{dx}{\sqrt{4 - x^2}} = \arcsin \frac{x}{2} + C$

52. $\int \frac{dx}{9 + x^2} = \frac{1}{3}\arctan \frac{x}{3} + C$

53. $\int \frac{dx}{\sqrt{25 - 16x^2}} = \frac{1}{4}\int \frac{4\, dx}{\sqrt{5^2 - (4x)^2}} = \frac{1}{4}\arcsin \frac{4x}{5} + C$

54. $\int \frac{dx}{4x^2 + 9} = \frac{1}{2}\int \frac{2\, dx}{(2x)^2 + 3^2} = \frac{1}{6}\arctan \frac{2x}{3} + C$

55. $\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2\, dx}{2x\sqrt{(2x)^2 - 3^2}} = \frac{1}{3}\operatorname{arcsec} \frac{2x}{3} + C$

56. $\int \frac{x^2\, dx}{\sqrt{1 - x^6}} = \frac{1}{3}\int \frac{3x^2\, dx}{\sqrt{1 - (x^3)^2}} = \frac{1}{3}\arcsin x^3 + C$

57. $\int \frac{x\, dx}{x^4 + 3} = \frac{1}{2}\int \frac{2x\, dx}{(x^2)^2 + 3} = \frac{1}{2}\frac{1}{\sqrt{3}}\arctan \frac{x^2}{\sqrt{3}} + C = \frac{\sqrt{3}}{6}\arctan \frac{x^2\sqrt{3}}{3} + C$

58. $\int \frac{dx}{x\sqrt{x^4 - 1}} = \frac{1}{2}\int \frac{2x\, dx}{x^2\sqrt{(x^2)^2 - 1}} = \frac{1}{2}\operatorname{arcsec} x^2 + C = \frac{1}{2}\arccos \frac{1}{x^2} + C$

59. $\int \frac{dx}{\sqrt{4 - (x + 2)^2}} = \arcsin \frac{x + 2}{2} + C$

60. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\, dx}{e^{2x} + 1} = \arctan e^x + C$

61. $\int \frac{3x^3 - 4x^2 + 3x}{x^2 + 1}\, dx = \int \left(3x - 4 + \frac{4}{x^2 + 1}\right) dx = \frac{3x^2}{2} - 4x + 4\arctan x + C$

62. $$\int \frac{\sec x \tan x\, dx}{9 + 4\sec^2 x} = \frac{1}{2}\int \frac{2\sec x \tan x\, dx}{3^2 + (2\sec x)^2} = \frac{1}{6}\arctan\frac{2\sec x}{3} + C$$

63. $$\int \frac{(x+3)\, dx}{\sqrt{1-x^2}} = \int \frac{x\, dx}{\sqrt{1-x^2}} + 3\int \frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + 3\arcsin x + C$$

64. $$\int \frac{(2x-7)\, dx}{x^2+9} = \int \frac{2x\, dx}{x^2+9} - 7\int \frac{dx}{x^2+9} = \ln(x^2+9) - \frac{7}{3}\arctan\frac{x}{3} + C$$

65. $$\int \frac{dy}{y^2+10y+30} = \int \frac{dy}{(y^2+10y+25)+5} = \int \frac{dy}{(y+5)^2+5} = \frac{\sqrt{5}}{5}\arctan\frac{(y+5)\sqrt{5}}{5} + C$$

66. $$\int \frac{dx}{\sqrt{20+8x-x^2}} = \int \frac{dx}{\sqrt{36-(x^2-8x+16)}} = \int \frac{dx}{\sqrt{36-(x-4)^2}} = \arcsin\frac{x-4}{6} + C$$

67. $$\int \frac{dx}{2x^2+2x+5} = \int \frac{2\, dx}{4x^2+4x+10} = \int \frac{2\, dx}{(2x+1)^2+9} = \frac{1}{3}\arctan\frac{2x+1}{3} + C$$

68. $$\int \frac{x+1}{x^2-4x+8}\, dx = \frac{1}{2}\int \frac{2x+2}{x^2-4x+8}\, dx = \frac{1}{2}\int \frac{(2x-4)+6}{x^2-4x+8}\, dx = \frac{1}{2}\int \frac{(2x-4)\, dx}{x^2-4x+8} + 3\int \frac{dx}{x^2-4x+8}$$
$$= \frac{1}{2}\int \frac{(2x-4)\, dx}{x^2-4x+8} + 3\int \frac{dx}{(x-2)^2+4} = \frac{1}{2}\ln(x^2-4x+8) + \frac{3}{2}\arctan\frac{x-2}{2} + C$$

The absolute-value sign is not needed here because $x^2 - 4x + 8 > 0$ for all values of x.

69. $$\int \frac{dx}{\sqrt{28-12x-x^2}} = \int \frac{dx}{\sqrt{64-(x^2+12x+36)}} = \int \frac{dx}{\sqrt{64-(x+6)^2}} = \arcsin\frac{x+6}{8} + C$$

70. $$\int \frac{x+3}{\sqrt{5-4x-x^2}}\, dx = -\frac{1}{2}\int \frac{-2x-6}{\sqrt{5-4x-x^2}}\, dx = -\frac{1}{2}\int \frac{(-2x-4)-2}{\sqrt{5-4x-x^2}}\, dx$$
$$= -\frac{1}{2}\int \frac{-2x-4}{\sqrt{5-4x-x^2}}\, dx + \int \frac{dx}{\sqrt{5-4x-x^2}}$$
$$= -\frac{1}{2}\int \frac{-2x-4}{\sqrt{5-4x-x^2}}\, dx + \int \frac{dx}{\sqrt{9-(x+2)^2}}$$
$$= -\sqrt{5-4x-x^2} + \arcsin\frac{x+2}{3} + C$$

71. $$\int \frac{2x+3}{9x^2-12x+8}\, dx = \frac{1}{9}\int \frac{18x+27}{9x^2-12x+8}\, dx = \frac{1}{9}\int \frac{(18x-12)+39}{9x^2-12x+8}\, dx$$
$$= \frac{1}{9}\int \frac{18x-12}{9x^2-12x+8}\, dx + \frac{13}{3}\int \frac{dx}{(3x-2)^2+4}$$
$$= \frac{1}{9}\ln(9x^2-12x+8) + \frac{13}{18}\arctan\frac{3x-2}{2} + C$$

72. $$\int \frac{x+2}{\sqrt{4x-x^2}}\, dx = -\frac{1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}\, dx = -\frac{1}{2}\int \frac{(-2x+4)-8}{\sqrt{4x-x^2}}\, dx$$
$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\, dx + 4\int \frac{dx}{\sqrt{4-(x-2)^2}} = -\sqrt{4x-x^2} + 4\arcsin\frac{x-2}{2} + C$$

FORMULAS 21 TO 24

In Problems 73 to 89, evaluate the integral at the left.

73. $\int \frac{dx}{x^2 - 1} = \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C$

74. $\int \frac{dx}{1 - x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$

75. $\int \frac{dx}{x^2 - 4} = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C$

76. $\int \frac{dx}{9 - x^2} = \frac{1}{6} \ln \left| \frac{3+x}{3-x} \right| + C$

77. $\int \frac{dx}{\sqrt{x^2+1}} = \ln (x + \sqrt{x^2+1}) + C$

78. $\int \frac{dx}{\sqrt{x^2-1}} = \ln |x + \sqrt{x^2-1}| + C$

79. $\int \frac{dx}{\sqrt{4x^2+9}} = \frac{1}{2} \int \frac{2\,dx}{\sqrt{(2x)^2+3^2}} = \frac{1}{2} \ln (2x + \sqrt{4x^2+9}) + C$

80. $\int \frac{dz}{\sqrt{9z^2-25}} = \frac{1}{3} \int \frac{3\,dz}{\sqrt{9z^2-25}} = \frac{1}{3} \ln |3z + \sqrt{9z^2-25}| + C$

81. $\int \frac{dx}{9x^2 - 16} = \frac{1}{3} \int \frac{3\,dx}{(3x)^2 - 16} = \frac{1}{24} \ln \left| \frac{3x-4}{3x+4} \right| + C$

82. $\int \frac{dy}{25 - 16y^2} = \frac{1}{4} \int \frac{4\,dy}{25 - (4y)^2} = \frac{1}{40} \ln \left| \frac{5+4y}{5-4y} \right| + C$

83. $\int \frac{dx}{x^2 + 6x + 8} = \int \frac{dx}{(x+3)^2 - 1} = \frac{1}{2} \ln \left| \frac{(x+3)-1}{(x+3)+1} \right| + C = \frac{1}{2} \ln \left| \frac{x+2}{x+4} \right| + C$

84. $\int \frac{dx}{4x - x^2} = \int \frac{dx}{4 - (x-2)^2} = \frac{1}{4} \ln \left| \frac{2+(x-2)}{2-(x-2)} \right| + C = \frac{1}{4} \ln \left| \frac{x}{4-x} \right| + C$

85. $\int \frac{ds}{\sqrt{4s+s^2}} = \int \frac{ds}{\sqrt{(s+2)^2-4}} = \ln |s + 2 + \sqrt{4s+s^2}| + C$

86. $\int \frac{x+2}{\sqrt{x^2+9}}\,dx = \frac{1}{2} \int \frac{2x+4}{\sqrt{x^2+9}}\,dx = \frac{1}{2} \int \frac{2x\,dx}{\sqrt{x^2+9}} + 2 \int \frac{dx}{\sqrt{x^2+9}}$

$= \sqrt{x^2+9} + 2 \ln (x + \sqrt{x^2+9}) + C$

87. $\int \frac{2x-3}{4x^2-11}\,dx = \frac{1}{4} \int \frac{8x-12}{4x^2-11}\,dx = \frac{1}{4} \int \frac{8x\,dx}{4x^2-11} - \frac{3}{2} \int \frac{2\,dx}{4x^2-11}$

$= \frac{1}{4} \ln |4x^2 - 11| - \frac{3\sqrt{11}}{44} \ln \left| \frac{2x-\sqrt{11}}{2x+\sqrt{11}} \right| + C$

88. $\int \frac{x+2}{\sqrt{x^2+2x-3}}\,dx = \frac{1}{2} \int \frac{2x+4}{\sqrt{x^2+2x-3}}\,dx = \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x-3}}\,dx + \int \frac{dx}{\sqrt{(x+1)^2-4}}$

$= \sqrt{x^2+2x-3} + \ln |x + 1 + \sqrt{x^2+2x-3}| + C$

89. $\displaystyle\int \frac{2-x}{4x^2+4x-3}\,dx = -\frac{1}{8}\int \frac{8x-16}{4x^2+4x-3}\,dx = -\frac{1}{8}\int \frac{8x+4}{4x^2+4x-3}\,dx + \frac{5}{2}\int \frac{dx}{(2x+1)^2-4}$

$$= -\frac{1}{8}\ln|4x^2+4x-3| + \frac{5}{16}\ln\left|\frac{2x-1}{2x+3}\right| + C$$

FORMULAS 25 TO 27

In Problems 90 to 95, evaluate the integral at the left.

90. $\displaystyle\int \sqrt{25-x^2}\,dx = \frac{1}{2}x\sqrt{25-x^2} + \frac{25}{2}\arcsin\frac{x}{5} + C$

91. $\displaystyle\int \sqrt{3-4x^2}\,dx = \frac{1}{2}\int (\sqrt{3-4x^2})(2\,dx) = \frac{1}{2}\left(\frac{2x}{2}\sqrt{3-4x^2} + \frac{3}{2}\arcsin\frac{2x}{\sqrt{3}}\right) + C$

$$= \frac{1}{2}x\sqrt{3-4x^2} + \frac{3}{4}\arcsin\frac{2x\sqrt{3}}{3} + C$$

92. $\displaystyle\int \sqrt{x^2-36}\,dx = \tfrac{1}{2}x\sqrt{x^2-36} - 18\ln|x+\sqrt{x^2-36}| + C$

93. $\displaystyle\int \sqrt{3x^2+5}\,dx = \frac{1}{\sqrt{3}}\int \sqrt{3x^2+5}\sqrt{3}\,dx = \frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}x\sqrt{3x^2+5} + \frac{5}{2}\ln(\sqrt{3}x+\sqrt{3x^2+5})\right] + C$

$$= \frac{1}{2}x\sqrt{3x^2+5} + \frac{5\sqrt{3}}{6}\ln(\sqrt{3}x+\sqrt{3x^2+5}) + C$$

94. $\displaystyle\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\arcsin\frac{x+1}{2} + C$

95. $\displaystyle\int \sqrt{4x^2-4x+5}\,dx = \frac{1}{2}\int (\sqrt{(2x-1)^2+4})(2\,dx)$

$$= \frac{1}{2}\left[\frac{2x-1}{2}\sqrt{4x^2-4x+5} + 2\ln(2x-1+\sqrt{4x^2-4x+5})\right] + C$$

$$= \frac{2x-1}{4}\sqrt{4x^2-4x+5} + \ln(2x-1+\sqrt{4x^2-4x+5}) + C$$

Supplementary Problems

Da scegliere quello compatibile

In Problems 96 to 200, evaluate the integral at the left.

96. $\displaystyle\int (4x^3+3x^2+2x+5)\,dx = x^4+x^3+x^2+5x+C$

97. $\displaystyle\int (3-2x-x^4)\,dx = 3x-x^2-\tfrac{1}{5}x^5+C$

98. $\displaystyle\int (2-3x+x^3)\,dx = 2x-\tfrac{3}{2}x^2+\tfrac{1}{4}x^4+C$

99. $\displaystyle\int (x^2-1)^2\,dx = x^5/5-2x^3/3+x+C$

100. $\int (\sqrt{x} - \frac{1}{2}x + 2/\sqrt{x})\,dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4x^{1/2} + C$

101. $\int (a+x)^3\,dx = \frac{1}{4}(a+x)^4 + C$

102. $\int (x-2)^{3/2}\,dx = \frac{2}{5}(x-2)^{5/2} + C$

103. $\int \frac{dx}{x^3} = -\frac{1}{2x^2} + C$

104. $\int \frac{dx}{(x-1)^3} = -\frac{1}{2(x-1)^2} + C$

105. $\int \frac{dx}{\sqrt{x+3}} = 2\sqrt{x+3} + C$

106. $\int \sqrt{3x-1}\,dx = \frac{2}{9}(3x-1)^{3/2} + C$

107. $\int \sqrt{2-3x}\,dx = -\frac{2}{9}(2-3x)^{3/2} + C$

108. $\int (2x^2+3)^{1/3}x\,dx = \frac{3}{16}(2x^2+3)^{4/3} + C$

109. $\int (x-1)^2 x\,dx = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + C$

110. $\int (x^2-1)x\,dx = \frac{1}{4}(x^2-1)^2 + C$

111. $\int \sqrt{1+y^4}\,y^3\,dy = \frac{1}{6}(1+y^4)^{3/2} + C$

112. $\int (x^3+3)x^2\,dx = \frac{1}{6}(x^3+3)^2 + C$

113. $\int (4-x^2)^2 x^2\,dx = \frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7 + C$

114. $\int \frac{dy}{(2-y)^3} = \frac{1}{2(2-y)^2} + C$

115. $\int \frac{x\,dx}{(x^2+4)^3} = -\frac{1}{4(x^2+4)^2} + C$

116. $\int (1-x^3)^2\,dx = x - \frac{1}{2}x^4 + \frac{1}{7}x^7 + C$

117. $\int (1-x^3)^2 x\,dx = \frac{1}{2}x^2 - \frac{2}{5}x^5 + \frac{1}{8}x^8 + C$

118. $\int (1-x^3)^2 x^2\,dx = -\frac{1}{9}(1-x^3)^3 + C$

119. $\int (x^2-x)^4(2x-1)\,dx = \frac{1}{5}(x^2-x)^5 + C$

120. $\int \frac{3t\,dt}{\sqrt[3]{t^2+3}} = \frac{9}{4}(t^2+3)^{2/3} + C$

121. $\int \frac{(x+1)\,dx}{\sqrt{x^2+2x-4}} = \sqrt{x^2+2x-4} + C$

122. $\int \frac{dx}{(a+bx)^{1/3}} = \frac{3}{2b}(a+bx)^{2/3} + C$

123. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx = \frac{2}{3}(1+\sqrt{x})^3 + C$

124. $\int \sqrt{x}\,(3-5x)\,dx = 2x^{3/2}(1-x) + C$

125. $\int \frac{(x+1)(x-2)}{\sqrt{x}}\,dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} - 4x^{1/2} + C$

126. $\int \frac{dx}{x-1} = \ln|x-1| + C$

127. $\int \frac{dx}{3x+1} = \frac{1}{3}\ln|3x+1| + C$

128. $\int \frac{3x\,dx}{x^2+2} = \frac{3}{2}\ln(x^2+2) + C$

129. $\int \frac{x^2\,dx}{1-x^3} = -\frac{1}{3}\ln|1-x^3| + C$

130. $\int \frac{x-1}{x+1}\,dx = x - 2\ln|x+1| + C$

131. $\int \frac{x^2+2x+2}{x+2}\,dx = \frac{1}{2}x^2 + 2\ln|x+2| + C$

132. $\int \frac{x+1}{x^2+2x+2}\,dx = \frac{1}{2}\ln(x^2+2x+2) + C$

133. $\int \left(\frac{dx}{2x-1} - \frac{dx}{2x+1}\right) = \ln\sqrt{\left|\frac{2x-1}{2x+1}\right|} + C$

134. $\int a^{4x}\,dx = \frac{1}{4}\frac{a^{4x}}{\ln a} + C$

135. $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$

136. $\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}e^{1/x^2} + C$

137. $\int e^{-x^2+2}x\,dx = -\frac{1}{2}e^{-x^2+2} + C$

138. $\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$

139. $\int (e^x+1)^2\,dx = \frac{1}{2}e^{2x} + 2e^x + x + C$

140. $\int (e^x - x^e)\,dx = e^x - \frac{x^{e+1}}{e+1} + C$

141. $\int (e^x+1)^2 e^x\,dx = \frac{1}{3}(e^x+1)^3 + C$

142. $\int \frac{e^{2x}}{e^{2x}+3}\,dx = \frac{1}{2}\ln(e^{2x}+3) + C$

143. $\int \left(e^x + \frac{1}{e^x}\right)^2 dx = \frac{1}{2}e^{2x} + 2x - \frac{1}{2e^{2x}} + C$

144. $\int \frac{e^x-1}{e^x+1}\,dx = \ln(e^x+1)^2 - x + C$

145. $\int \frac{e^{2x}-1}{e^{2x}+3}\,dx = \ln(e^{2x}+3)^{2/3} - \frac{1}{3}x + C$

146. $\int \frac{dx}{\sqrt{x}(1-\sqrt{x})} = \ln\frac{C}{(1-\sqrt{x})^2},\ C>0$

147. $\int \frac{dx}{x+x^{1/3}} = \frac{3}{2}\ln C(x^{2/3}+1),\ C>0$

148. $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

149. $\int \cos\frac{1}{2}x\,dx = 2\sin\frac{1}{2}x + C$

150. $\int \sec 3x\tan 3x\,dx = \frac{1}{3}\sec 3x + C$

151. $\int \csc^2 2x\,dx = -\frac{1}{2}\cot 2x + C$

152. $\int x\sec^2 x^2\,dx = \frac{1}{2}\tan x^2 + C$

153. $\int \tan^2 x\,dx = \tan x - x + C$

154. $\int \tan\frac{1}{2}x\,dx = 2\ln|\sec\frac{1}{2}x| + C$

155. $\int \csc 3x\,dx = \frac{1}{3}\ln|\csc 3x - \cot 3x| + C$

156. $\int b\sec ax\tan ax\,dx = \frac{b}{a}\sec ax + C$

157. $\int (\cos x - \sin x)^2\,dx = x + \frac{1}{2}\cos 2x + C$

158. $\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax + C$

$= -\frac{1}{2a}\cos^2 ax + C' = -\frac{1}{4a}\cos 2ax + C''$

159. $\int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

160. $\int \cos^4 x\sin x\,dx = -\frac{1}{5}\cos^5 x + C$

161. $\int \tan^5 x\sec^2 x\,dx = \frac{1}{6}\tan^6 x + C$

162. $\int \cot^4 3x\csc^2 3x\,dx = -\frac{1}{15}\cot^5 3x + C$

163. $\int \frac{dx}{1-\sin\frac{1}{2}x} = 2(\tan\frac{1}{2}x + \sec\frac{1}{2}x) + C$

164. $\int \frac{dx}{1+\cos 3x} = \frac{1-\cos 3x}{3\sin 3x} + C$

165. $\int \frac{dx}{1+\sec ax} = x + \frac{1}{a}(\cot ax - \csc ax) + C$

166. $\int \sec^2\frac{x}{a}\tan\frac{x}{a}\,dx = \frac{1}{2}a\tan^2\frac{x}{a} + C$

167. $\int \frac{\sec^2 3x}{\tan 3x}\,dx = \frac{1}{3}\ln|\tan 3x| + C$

168. $\int \frac{\sec^5 x}{\csc x}\,dx = \frac{1}{4}\sec^4 x + C$

169. $\int e^{\tan 2x}\sec^2 2x\,dx = \frac{1}{2}e^{\tan 2x} + C$

170. $\int e^{2\sin 3x}\cos 3x\,dx = \frac{1}{6}e^{2\sin 3x} + C$

171. $\int \frac{dx}{\sqrt{5-x^2}} = \arcsin\frac{x\sqrt{5}}{5} + C$

172. $\int \frac{dx}{5+x^2} = \frac{\sqrt{5}}{5}\arctan\frac{x\sqrt{5}}{5} + C$

173. $\int \frac{dx}{x\sqrt{x^2-5}} = \frac{\sqrt{5}}{5}\operatorname{arcsec}\frac{x\sqrt{5}}{5} + C$

174. $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}} = \arcsin e^x + C$

175. $\int \frac{e^{2x}\,dx}{1+e^{4x}} = \frac{1}{2}\arctan e^{2x} + C$

176. $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\arcsin\frac{3x}{2} + C$

177. $\int \frac{dx}{9x^2+4} = \frac{1}{6}\arctan\frac{3x}{2} + C$

178. $\int \frac{\sin 8x}{9+\sin^4 4x}\,dx = \frac{1}{12}\arctan\frac{\sin^2 4x}{3} + C$

179. $\int \frac{\sec^2 x\,dx}{\sqrt{1-4\tan^2 x}} = \frac{1}{2}\arcsin(2\tan x) + C$

180. $\int \frac{dx}{x\sqrt{4-9\ln^2 x}} = \frac{1}{3}\arcsin\ln x^{3/2} + C$

181. $\int \frac{2x^4 - x^2}{2x^2+1}\,dx = \frac{1}{3}x^3 - x + \frac{\sqrt{2}}{2}\arctan x\sqrt{2} + C$

182. $\int \frac{\cos 2x\,dx}{\sin^2 2x + 8} = \frac{\sqrt{2}}{8}\arctan\frac{\sin 2x}{2\sqrt{2}} + C$

183. $\int \frac{(2x-3)\,dx}{x^2+6x+13} = \int \frac{(2x+6)\,dx}{x^2+6x+13} - 9\int \frac{dx}{x^2+6x+13} = \ln(x^2+6x+13) - \frac{9}{2}\arctan\frac{x+3}{2} + C$

184. $\int \frac{(x-1)\,dx}{3x^2-4x+3} = \frac{1}{6}\int \frac{(6x-4)\,dx}{3x^2-4x+3} - \int \frac{dx}{9x^2-12x+9} = \frac{1}{6}\ln(3x^2-4x+3) - \frac{\sqrt{5}}{15}\arctan\frac{3x-2}{\sqrt{5}} + C$

185. $\int \frac{x\,dx}{\sqrt{27+6x-x^2}} = -\sqrt{27+6x-x^2} + 3\arcsin\frac{x-3}{6} + C$

186. $\int \frac{(5-4x)\,dx}{\sqrt{12x-4x^2-8}} = \sqrt{12x-4x^2-8} - \frac{1}{2}\arcsin(2x-3) + C$

187. $\int \frac{dx}{x^2-4} = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

188. $\int \frac{dx}{4x^2-9} = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$

189. $\int \frac{dx}{9-x^2} = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

190. $\int \frac{dx}{25-9x^2} = \frac{1}{30}\ln\left|\frac{3x+5}{3x-5}\right| + C$

191. $\int \frac{dx}{\sqrt{x^2+4}} = \ln(x+\sqrt{x^2+4}) + C$

192. $\int \frac{dx}{\sqrt{4x^2-25}} = \frac{1}{2}\ln|2x+\sqrt{4x^2-25}| + C$

193. $\int \sqrt{16-9x^2}\,dx = \frac{1}{2}x\sqrt{16-9x^2} + \frac{8}{3}\arcsin\frac{3x}{4} + C$

194. $\int \sqrt{x^2-16}\,dx = \frac{1}{2}x\sqrt{x^2-16} - 8\ln|x+\sqrt{x^2-16}| + C$

195. $\int \sqrt{4x^2+9}\,dx = \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\ln(2x+\sqrt{4x^2+9}) + C$

196. $\int \sqrt{x^2-2x-3}\,dx = \frac{1}{2}(x-1)\sqrt{x^2-2x-3} - 2\ln|x-1+\sqrt{x^2-2x-3}| + C$

197. $\int \sqrt{12+4x-x^2}\,dx = \frac{1}{2}(x-2)\sqrt{12+4x-x^2} + 8\arcsin\frac{1}{4}(x-2) + C$

198. $\int \sqrt{x^2+4x}\,dx = \frac{1}{2}(x+2)\sqrt{x^2+4x} - 2\ln|x+2+\sqrt{x^2+4x}| + C$

199. $\int \sqrt{x^2-8x}\,dx = \frac{1}{2}(x-4)\sqrt{x^2-8x} - 8\ln|x-4+\sqrt{x^2-8x}| + C$

200. $\int \sqrt{6x-x^2}\,dx = \frac{1}{2}(x-3)\sqrt{6x-x^2} + \frac{9}{2}\arcsin\frac{x-3}{3} + C$

Chapter 31

Integration by Parts

INTEGRATION BY PARTS. When u and v are differentiable functions of x,

$$d(uv) = u\,dv + v\,du$$

or

$$u\,dv = d(uv) - v\,du$$

and

$$\int u\,dv = uv - \int v\,du \tag{31.1}$$

When (*31.1*) is to be used in a required integration, the given integral must be separated into two parts, one part being u and the other part, together with dx, being dv. (For this reason, integration by use of (*31.1*) is called *integration by parts*.) Two general rules can be stated:

1. The part selected as dv must be readily integrable.
2. $\int v\,du$ must not be more complex than $\int u\,dv$.

EXAMPLE 1: Find $\int x^3 e^{x^2}\,dx$.

Take $u = x^2$ and $dv = e^{x^2} x\,dx$; then $du = 2x\,dx$ and $v = \frac{1}{2}e^{x^2}$. Now by (*31.1*),

$$\int x^3 e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \int x e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \tfrac{1}{2}e^{x^2} + C$$

EXAMPLE 2: Find $\int \ln(x^2+2)\,dx$.

Take $u = \ln(x^2+2)$ and $dv = dx$; then $du = \dfrac{2x\,dx}{x^2+2}$ and $v = x$. By (*31.1*),

$$\int \ln(x^2+2)\,dx = x\ln(x^2+2) - \int \frac{2x^2\,dx}{x^2+2} = x\ln(x^2+2) - \int\left(2 - \frac{4}{x^2+2}\right)dx$$

$$= x\ln(x^2+2) - 2x + 2\sqrt{2}\arctan\frac{x}{\sqrt{2}} + C$$

(See Problems 1 to 10.)

(NO) Formula necessaria

REDUCTION FORMULAS. The labor involved in successive applications of integration by parts to evaluate an integral (see Problem 9) may be materially reduced by the use of *reduction formulas*. In general, a reduction formula yields a new integral of the same form as the original but with an exponent increased or reduced. A reduction formula succeeds if ultimately it produces an integral that can be evaluated. Among the reduction formulas are:

$$\int \frac{dx}{(a^2 \pm x^2)^m} = \frac{1}{a^2}\left[\frac{x}{(2m-2)(a^2 \pm x^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{dx}{(a^2 \pm x^2)^{m-1}}\right],\quad m \neq 1 \tag{31.2}$$

$$\int (a^2 \pm x^2)^m\,dx = \frac{x(a^2 \pm x^2)^m}{2m+1} + \frac{2ma^2}{2m+1}\int (a^2 \pm x^2)^{m-1}\,dx,\quad m \neq -1/2 \tag{31.3}$$

$$\int \frac{dx}{(x^2 - a^2)^m} = -\frac{1}{a^2}\left[\frac{x}{(2m-2)(x^2 - a^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{dx}{(x^2 - a^2)^{m-1}}\right],\quad m \neq 1 \tag{31.4}$$

$$\int (x^2 - a^2)^m\,dx = \frac{x(x^2 - a^2)^m}{2m+1} - \frac{2ma^2}{2m+1}\int (x^2 - a^2)^{m-1}\,dx,\quad m \neq -1/2 \tag{31.5}$$

$$\int x^m e^{ax}\,dx = \frac{1}{a}x^m e^{ax} - \frac{m}{a}\int x^{m-1} e^{ax}\,dx \tag{31.6}$$

$$\int \sin^m x\,dx = -\frac{\sin^{m-1}x\cos x}{m} + \frac{m-1}{m}\int \sin^{m-2}x\,dx \tag{31.7}$$

$$\int \cos^m x\,dx = \frac{\cos^{m-1}x\sin x}{m} + \frac{m-1}{m}\int \cos^{m-2}x\,dx \tag{31.8}$$

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx\,,\quad m \neq -n \tag{31.9}$$

$$\int x^m \sin bx\,dx = -\frac{x^m}{b}\cos bx + \frac{m}{b}\int x^{m-1}\cos bx\,dx \tag{31.10}$$

$$\int x^m \cos bx\,dx = \frac{x^m}{b}\sin bx - \frac{m}{b}\int x^{m-1}\sin bx\,dx \tag{31.11}$$

(See Problem 11.)

Solved Problems

Solo esercizi semplici

1. Find $\int x\sin x\,dx$.

We have three choices: (*a*) $u = x\sin x$, $dv = dx$; (*b*) $u = \sin x$, $dv = x\,dx$; (*c*) $u = x$, $dv = \sin x\,dx$.

(*a*) Let $u = x\sin x$, $dv = dx$. Then $du = (\sin x + x\cos x)\,dx$, $v = x$, and

$$\int x\sin x\,dx = x\cdot x\sin x - \int x(\sin x + x\cos x)\,dx$$

The resulting integral is not as simple as the original, and this choice is discarded.

(*b*) Let $u = \sin x$, $dv = x\,dx$. Then $du = \cos x\,dx$, $v = \frac{1}{2}x^2$, and

$$\int x\sin x\,dx = \tfrac{1}{2}x^2\sin x - \int \tfrac{1}{2}x^2\cos x\,dx$$

The resulting integral is not as simple as the original, and this choice too is discarded.

(*c*) Let $u = x$, $dv = \sin x\,dx$. Then $du = dx$, $v = -\cos x$, and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

2. Find $\int xe^x\,dx$.

Let $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

3. Find $\int x^2\ln x\,dx$.

Let $u = \ln x$, $dv = x^2\,dx$. Then $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

4. Find $\int x\sqrt{1+x}\,dx$.

Let $u = x$, $dv = \sqrt{1+x}\,dx$. Then $du = dx$, $v = \frac{2}{3}(1+x)^{3/2}$, and

$$\int x\sqrt{1+x}\,dx = \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\int (1+x)^{3/2}\,dx = \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + C$$

5. Find $\int \arcsin x\,dx$.

Let $u = \arcsin x$, $dv = dx$. Then $du = \frac{dx}{\sqrt{1-x^2}}$, $v = x$, and

$$\int \arcsin x\,dx = x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C$$

6. Find $\int \sin^2 x\,dx$.

Let $u = \sin x$, $dv = \sin x\,dx$. Then $du = \cos x\,dx$, $v = -\cos x$, and

$$\int \sin^2 x\,dx = -\sin x \cos x + \int \cos^2 x\,dx = -\sin x \cos x + \int (1 - \sin^2 x)\,dx$$

$$= -\tfrac{1}{2}\sin 2x + \int dx - \int \sin^2 x\,dx$$

Hence $\quad 2\int \sin^2 x\,dx = -\frac{1}{2}\sin 2x + x + C' \quad$ and $\quad \int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

7. Find $\int \sec^3 x\,dx$.

Let $u = \sec x$, $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$, $v = \tan x$, and

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

Then $\quad 2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C'$

and $\quad \int \sec^3 x\,dx = \frac{1}{2}\{\sec x \tan x + \ln|\sec x + \tan x|\} + C$

8. Find $\int x^2 \sin x\,dx$.

Let $u = x^2$, $dv = \sin x\,dx$. Then $du = 2x\,dx$, $v = -\cos x$, and

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2\int x \cos x\,dx$$

For the resulting integral, let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$, $v = \sin x$, and

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2\left(x \sin x - \int \sin x\,dx\right) = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

9. Find $\int x^3 e^{2x}\,dx$.

Let $u = x^3$, $dv = e^{2x}\,dx$. Then $du = 3x^2\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{2}\int x^2e^{2x}\,dx$$

For the resulting integral, let $u = x^2$ and $dv = e^{2x}\,dx$. Then $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{2}\left(\tfrac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx\right) = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{2}\int xe^{2x}\,dx$$

For the resulting integral, let $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{2}\left(\tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,dx\right) = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + C$$

10. Find reduction formulas for (*a*) $\int \frac{x^2\,dx}{(a^2 \pm x^2)^m}$ and (*b*) $\int x^2(a^2 \pm x^2)^{m-1}\,dx$.

(*a*) Take $u = x$, $dv = \frac{x\,dx}{(a^2 \pm x^2)^m}$; then $du = dx$, $v = \frac{\mp 1}{(2m-2)(a^2 \pm x^2)^{m-1}}$, and

$$\int \frac{x^2\,dx}{(a^2 \pm x^2)^m} = \frac{\mp x}{(2m-2)(a^2 \pm x^2)^{m-1}} \pm \frac{1}{2m-2}\int \frac{dx}{(a^2 \pm x^2)^{m-1}}$$

(*b*) Take $u = x$, $dv = x(a^2 \pm x^2)^{m-1}\,dx$; then $du = dx$, $v = \frac{\pm 1}{2m}(a^2 \pm x^2)^m$, and

$$\int x^2(a^2 \pm x^2)^{m-1}\,dx = \frac{\pm x}{2m}(a^2 \pm x^2)^m \mp \frac{1}{2m}\int (a^2 \pm x^2)^m\,dx$$

11. Find: (*a*) $\int \frac{dx}{(1 + x^2)^{5/2}}$ and (*b*) $\int (9 + x^2)^{3/2}\,dx$.

(*a*) Since (*31.2*) reduces the exponent in the denominator by 1, we use this formula twice to obtain

$$\int \frac{dx}{(1 + x^2)^{5/2}} = \frac{x}{3(1 + x^2)^{3/2}} + \frac{2}{3}\int \frac{dx}{(1 + x^2)^{3/2}} = \frac{x}{3(1 + x^2)^{3/2}} + \frac{2}{3}\frac{x}{(1 + x^2)^{1/2}} + C$$

(*b*) Using (*31.3*), we obtain

$$\int (9 + x^2)^{3/2}\,dx = \tfrac{1}{4}x(9 + x^2)^{3/2} + \tfrac{27}{4}\int (9 + x^2)^{1/2}\,dx$$
$$= \tfrac{1}{4}x(9 + x^2)^{3/2} + \tfrac{27}{8}[x(9 + x^2)^{1/2} + 9\ln(x + \sqrt{9 + x^2})] + C$$

12. Derive reduction formula (*31.7*): $\int \sin^m x\,dx = -\frac{\sin^{m-1} x \cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x\,dx$.

We use integration by parts: Let $u = \sin^{m-1} x$ and $dv = \sin x\,dx$; then $du = (m-1)\sin^{m-2} x \cos x\,dx$, $v = -\cos x$, and

$$\int \sin^m x\,dx = -\cos x \sin^{m-1} x + (m-1)\int \sin^{m-2} x \cos^2 x\,dx$$
$$= -\cos x \sin^{m-1} x + (m-1)\int (\sin^{m-2} x)(1 - \sin^2 x)\,dx$$
$$= -\cos x \sin^{m-1} x + (m-1)\int \sin^{m-2} x\,dx - (m-1)\int \sin^m x\,dx$$

Hence,
$$m\int \sin^m x\,dx = -\cos x \sin^{m-1} x + (m-1)\int \sin^{m-2} x\,dx$$

and division by m yields (*31.7*).

Supplementary Problems

Scegliere quella semplice

In Problems 13 to 29 and 32 to 40 evaluate the indefinite integral at left.

13. $\int x \cos x\, dx = x \sin x + \cos x + C$

14. $\int x \sec^2 3x\, dx = \frac{1}{3}x \tan 3x - \frac{1}{9} \ln|\sec 3x| + C$

15. $\int \arccos 2x\, dx = x \arccos 2x - \frac{1}{2}\sqrt{1-4x^2} + C$

16. $\int \arctan x\, dx = x \arctan x - \ln \sqrt{1+x^2} + C$

17. $\int x^2\sqrt{1-x}\, dx = -\frac{2}{105}(1-x)^{3/2}(15x^2+12x+8) + C$

18. $\int \frac{xe^x\, dx}{(1+x)^2} = \frac{e^x}{1+x} + C$

19. $\int x \arctan x\, dx = \frac{1}{2}(x^2+1) \arctan x - \frac{1}{2}x + C$

20. $\int x^2 e^{-3x}\, dx = -\frac{1}{3}e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C$

21. $\int \sin^3 x\, dx = -\frac{2}{3}\cos^3 x - \sin^2 x \cos x + C$

22. $\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

23. $\int \frac{x\, dx}{\sqrt{a+bx}} = \frac{2(bx-2a)\sqrt{a+bx}}{3b^2} + C$

24. $\int \frac{x^2\, dx}{\sqrt{1+x}} = \frac{2}{15}(3x^2-4x+8)\sqrt{1+x} + C$

25. $\int x \arcsin x^2\, dx = \frac{1}{2}x^2 \arcsin x^2 + \frac{1}{2}\sqrt{1-x^4} + C$

26. $\int \sin x \sin 3x\, dx = \frac{1}{8}\sin 3x \cos x - \frac{3}{8}\sin x \cos 3x + C$

27. $\int \sin(\ln x)\, dx = \frac{1}{2}x(\sin \ln x - \cos \ln x) + C$

28. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2+b^2} + C$

29. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C$

30. (*a*) Write $\int \frac{a^2\, dx}{(a^2 \pm x^2)^m} = \int \frac{(a^2 \pm x^2) \mp x^2}{(a^2 \pm x^2)^m}\, dx = \int \frac{dx}{(a^2 \pm x^2)^{m-1}} \mp \int \frac{x^2\, dx}{(a^2 \pm x^2)^m}$ and use the result of Problem 10(*a*) to obtain (*31.2*).

(*b*) Write $\int (a^2 \pm x^2)^m\, dx = a^2 \int (a^2 \pm x^2)^{m-1}\, dx \pm \int x^2(a^2 \pm x^2)^{m-1}\, dx$ and use the result of Problem 10(*b*) to obtain (*31.3*).

31. Derive reduction formulas (*31.4*) to (*31.11*).

32. $\int \frac{dx}{(1-x^2)^3} = \frac{x(5-3x^2)}{8(1-x^2)^2} + \frac{3}{16} \ln\left|\frac{1+x}{1-x}\right| + C$

33. $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}} = \frac{x}{4(4+x^2)^{1/2}} + C$

34. $\displaystyle\int (4-x^2)^{3/2}\,dx = \tfrac{1}{4}x(10-x^2)\sqrt{4-x^2} + 6\arcsin\tfrac{1}{2}x + C$

35. $\displaystyle\int \frac{dx}{(x^2-16)^3} = \frac{1}{2048}\left[\frac{x(3x^2-80)}{(x^2-16)^2} + \frac{3}{8}\ln\left|\frac{x-4}{x+4}\right|\right] + C$

36. $\displaystyle\int (x^2-1)^{5/2}\,dx = \tfrac{1}{48}x(8x^4-26x^2+33)\sqrt{x^2-1} - \tfrac{5}{16}\ln|x+\sqrt{x^2-1}| + C$

37. $\displaystyle\int \sin^4 x\,dx = \tfrac{3}{8}x - \tfrac{3}{8}\sin x\cos x - \tfrac{1}{4}\sin^3 x\cos x + C$

38. $\displaystyle\int \cos^5 x\,dx = \tfrac{1}{15}(3\cos^4 x + 4\cos^2 x + 8)\sin x + C$

39. $\displaystyle\int \sin^3 x\cos^2 x\,dx = -\tfrac{1}{5}\cos^3 x\,(\sin^2 x + \tfrac{2}{3}) + C$

40. $\displaystyle\int \sin^4 x\cos^5 x\,dx = \tfrac{1}{9}\sin^5 x\,(\cos^4 x + \tfrac{4}{7}\cos^2 x + \tfrac{8}{35}) + C$

An alternative procedure for some of the more tedious problems of this section can be found by noting (see Problem 9) that in

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + C \qquad (1)$$

the terms on the right, apart from the coefficients, are the different terms obtained by repeated differentiations of the integrand x^3e^{2x}. Thus, we may write at once

$$\int x^3e^{2x}\,dx = Ax^3e^{2x} + Bx^2e^{2x} + Dxe^{2x} + Ee^{2x} + C \qquad (2)$$

and from it obtain by differentiation

$$x^3e^{2x} = 2Ax^3e^{2x} + (3A+2B)x^2e^{2x} + (2B+2D)xe^{2x} + (D+2E)e^{2x}$$

Equating coefficients, we have

$$2A = 1 \qquad 3A+2B=0 \qquad 2B+2D=0 \qquad D+2E=0$$

so that $A=\tfrac{1}{2}$, $B=-\tfrac{3}{2}A=-\tfrac{3}{4}$, $D=-B=\tfrac{3}{4}$, $E=-\tfrac{1}{2}D=-\tfrac{3}{8}$. Substituting for A, B, D, E in (2), we obtain (1).

This procedure may be used for finding $\int f(x)\,dx$ whenever repeated differentiation of $f(x)$ yields only a finite number of different terms.

41. Find $\int e^{2x}\cos 3x\,dx = \tfrac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C$, using

$$\int e^{2x}\cos 3x\,dx = Ae^{2x}\sin 3x + Be^{2x}\cos 3x + C$$

42. Find $\int e^{3x}(2\sin 4x - 5\cos 4x)\,dx = \tfrac{1}{25}e^{3x}(-14\sin 4x - 23\cos 4x) + C$, using

$$\int e^{3x}(2\sin 4x - 5\cos 4x)\,dx = Ae^{3x}\sin 4x + Be^{3x}\cos 4x + C$$

43. Find $\int \sin 3x\cos 2x\,dx = -\tfrac{1}{5}(2\sin 3x\sin 2x + 3\cos 3x\cos 2x) + C$, using

$$\int \sin 3x\cos 2x\,dx = A\sin 3x\sin 2x + B\cos 3x\cos 2x + D\cos 3x\sin 2x + E\sin 3x\cos 2x + C$$

44. Find $\displaystyle\int e^{3x}x^2\sin x\,dx = \frac{e^{3x}}{250}[25x^2(3\sin x - \cos x) - 10x(4\sin x - 3\cos x) + 9\sin x - 13\cos x] + C.$

Chapter 32

Trigonometric Integrals

THE FOLLOWING IDENTITIES are employed to find some of the trigonometric integrals of this chapter:

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin x \cos x = \frac{1}{2} \sin 2x$
7. $\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$
8. $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
9. $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$
10. $1 - \cos x = 2 \sin^2 \frac{1}{2}x$
11. $1 + \cos x = 2 \cos^2 \frac{1}{2}x$
12. $1 \pm \sin x = 1 \pm \cos(\frac{1}{2}\pi - x)$

TWO SPECIAL SUBSTITUTION RULES are useful in a few simple cases:

1. For $\int \sin^m x \cos^n x\, dx$: If m is odd, substitute $u = \cos x$. If n is odd, substitute $u = \sin x$.
2. For $\int \tan^m x \sec^n x\, dx$: If n is even, substitute $u = \tan x$. If m is odd, substitute $u = \sec x$.

Solved Problems

SINES AND COSINES

In Problems 1 to 17, evaluate the integral at the left.

1. $\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

2. $\int \cos^2 3x\, dx = \int \frac{1}{2}(1 + \cos 6x)\, dx = \frac{1}{2}x + \frac{1}{12}\sin 6x + C$

3. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1 - \cos^2 x)\sin x\, dx = -\cos x + \frac{1}{3}\cos^3 x + C$

This solution is equivalent to using the substitution $u = \cos x$, $du = -\sin x\, dx$, as follows:

$$\int \sin^3 x\, dx = -\int (1 - u^2)\, du = -u + \tfrac{1}{3}u^3 + C = -\cos x + \tfrac{1}{3}\cos^3 x + C$$

4.
$$\begin{aligned}\int \cos^5 x\, dx &= \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx\\ &= \int \cos x\, dx - 2\int \sin^2 x \cos x\, dx + \int \sin^4 x \cos x\, dx\\ &= \sin x - \tfrac{2}{3}\sin^3 x + \tfrac{1}{5}\sin^5 x + C\end{aligned}$$

This amounts to the use of the substitution $u = \sin x$. We have also used *(30.2)*.

5. $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx = \int \sin^2 x\,(1 - \sin^2 x) \cos x\, dx$

$$= \int \sin^2 x \cos x\, dx - \int \sin^4 x \cos x\, dx = \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C$$

6. $\int \cos^4 2x \sin^3 2x\, dx = \int \cos^4 2x \sin^2 2x \sin 2x\, dx = \int \cos^4 2x\,(1 - \cos^2 2x) \sin 2x\, dx$

$$= \int \cos^4 2x \sin 2x\, dx - \int \cos^6 2x \sin 2x\, dx = -\tfrac{1}{10}\cos^5 2x + \tfrac{1}{14}\cos^7 2x + C$$

7. $\int \sin^3 3x \cos^5 3x\, dx = \int (1 - \cos^2 3x) \cos^5 3x \sin 3x\, dx$

$$= \int \cos^5 3x \sin 3x\, dx - \int \cos^7 3x \sin 3x\, dx = -\tfrac{1}{18}\cos^6 3x + \tfrac{1}{24}\cos^8 3x + C$$

or $\int \sin^3 3x \cos^5 3x\, dx = \int \sin^3 3x\,(1 - \sin^2 3x)^2 \cos 3x\, dx$

$$= \int \sin^3 3x \cos 3x\, dx - 2\int \sin^5 3x \cos 3x\, dx + \int \sin^7 3x \cos 3x\, dx$$

$$= \tfrac{1}{12}\sin^4 3x - \tfrac{1}{9}\sin^6 3x + \tfrac{1}{24}\sin^8 3x + C$$

8. $\int \cos^3 \frac{x}{3}\, dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cos \frac{x}{3}\, dx = 3 \sin \frac{x}{3} - \sin^3 \frac{x}{3} + C$

9. $\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx = \tfrac{1}{4}\int (1 - \cos 2x)^2\, dx$

$$= \tfrac{1}{4}\int dx - \tfrac{1}{2}\int \cos 2x\, dx + \tfrac{1}{4}\int \cos^2 2x\, dx$$

$$= \tfrac{1}{4}\int dx - \tfrac{1}{2}\int \cos 2x\, dx + \tfrac{1}{8}\int (1 + \cos 4x)\, dx$$

$$= \tfrac{1}{4}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x + C = \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

10. $\int \sin^2 x \cos^2 x\, dx = \tfrac{1}{4}\int \sin^2 2x\, dx = \tfrac{1}{8}\int (1 - \cos 4x)\, dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$

11. $\int \sin^4 3x \cos^2 3x\, dx = \int (\sin^2 3x \cos^2 3x) \sin^2 3x\, dx = \tfrac{1}{8}\int \sin^2 6x\,(1 - \cos 6x)\, dx$

$$= \tfrac{1}{8}\int \sin^2 6x\, dx - \tfrac{1}{8}\int \sin^2 6x \cos 6x\, dx$$

$$= \tfrac{1}{16}\int (1 - \cos 12x)\, dx - \tfrac{1}{8}\int \sin^2 6x \cos 6x\, dx$$

$$= \tfrac{1}{16}x - \tfrac{1}{192}\sin 12x - \tfrac{1}{144}\sin^3 6x + C$$

12. $\int \sin 3x \sin 2x\, dx = \int \tfrac{1}{2}[\cos(3x - 2x) - \cos(3x + 2x)]\, dx = \tfrac{1}{2}\int (\cos x - \cos 5x)\, dx$

$$= \tfrac{1}{2}\sin x - \tfrac{1}{10}\sin 5x + C$$

13. $\int \sin 3x \cos 5x\, dx = \int \tfrac{1}{2}[\sin(3x - 5x) + \sin(3x + 5x)]\, dx = \tfrac{1}{4}\cos 2x - \tfrac{1}{16}\cos 8x + C$

14. $\int \cos 4x \cos 2x\, dx = \frac{1}{2}\int (\cos 2x + \cos 6x)\, dx = \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C$

15. $\int \sqrt{1-\cos x}\, dx = \sqrt{2}\int \sin \frac{1}{2}x\, dx = -2\sqrt{2}\cos \frac{1}{2}x + C$

16. $\int (1+\cos 3x)^{3/2}\, dx = 2\sqrt{2}\int \cos^3 \frac{3}{2}x\, dx = 2\sqrt{2}\int (1-\sin^2 \frac{3}{2}x)\cos \frac{3}{2}x\, dx$

$$= 2\sqrt{2}\left(\tfrac{2}{3}\sin \tfrac{3}{2}x - \tfrac{2}{9}\sin^3 \tfrac{3}{2}x\right) + C$$

17. $\int \frac{dx}{\sqrt{1-\sin 2x}} = \int \frac{dx}{\sqrt{1-\cos(\frac{1}{2}\pi - 2x)}} = \frac{\sqrt{2}}{2}\int \frac{dx}{\sin(\frac{1}{4}\pi - x)} = \frac{\sqrt{2}}{2}\int \csc(\frac{1}{4}\pi - x)\, dx$

$$= -\frac{\sqrt{2}}{2}\ln|\csc(\tfrac{1}{4}\pi - x) - \cot(\tfrac{1}{4}\pi - x)| + C$$

TANGENTS, SECANTS, COTANGENTS, COSECANTS

Evaluate the integral at the left.

18. $\int \tan^4 x\, dx = \int \tan^2 x \tan^2 x\, dx = \int \tan^2 x(\sec^2 x - 1)\, dx = \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx = \tfrac{1}{3}\tan^3 x - \tan x + x + C$$

19. $\int \tan^5 x\, dx = \int \tan^3 x \tan^2 x\, dx = \int \tan^3 x(\sec^2 x - 1)\, dx$

$$= \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx = \int \tan^3 x \sec^2 x\, dx - \int \tan x(\sec^2 x - 1)\, dx$$

$$= \tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x + \ln|\sec x| + C$$

20. $\int \sec^4 2x\, dx = \int \sec^2 2x \sec^2 2x\, dx = \int \sec^2 2x(1+\tan^2 2x)\, dx$

$$= \int \sec^2 2x\, dx + \int \tan^2 2x \sec^2 2x\, dx = \tfrac{1}{2}\tan 2x + \tfrac{1}{6}\tan^3 2x + C$$

21. $\int \tan^3 3x \sec^4 3x\, dx = \int \tan^3 3x(1+\tan^2 3x)\sec^2 3x\, dx$

$$= \int \tan^3 3x \sec^2 3x\, dx + \int \tan^5 3x \sec^2 3x\, dx = \tfrac{1}{12}\tan^4 3x + \tfrac{1}{18}\tan^6 3x + C$$

22. $\int \tan^2 x \sec^3 x\, dx = \int (\sec^2 x - 1)\sec^3 x\, dx = \int \sec^5 x\, dx - \int \sec^3 x\, dx$

$$= \tfrac{1}{4}\sec^3 x \tan x - \tfrac{1}{8}\sec x \tan x - \tfrac{1}{8}\ln|\sec x + \tan x| + C \quad \text{(integrating by parts)}$$

23. $\int \tan^3 2x \sec^3 2x\, dx = \int (\tan^2 2x \sec^2 2x)(\sec 2x \tan 2x\, dx)$

$$= \int (\sec^2 2x - 1)(\sec^2 2x)(\sec 2x \tan 2x\, dx)$$

$$= \int (\sec^4 2x)(\sec 2x \tan 2x\, dx) - \int (\sec^2 2x)(\sec 2x \tan 2x\, dx)$$

$$= \tfrac{1}{10}\sec^5 2x - \tfrac{1}{6}\sec^3 2x + C$$

24. $\int \cot^3 2x\,dx = \int \cot 2x\,(\csc^2 2x - 1)\,dx = -\frac{1}{4}\cot^2 2x + \frac{1}{2}\ln|\csc 2x| + C$

25. $\int \cot^4 3x\,dx = \int \cot^2 3x\,(\csc^2 3x - 1)\,dx = \int \cot^2 3x \csc^2 3x\,dx - \int \cot^2 3x\,dx$

$= \int \cot^2 3x \csc^2 3x\,dx - \int (\csc^2 3x - 1)\,dx = -\frac{1}{9}\cot^3 3x + \frac{1}{3}\cot 3x + x + C$

26. $\int \csc^6 x\,dx = \int \csc^2 x(1 + \cot^2 x)^2\,dx = \int \csc^2 x\,dx + 2\int \cot^2 x \csc^2 x\,dx + \int \cot^4 x \csc^2 x\,dx$

$= -\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C$

27. $\int \cot 3x \csc^4 3x\,dx = \int \cot 3x\,(1 + \cot^2 3x)\csc^2 3x\,dx$

$= \int \cot 3x \csc^2 3x\,dx + \int \cot^3 3x \csc^2 3x\,dx = -\frac{1}{6}\cot^2 3x - \frac{1}{12}\cot^4 3x + C$

28. $\int \cot^3 x \csc^5 x\,dx = \int (\cot^2 x \csc^4 x)(\csc x \cot x\,dx) = \int (\csc^2 x - 1)(\csc^4 x)(\csc x \cot x\,dx)$

$= \int (\csc^6 x)(\csc x \cot x\,dx) - \int (\csc^4 x)(\csc x \cot x\,dx) = -\frac{1}{7}\csc^7 x + \frac{1}{5}\csc^5 x + C$

Supplementary Problems

In Problems 29 to 56, evaluate the integral at the left.

29. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

30. $\int \sin^3 2x\,dx = \frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + C$

31. $\int \sin^4 2x\,dx = \frac{3}{8}x - \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$

32. $\int \cos^4 \frac{1}{2}x\,dx = \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$

33. $\int \sin^7 x\,dx = \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + C$

34. $\int \cos^6 \frac{1}{2}x\,dx = \frac{5}{16}x + \frac{1}{2}\sin x + \frac{3}{32}\sin 2x - \frac{1}{24}\sin^3 x + C$

35. $\int \sin^2 x \cos^5 x\,dx = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

36. $\int \sin^3 x \cos^2 x\,dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

37. $\int \sin^3 x \cos^3 x\,dx = \frac{1}{48}\cos^3 2x - \frac{1}{16}\cos 2x + C$

38. $\int \sin^4 x \cos^4 x\, dx = \frac{1}{128}(3x - \sin 4x + \frac{1}{8}\sin 8x) + C$

39. $\int \sin 2x \cos 4x\, dx = \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

40. $\int \cos 3x \cos 2x\, dx = \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$

41. $\int \sin 5x \sin x\, dx = \frac{1}{8}\sin 4x - \frac{1}{12}\sin 6x + C$

42. $\int \frac{\cos^3 x\, dx}{1 - \sin x} = \sin x + \frac{1}{2}\sin^2 x + C$

43. $\int \frac{\cos^{2/3} x}{\sin^{8/3} x}\, dx = -\frac{3}{5}\cot^{5/3} x + C$

44. $\int \frac{\cos^3 x}{\sin^4 x}\, dx = \csc x - \frac{1}{3}\csc^3 x + C$

45. $\int x(\cos^3 x^2 - \sin^3 x^2)\, dx = \frac{1}{12}(\sin x^2 + \cos x^2)(4 + \sin 2x^2) + C$

46. $\int \tan^3 x\, dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

47. $\int \tan^3 3x \sec 3x\, dx = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

48. $\int \tan^{3/2} x \sec^4 x\, dx = \frac{2}{5}\tan^{5/2} x + \frac{2}{9}\tan^{9/2} x + C$

49. $\int \tan^4 x \sec^4 x\, dx = \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

50. $\int \csc^4 2x\, dx = -\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

51. $\int \cot^3 x\, dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

52. $\int \left(\frac{\sec x}{\tan x}\right)^4 dx = -\frac{1}{3\tan^3 x} - \frac{1}{\tan x} + C$

53. $\int \cot^3 x \csc^4 x\, dx = -\frac{1}{4}\cot^4 x - \frac{1}{6}\cot^6 x + C$

54. $\int \frac{\cot^3 x}{\csc x}\, dx = -\sin x - \csc x + C$

55. $\int \cot^3 x \csc^3 x\, dx = -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$

56. $\int \tan x \sqrt{\sec x}\, dx = 2\sqrt{\sec x} + C$

57. Use integration by parts to derive the reduction formulas

$$\int \sec^m u\, du = \frac{1}{m-1}\sec^{m-2} u \tan u + \frac{m-2}{m-1}\int \sec^{m-2} u\, du$$

and

$$\int \csc^m u\, du = -\frac{1}{m-1}\csc^{m-2} u \cot u + \frac{m-2}{m-1}\int \csc^{m-2} u\, du$$

Use the reduction formulas of Problem 57 to evaluate the left-hand integral in Problems 58 to 60.

58. $\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

59. $\int \csc^5 x\, dx = -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x + \frac{3}{8}\ln|\csc x - \cot x| + C$

60. $\int \sec^6 x\, dx = \frac{1}{5}\sec^4 x \tan x + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + C = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

Chapter 33

Trigonometric Substitutions

SOME INTEGRATIONS may be simplified with the following substitutions:

1. If an integrand contains $\sqrt{a^2 - x^2}$, substitute $x = a \sin z$.
2. If an integrand contains $\sqrt{a^2 + x^2}$, substitute $x = a \tan z$.
3. If an integrand contains $\sqrt{x^2 - a^2}$, substitute $x = a \sec z$.

More generally, an integrand that contains one of the forms $\sqrt{a^2 - b^2x^2}$, $\sqrt{a^2 + b^2x^2}$, or $\sqrt{b^2x^2 - a^2}$ but no other irrational factor may be transformed into another involving trigonometric functions of a new variable as follows:

For	Use	To obtain
$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b} \sin z$	$a\sqrt{1 - \sin^2 z} = a \cos z$
$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b} \tan z$	$a\sqrt{1 + \tan^2 z} = a \sec z$
$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b} \sec z$	$a\sqrt{\sec^2 z - 1} = a \tan z$

In each case, integration yields an expression in the variable z. The corresponding expression in the original variable may be obtained by the use of a right triangle as shown in the solved problems that follow.

Solved Problems

1. Find $\int \frac{dx}{x^2\sqrt{4 + x^2}}$.

Let $x = 2 \tan z$, so that x and z are related as in Fig. 33-1. Then $dx = 2 \sec^2 z\, dz$ and $\sqrt{4 + x^2} = 2 \sec z$, and

$$\int \frac{dx}{x^2\sqrt{4 + x^2}} = \int \frac{2 \sec^2 z\, dz}{(4 \tan^2 z)(2 \sec z)} = \frac{1}{4} \int \frac{\sec z}{\tan^2 z}\, dz = \frac{1}{4} \int \sin^{-2} z \cos z\, dz$$

$$= -\frac{1}{4 \sin z} + C = -\frac{\sqrt{4 + x^2}}{4x} + C$$

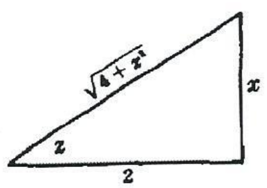

Fig. 33-1

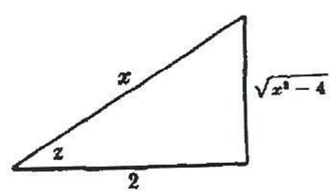

Fig. 33-2

2. Find $\int \frac{x^2}{\sqrt{x^2-4}}\,dx$.

Let $x = 2\sec z$, so that x and z are related as in Fig. 33-2. Then $dx = 2\sec z \tan z\,dz$ and $\sqrt{x^2-4} = 2\tan z$, and

$$\int \frac{x^2}{\sqrt{x^2-4}}\,dx = \int \frac{4\sec^2 z}{2\tan z}(2\sec z\tan z\,dz) = 4\int \sec^3 z\,dz$$
$$= 2\sec z\tan z + 2\ln|\sec z + \tan z| + C'$$
$$= \tfrac{1}{2}x\sqrt{x^2-4} + 2\ln|x+\sqrt{x^2-4}| + C$$

3. Find $\int \frac{\sqrt{9-4x^2}}{x}\,dx$.

Let $x = \frac{3}{2}\sin z$ (see Fig. 33-3); then $dx = \frac{3}{2}\cos z\,dz$ and $\sqrt{9-4x^2} = 3\cos z$, and

$$\int \frac{\sqrt{9-4x^2}}{x}\,dx = \int \frac{3\cos z}{\frac{3}{2}\sin z}\left(\frac{3}{2}\cos z\,dz\right) = 3\int \frac{\cos^2 z}{\sin z}\,dz = 3\int \frac{1-\sin^2 z}{\sin z}\,dx$$
$$= 3\int \csc z\,dz - 3\int \sin z\,dz = 3\ln|\csc z - \cot z| + 3\cos z + C'$$
$$= 3\ln\left|\frac{3-\sqrt{9-4x^2}}{x}\right| + \sqrt{9-4x^2} + C$$

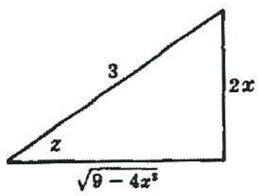

Fig. 33-3

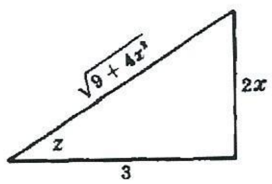

Fig. 33-4

4. Find $\int \frac{dx}{x\sqrt{9+4x^2}}$.

Let $x = \frac{3}{2}\tan z$ (see Fig. 33-4); then $dx = \frac{3}{2}\sec^2 z\,dz$ and $\sqrt{9+4x^2} = 3\sec z$, and

$$\int \frac{dx}{x\sqrt{9+4x^2}} = \int \frac{\frac{3}{2}\sec^2 z\,dz}{(\frac{3}{2}\tan z)(3\sec z)} = \frac{1}{3}\int \csc z\,dz = \frac{1}{3}\ln|\csc z - \cot z| + C'$$
$$= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{x}\right| + C$$

5. Find $\int \frac{(16-9x^2)^{3/2}}{x^6}\,dx$.

Let $x = \frac{4}{3}\sin z$ (see Fig. 33-5); then $dx = \frac{4}{3}\cos z\,dz$ and $\sqrt{16-9x^2} = 4\cos z$, and

$$\int \frac{(16-9x^2)^{3/2}}{x^6}\,dx = \int \frac{(64\cos^3 z)(\frac{4}{3}\cos z\,dz)}{\frac{4096}{729}\sin^6 z} = \frac{243}{16}\int \frac{\cos^4 z}{\sin^6 z}\,dz = \frac{243}{16}\int \cot^4 z\csc^2 z\,dz$$
$$= -\frac{243}{80}\cot^5 z + C = -\frac{243}{80}\frac{(16-9x^2)^{5/2}}{243x^5} + C = -\frac{1}{80}\frac{(16-9x^2)^{5/2}}{x^5} + C$$

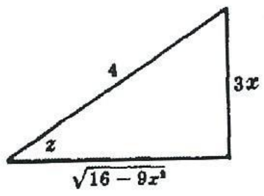

Fig. 33-5

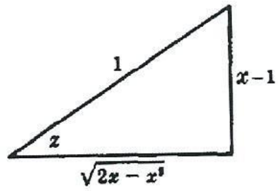

Fig. 33-6

6. Find $\int \frac{x^2\,dx}{\sqrt{2x-x^2}} = \int \frac{x^2\,dx}{\sqrt{1-(x-1)^2}}$.

Let $x-1 = \sin z$ (see Fig. 33-6); then $dx = \cos z\,dz$ and $\sqrt{2x-x^2} = \cos z$, and

$$\int \frac{x^2\,dx}{\sqrt{2x-x^2}} = \int \frac{(1+\sin z)^2}{\cos z}\cos z\,dz = \int (1+\sin z)^2\,dz = \int \left(\frac{3}{2} + 2\sin z - \frac{1}{2}\cos 2z\right) dz$$

$$= \frac{3}{2}z - 2\cos z - \frac{1}{4}\sin 2z + C = \frac{3}{2}\arcsin(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + C$$

$$= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}(x+3)\sqrt{2x-x^2} + C$$

7. Find $\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{dx}{[4(x-3)^2-9]^{3/2}}$.

Let $x-3 = \frac{3}{2}\sec z$ (see Fig. 33-7); then $dx = \frac{3}{2}\sec z\tan z\,dz$ and $\sqrt{4x^2-24x+27} = 3\tan z$, and

$$\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{\frac{3}{2}\sec z\tan z\,dz}{27\tan^3 z} = \frac{1}{18}\int \sin^{-2} z\cos z\,dz$$

$$= -\frac{1}{18}\csc z + C = -\frac{1}{9}\frac{x-3}{\sqrt{4x^2-24x+27}} + C$$

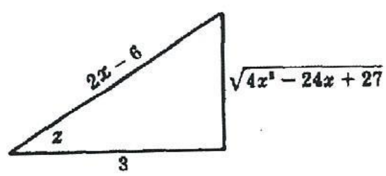

Fig. 33-7

Supplementary Problems

In Problems 8 to 22, integrate to obtain the given result.

8. $\int \frac{dx}{(4-x^2)^{3/2}} = \frac{x}{4\sqrt{4-x^2}} + C$

9. $\int \frac{\sqrt{25-x^2}}{x}\,dx = 5\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + C$

10. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

11. $\int \sqrt{x^2+4}\,dx = \frac{1}{2}x\sqrt{x^2+4} + 2\ln(x+\sqrt{x^2+4}) + C$

12. $\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + C$

13. $\int \sqrt{x^2-4}\,dx = \frac{1}{2}x\sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + C$

14. $\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} + \frac{a}{2}\ln\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a} + C$

15. $\int \frac{x^2\,dx}{(4-x^2)^{5/2}} = \frac{x^3}{12(4-x^2)^{3/2}} + C$

16. $\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$

17. $\int \frac{dx}{x^2\sqrt{9-x^2}} = -\frac{\sqrt{9-x^2}}{9x} + C$

18. $\int \frac{x^2\,dx}{\sqrt{x^2-16}} = \frac{1}{2}x\sqrt{x^2-16} + 8\ln|x+\sqrt{x^2-16}| + C$

19. $\int x^3\sqrt{a^2-x^2}\,dx = \frac{1}{5}(a^2-x^2)^{5/2} - \frac{a^2}{3}(a^2-x^2)^{3/2} + C$

20. $\int \frac{dx}{\sqrt{x^2-4x+13}} = \ln(x-2+\sqrt{x^2-4x+13}) + C$

21. $\int \frac{dx}{(4x-x^2)^{3/2}} = \frac{x-2}{4\sqrt{4x-x^2}} + C$

22. $\int \frac{dx}{(9+x^2)^2} = \frac{1}{54}\arctan\frac{x}{3} + \frac{x}{18(9+x^2)} + C$

In Problems 23 and 24, integrate by parts and apply the method of this chapter.

23. $\int x\arcsin x\,dx = \frac{1}{4}(2x^2-1)\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C$

24. $\int x\arccos x\,dx = \frac{1}{4}(2x^2-1)\arccos x - \frac{1}{4}x\sqrt{1-x^2} + C$

Chapter 34

Integration by Partial Fractions (S1)

A POLYNOMIAL IN x is a function of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where the a's are constants, $a_0 \neq 0$, and n, called the *degree* of the polynomial, is a nonnegative integer.

If two polynomials of the same degree are equal for all values of the variable, then the coefficients of the like powers of the variable in the two polynomials are equal.

Every polynomial with real coefficients can be expressed (at least, theoretically) as a product of real linear factors of the form $ax + b$ and real irreducible quadratic factors of the form $ax^2 + bx + c$. (A polynomial of degree 1 or greater is said to be *irreducible* if it cannot be factored into polynomials of lower degree.) By the quadratic formula, $ax^2 + bx + c$ is irreducible if and only if $b^2 - 4ac < 0$. (In that case, the roots of $ax^2 + bx + c = 0$ are not real.)

EXAMPLE 1: (*a*) $x^2 - x + 1$ is irreducible, since $(-1)^2 - 4(1)(1) = -3 < 0$.
(*b*) $x^2 - x - 1$ is not irreducible, since $(-1)^2 - 4(1)(-1) = 5 > 0$. In fact, $x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$.

A FUNCTION $F(x) = f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials, is called a *rational fraction*.

If the degree of $f(x)$ is less than the degree of $g(x)$, $F(x)$ is called *proper*; otherwise, $F(x)$ is called *improper*.

An improper rational fraction can be expressed as the sum of a polynomial and a proper rational fraction. Thus, $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$.

Every proper rational fraction can be expressed (at least, theoretically) as a sum of simpler fractions (*partial fractions*) whose denominators are of the form $(ax + b)^n$ and $(ax^2 + bx + c)^n$, n being a positive integer. Four cases, depending upon the nature of the factors of the denominator, arise.

CASE I: DISTINCT LINEAR FACTORS. To each linear factor $ax + b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be determined. (See Problems 1 and 2.)

CASE II: REPEATED LINEAR FACTORS. To each linear factor $ax + b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

where the A's are constants to be determined. (See Problems 3 and 4.)

CASE III: DISTINCT QUADRATIC FACTORS. To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined. (See Problems 5 and 6.)

CASE IV: REPEATED QUADRATIC FACTORS. To each irreducible quadratic factor $ax^2 + bx + c$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where the A's and B's are constants to be determined. (See Problems 7 and 8.)

Solved Problems

1. Find $\int \frac{dx}{x^2 - 4}$.

We factor the denominator into $(x-2)(x+2)$ and write $\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$. Clearing of fractions yields

$$1 = A(x+2) + B(x-2) \qquad (1)$$

or

$$1 = (A+B)x + (2A - 2B) \qquad (2)$$

We can determine the constants by either of two methods.

General method: Equate coefficients of like powers of x in (2) and solve simultaneously for the constants. Thus, $A + B = 0$ and $2A - 2B = 1$; $A = \frac{1}{4}$ and $B = -\frac{1}{4}$.

Short method: Substitute in (1) the values $x = 2$ and $x = -2$ to obtain $1 = 4A$ and $1 = -4B$; then $A = \frac{1}{4}$ and $B = -\frac{1}{4}$, as before. (Note that the values of x used are those for which the denominators of the partial fractions become 0.)

By either method, we have $\frac{1}{x^2-4} = \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}$. Then

$$\int \frac{dx}{x^2-4} = \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2} = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

2. Find $\int \frac{(x+1)\,dx}{x^3 + x^2 - 6x}$.

Factoring yields $x^3 + x^2 - 6x = x(x-2)(x+3)$. Then $\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$ and

$$x + 1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2) \qquad (1)$$

$$x + 1 = (A+B+C)x^2 + (A + 3B - 2C)x - 6A \qquad (2)$$

General method: We solve simultaneously the system of equations

$$A + B + C = 0 \qquad A + 3B - 2C = 1 \qquad -6A = 1$$

to obtain $A = -\frac{1}{6}$, $B = \frac{3}{10}$, and $C = -\frac{2}{15}$.

Short method: We substitute in (1) the values $x = 0$, $x = 2$, and $x = -3$ to obtain $1 = -6A$ or $A = -1/6$, $3 = 10B$ or $B = 3/10$, and $-2 = 15C$ or $C = -2/15$.

By either method,

$$\int \frac{(x+1)\,dx}{x^3+x^2-6x} = -\frac{1}{6}\int\frac{dx}{x} + \frac{3}{10}\int\frac{dx}{x-2} - \frac{2}{15}\int\frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C = \ln\frac{|x-2|^{3/10}}{|x|^{1/6}|x+3|^{2/15}} + C$$

3. Find $\int \frac{(3x+5)\,dx}{x^3 - x^2 - x + 1}$.

$x^3 - x^2 - x + 1 = (x+1)(x-1)^2$. Hence, $\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ and

$$3x + 5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

For $x = -1$, $2 = 4A$ and $A = \frac{1}{2}$. For $x = 1$, $8 = 2C$ and $C = 4$. To determine the remaining constant, we use any other value of x, say $x = 0$; for $x = 0$, $5 = A - B + C$ and $B = -\frac{1}{2}$. Thus,

$$\int \frac{3x+5}{x^3 - x^2 - x + 1}\,dx = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| - \frac{4}{x-1} + C = -\frac{4}{x-1} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

4. Find $\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}\,dx$.

The integrand is an improper fraction. By division,

$$\frac{x^4 - x^3 - x - 1}{x^3 - x^2} = x - \frac{x+1}{x^3 - x^2} = x - \frac{x+1}{x^2(x-1)}$$

We write $\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ and obtain

$$x + 1 = Ax(x-1) + B(x-1) + Cx^2$$

For $x = 0$, $1 = -B$ and $B = -1$. For $x = 1$, $2 = C$. For $x = 2$, $3 = 2A + B + 4C$ and $A = -2$. Thus,

$$\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}\,dx = \int x\,dx + 2\int \frac{dx}{x} + \int \frac{dx}{x^2} - 2\int \frac{dx}{x-1}$$

$$= \frac{1}{2}x^2 + 2\ln|x| - \frac{1}{x} - 2\ln|x-1| + C = \frac{1}{2}x^2 - \frac{1}{x} + 2\ln\left|\frac{x}{x-1}\right| + C$$

5. Find $\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\,dx$.

$x^4 + 3x^2 + 2 = (x^2+1)(x^2+2)$. We write $\frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$ and obtain

$$x^3 + x^2 + x + 2 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$
$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$$

Hence $A + C = 1$, $B + D = 1$, $2A + C = 1$, and $2B + D = 2$. Solving simultaneously yields $A = 0$, $B = 1$, $C = 1$, $D = 0$. Thus,

$$\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\,dx = \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{x^2+2} = \arctan x + \frac{1}{2}\ln(x^2+2) + C$$

6. Solve the equation $\int \frac{x^2\,dx}{a^4 - x^4} = \int k\,dt$, which occurs in physical chemistry.

We write $\frac{x^2}{a^4 - x^4} = \frac{A}{a-x} + \frac{B}{a+x} + \frac{Cx+D}{a^2+x^2}$. Then

$$x^2 = A(a+x)(a^2+x^2) + B(a-x)(a^2+x^2) + (Cx+D)(a-x)(a+x)$$

For $x = a$, $a^2 = 4Aa^3$ and $A = 1/4a$. For $x = -a$, $a^2 = 4Ba^3$ and $B = 1/4a$. For $x = 0$, $0 = Aa^3 + Ba^3 + Da^2 = a^2/2 + Da^2$ and $D = -\frac{1}{2}$. For $x = 2a$, $4a^2 = 15Aa^3 - 5Ba^3 - 6Ca^3 - 3Da^2$ and $C = 0$. Thus,

$$\int \frac{x^2\,dx}{a^4 - x^4} = \frac{1}{4a}\int \frac{dx}{a-x} + \frac{1}{4a}\int \frac{dx}{a+x} - \frac{1}{2}\int \frac{dx}{a^2+x^2}$$

$$= -\frac{1}{4a}\ln|a-x| + \frac{1}{4a}\ln|a+x| - \frac{1}{2a}\arctan\frac{x}{a} + C$$

so that

$$\int k\,dt = kt = \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\arctan\frac{x}{a} + C$$

7. Find $\int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}\,dx$.

We write $\frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}$. Then

$$x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4 = (Ax+B)(x^2+2)^2 + (Cx+D)(x^2+2) + Ex + F$$

$$= Ax^5 + Bx^4 + (4A+C)x^3 + (4B+D)x^2 + (4A+2C+E)x + (4B+2D+F)$$

from which $A = 1$, $B = -1$, $C = 0$, $D = 0$, $E = 4$, $F = 0$. Thus the given integral is equal to

$$\int \frac{x-1}{x^2+2}\,dx + 4\int \frac{x\,dx}{(x^2+2)^3} = \frac{1}{2}\ln(x^2+2) - \frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}} - \frac{1}{(x^2+2)^2} + C$$

8. Find $\int \frac{2x^2+3}{(x^2+1)^2}\,dx$.

We write $\frac{2x^2+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$. Then

$$2x^2 + 3 = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

from which $A = 0$, $B = 2$, $A + C = 0$, $B + D = 3$. Thus $A = 0$, $B = 2$, $C = 0$, $D = 1$ and

$$\int \frac{2x^2+3}{(x^2+1)^2}\,dx = \int \frac{2\,dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$$

For the second integral on the right, let $x = \tan z$. Then

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2 z\,dz}{\sec^4 z} = \int \cos^2 z\,dz = \frac{1}{2}z + \frac{1}{4}\sin 2z + C$$

and

$$\int \frac{2x^2+3}{(x^2+1)^2}\,dx = 2\arctan x + \frac{1}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C = \frac{5}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C$$

Supplementary Problems

In Problems 9 to 27, evaluate the integral at the left.

9. $\int \frac{dx}{x^2-9} = \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$

10. $\int \frac{dx}{x^2+7x+6} = \frac{1}{5}\ln\left|\frac{x+1}{x+6}\right| + C$

11. $\int \frac{x\,dx}{x^2-3x-4} = \frac{1}{5}\ln|(x+1)(x-4)^4| + C$

12. $\int \frac{x^2+3x-4}{x^2-2x-8}\,dx = x + \ln|(x+2)(x-4)^4| + C$

13. $\int \frac{x^2-3x-1}{x^3+x^2-2x}\,dx = \ln\left|\frac{x^{1/2}(x+2)^{3/2}}{x-1}\right| + C$

14. $\int \frac{x\,dx}{(x-2)^2} = \ln|x-2| - \frac{2}{x-2} + C$

15. $\int \frac{x^4}{(1-x)^3}\,dx = -\frac{1}{2}x^2 - 3x - \ln(1-x)^6 - \frac{4}{1-x} + \frac{1}{2(1-x)^2} + C$

16. $\int \frac{dx}{x^3+x} = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

17. $\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}\,dx = \ln\sqrt{x^2+3} + \arctan x + C$

18. $\int \frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}\,dx = \frac{1}{2}x^2 + \ln\left|\frac{x}{\sqrt{x^2-2x+3}}\right| + C$

19. $\int \frac{2x^3\,dx}{(x^2+1)^2} = \ln(x^2+1) + \frac{1}{x^2+1} + C$

20. $\int \frac{2x^3+x^2+4}{(x^2+4)^2}\,dx = \ln(x^2+4) + \frac{1}{2}\arctan\frac{1}{2}x + \frac{4}{x^2+4} + C$

21. $\int \frac{x^3+x-1}{(x^2+1)^2}\,dx = \ln\sqrt{x^2+1} - \frac{1}{2}\arctan x - \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C$

22. $\int \frac{x^4+8x^3-x^2+2x+1}{(x^2+x)(x^3+1)}\,dx = \ln\left|\frac{x^3-x^2+x}{(x+1)^2}\right| - \frac{3}{x+1} + \frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$

23. $\int \frac{x^3+x^2-5x+15}{(x^2+5)(x^2+2x+3)}\,dx = \ln\sqrt{x^2+2x+3} + \frac{5}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} - \sqrt{5}\arctan\frac{x}{\sqrt{5}} + C$

24. $\int \frac{x^6+7x^5+15x^4+32x^3+23x^2+25x-3}{(x^2+x+2)^2(x^2+1)^2}\,dx = \frac{1}{x^2+x+2} - \frac{3}{x^2+1} + \ln\frac{x^2+1}{x^2+x+2} + C$

25. $\int \frac{dx}{e^{2x}-3e^x} = \frac{1}{3e^x} + \frac{1}{9}\ln\left|\frac{e^x-3}{e^x}\right| + C$ (*Hint*: Let $e^x = u$.)

26. $\int \frac{\sin x\,dx}{\cos x\,(1+\cos^2 x)} = \ln\left|\frac{\sqrt{1+\cos^2 x}}{\cos x}\right| + C$ (*Hint*: Let $\cos x = u$.)

27. $\int \frac{(2+\tan^2\theta)\sec^2\theta\,d\theta}{1+\tan^3\theta} = \ln|1+\tan\theta| + \frac{2}{\sqrt{3}}\arctan\frac{2\tan\theta-1}{\sqrt{3}} + C$

Chapter 35

Miscellaneous Substitutions

IF AN INTEGRAND IS RATIONAL except for a radical of the form

1. $\sqrt[n]{ax+b}$, then the substitution $ax+b=z^n$ will replace it with a rational integrand.
2. $\sqrt{q+px+x^2}$, then the substitution $q+px+x^2=(z-x)^2$ will replace it with a rational integrand.
3. $\sqrt{q+px-x^2}=\sqrt{(\alpha+x)(\beta-x)}$, then the substitution $q+px-x^2=(\alpha+x)^2z^2$ or $q+px-x^2=(\beta-x)^2z^2$ will replace it with a rational integrand.

(See Problems 1 to 5.)

THE SUBSTITUTION $x=2\arctan z$ will replace any rational function of $\sin x$ and $\cos x$ with a rational function of z, since

$$\sin x=\frac{2z}{1+z^2}\qquad \cos x=\frac{1-z^2}{1+z^2}\qquad \text{and}\qquad dx=\frac{2\,dz}{1+z^2}$$

(The first and second of these relations are obtained from Fig. 35-1, and the third by differentiating $x=2\arctan z$.) After integrating, use $z=\tan\frac{1}{2}x$ to return to the original variable. (See Problems 6 to 10.)

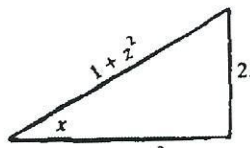

Fig. 35-1

EFFECTIVE SUBSTITUTIONS are often suggested by the form of the integrand. (See Problems 11 and 12.)

Solved Problems

1. Find $\int \frac{dx}{x\sqrt{1-x}}$.

Let $1-x=z^2$. Then $x=1-z^2$, $dx=-2z\,dz$, and

$$\int\frac{dx}{x\sqrt{1-x}}=\int\frac{-2z\,dz}{(1-z^2)z}=-2\int\frac{dz}{1-z^2}=-\ln\left|\frac{1+z}{1-z}\right|+C=\ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right|+C$$

2. Find $\int \frac{dx}{(x-2)\sqrt{x+2}}$.

Let $x + 2 = z^2$. Then $x = z^2 - 2$, $dx = 2z\,dz$, and

$$\int \frac{dx}{(x-2)\sqrt{x+2}} = \int \frac{2z\,dz}{z(z^2-4)} = 2\int \frac{dz}{z^2-4} = \frac{1}{2}\ln\left|\frac{z-2}{z+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right| + C$$

3. Find $\int \frac{dx}{x^{1/2} - x^{1/4}}$.

Let $x = z^4$. Then $dx = 4z^3\,dz$ and

$$\int \frac{dx}{x^{1/2} - x^{1/4}} = \int \frac{4z^3\,dz}{z^2 - z} = 4\int \frac{z^2}{z-1}\,dz = 4\int \left(z + 1 + \frac{1}{z-1}\right) dz$$

$$= 4(\tfrac{1}{2}z^2 + z + \ln|z-1|) + C = 2\sqrt{x} + 4\sqrt[4]{x} + \ln(\sqrt[4]{x} - 1)^4 + C$$

4. Find $\int \frac{dx}{x\sqrt{x^2+x+2}}$.

Let $x^2 + x + 2 = (z-x)^2$. Then

$$x = \frac{z^2-2}{1+2z} \qquad dx = \frac{2(z^2+z+2)\,dz}{(1+2z)^2} \qquad \sqrt{x^2+x+2} = \frac{z^2+z+2}{1+2z}$$

and

$$\int \frac{dx}{x\sqrt{x^2+x+2}} = \int \frac{\frac{2(z^2+z+2)}{(1+2z)^2}}{\frac{z^2-2}{1+2z}\,\frac{z^2+z+2}{1+2z}}\,dz = 2\int \frac{dz}{z^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{x^2+x+2}+x-\sqrt{2}}{\sqrt{x^2+x+2}+x+\sqrt{2}}\right| + C$$

5. Find $\int \frac{x\,dx}{(5-4x-x^2)^{3/2}}$.

Let $5 - 4x - x^2 = (5+x)(1-x) = (1-x)^2z^2$. Then

$$x = \frac{z^2-5}{1+z^2} \qquad dx = \frac{12z\,dz}{(1+z^2)^2} \qquad \sqrt{5-4x-x^2} = (1-x)z = \frac{6z}{1+z^2}$$

and

$$\int \frac{x\,dx}{(5-4x-x^2)^{3/2}} = \int \frac{\frac{z^2-5}{1+z^2}\,\frac{12z}{(1+z^2)^2}}{\frac{216z^3}{(1+z^2)^3}}\,dz = \frac{1}{18}\int \left(1 - \frac{5}{z^2}\right) dz$$

$$= \frac{1}{18}\left(z + \frac{5}{z}\right) + C = \frac{5-2x}{9\sqrt{5-4x-x^2}} + C$$

In Problems 6 to 10, evaluate the integral at the left.

6. $$\int \frac{dx}{1+\sin x - \cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{1 + \frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2}} = \int \frac{dz}{z(1+z)} = \ln|z| - \ln|1+z| + C$$

$$= \ln\left|\frac{z}{1+z}\right| + C = \ln\left|\frac{\tan\frac{1}{2}x}{1+\tan\frac{1}{2}x}\right| + C$$

7. $$\int \frac{dx}{3-2\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{3-2\frac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{1+5z^2} = \frac{2\sqrt{5}}{5}\arctan z\sqrt{5} + C$$
$$= \frac{2\sqrt{5}}{5}\arctan(\sqrt{5}\tan\tfrac{1}{2}x) + C$$

8. $$\int \sec x\,dx = \int \frac{1+z^2}{1-z^2}\frac{2\,dz}{1+z^2} = 2\int \frac{dz}{1-z^2} = \ln\left|\frac{1+z}{1-z}\right| + C = \ln\left|\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right| + C$$
$$= \ln\left|\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right| + C$$

9. $$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = 2\int \frac{dz}{3+z^2} = \frac{2}{\sqrt{3}}\arctan\frac{z}{\sqrt{3}} + C$$
$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\tan\frac{1}{2}x\right) + C$$

10. $$\int \frac{dx}{5+4\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{5+4\frac{2z}{1+z^2}} = \int \frac{2\,dz}{5+8z+5z^2} = \frac{2}{5}\int \frac{dz}{(z+\frac{4}{5})^2+\frac{9}{25}}$$
$$= \frac{2}{3}\arctan\frac{z+\frac{4}{5}}{\frac{3}{5}} + C = \frac{2}{3}\arctan\frac{5\tan\frac{1}{2}x+4}{3} + C$$

11. Use the substitution $1-x^3=z^2$ to find $\int x^5\sqrt{1-x^3}\,dx$.

The substitution yields $x^3 = 1-z^2$, $3x^2\,dx = -2z\,dz$, and
$$\int x^5\sqrt{1-x^3}\,dx = \int x^3\sqrt{1-x^3}\,(x^2\,dx) = \int (1-z^2)z(-\tfrac{2}{3}z\,dz) = -\frac{2}{3}\int (1-z^2)z^2\,dz$$
$$= -\frac{2}{3}\left(\frac{z^3}{3}-\frac{z^5}{5}\right) + C = -\frac{2}{45}(1-x^3)^{3/2}(2+3x^3) + C$$

12. Use $x=\dfrac{1}{z}$ to find $\displaystyle\int \frac{\sqrt{x-x^2}}{x^4}\,dx$.

The substitution yields $dx = -dz/z^2$, $\sqrt{x-x^2} = \sqrt{z-1}/z$, and
$$\int \frac{\sqrt{x-x^2}}{x^4}\,dx = \int \frac{\frac{\sqrt{z-1}}{z}\left(-\frac{dz}{z^2}\right)}{1/z^4} = -\int z\sqrt{z-1}\,dz$$

Let $z-1=s^2$. Then
$$-\int z\sqrt{z-1}\,dz = -\int (s^2+1)(s)(2s\,ds) = -2\left(\frac{s^5}{5}+\frac{s^3}{3}\right) + C$$
$$= -2\left[\frac{(z-1)^{5/2}}{5} + \frac{(z-1)^{3/2}}{3}\right] + C = -2\left[\frac{(1-x)^{5/2}}{5x^{5/2}} + \frac{(1-x)^{3/2}}{3x^{3/2}}\right] + C$$

13. Find $\displaystyle\int \frac{dx}{x^{1/2}+x^{1/3}}$.

Let $u = x^{1/6}$, so that $x = u^6$, $dx = 6u^5\,du$, $x^{1/2} = u^3$, and $x^{1/3} = u^2$. Then we obtain

$$\int \frac{6u^5\,du}{u^3 + u^2} = 6\int \frac{u^3}{u+1}\,du = 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right) du = 6\left(\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right) + C$$
$$= 2x^{1/2} - 3x^{1/3} + x^{1/6} - \ln|x^{1/6} + 1| + C$$

Supplementary Problems

In Problems 14 to 39, evaluate the integral at the left.

14. $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx = 2\sqrt{x} - 2\arctan\sqrt{x} + C$

15. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\ln(1+\sqrt{x}) + C$

16. $\displaystyle\int \frac{dx}{3+\sqrt{x+2}} = 2\sqrt{x+2} - 6\ln(3+\sqrt{x+2}) + C$

17. $\displaystyle\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}\,dx = -x + \frac{4}{3}\left\{\sqrt{3x+2} - \ln(1+\sqrt{3x+2})\right\} + C$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2-x+1}} = \ln|2\sqrt{x^2-x+1} + 2x - 1| + C$

19. $\displaystyle\int \frac{dx}{x\sqrt{x^2+x-1}} = 2\arctan(\sqrt{x^2+x-1} + x) + C$

20. $\displaystyle\int \frac{dx}{\sqrt{6+x-x^2}} = \arcsin\frac{2x-1}{5} + C$

21. $\displaystyle\int \frac{\sqrt{4x-x^2}}{x^3}\,dx = -\frac{(4x-x^2)^{3/2}}{6x^3} + C$

22. $\displaystyle\int \frac{dx}{(x+1)^{1/2} + (x+1)^{1/4}} = 2(x+1)^{1/2} - 4(x+1)^{1/4} + 4\ln(1+(x+1)^{1/4}) + C$

23. $\displaystyle\int \frac{dx}{2+\sin x} = \frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{1}{2}x + 1}{\sqrt{3}} + C$

24. $\displaystyle\int \frac{dx}{1-2\sin x} = \frac{\sqrt{3}}{3}\ln\left|\frac{\tan\frac{1}{2}x - 2 - \sqrt{3}}{\tan\frac{1}{2}x - 2 + \sqrt{3}}\right| + C$

25. $\displaystyle\int \frac{dx}{3+5\sin x} = \frac{1}{4}\ln\left|\frac{3\tan\frac{1}{2}x + 1}{\tan\frac{1}{2}x + 3}\right| + C$

26. $\displaystyle\int \frac{dx}{\sin x - \cos x - 1} = \ln|\tan\tfrac{1}{2}x - 1| + C$

27. $\displaystyle\int \frac{dx}{5+3\sin x} = \frac{1}{2}\arctan\frac{5\tan\frac{1}{2}x + 3}{4} + C$

28. $\displaystyle\int \frac{\sin x\,dx}{1+\sin^2 x} = \frac{\sqrt{2}}{4}\ln\left|\frac{\tan^2\frac{1}{2}x + 3 - 2\sqrt{2}}{\tan^2\frac{1}{2}x + 3 + 2\sqrt{2}}\right| + C$

29. $\displaystyle\int \frac{dx}{1+\sin x+\cos x} = \ln|1 + \tan\tfrac{1}{2}x| + C$

30. $\displaystyle\int \frac{dx}{2-\cos x} = \frac{2}{\sqrt{3}}\arctan(\sqrt{3}\tan\tfrac{1}{2}x) + C$

31. $\displaystyle\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

32. $\displaystyle\int \frac{dx}{x\sqrt{3x^2+2x-1}} = -\arcsin\frac{1-x}{2x} + C$ (*Hint*: Let $x = 1/z$.)

33. $\displaystyle\int \frac{(e^x-2)e^x}{e^x+1}\,dx = e^x - 3\ln(e^x+1) + C$ (*Hint*: Let $e^x + 1 = z$.)

34. $\displaystyle\int \frac{\sin x \cos x}{1-\cos x}\,dx = \cos x + \ln(1-\cos x) + C$ (*Hint*: Let $\cos x = z$.)

35. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4x} + C$ (*Hint*: Let $x = 2/z$.)

36. $\displaystyle\int \frac{dx}{x^2(4+x^2)} = -\frac{1}{4x} + \frac{1}{8}\arctan\frac{2}{x} + C$

37. $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx = \tfrac{4}{5}(1+\sqrt{x})^{5/2} - \tfrac{4}{3}(1+\sqrt{x})^{3/2} + C$

38. $\displaystyle\int \frac{dx}{3(1-x^2)-(5+4x)\sqrt{1-x^2}} = \frac{2\sqrt{1+x}}{3\sqrt{1+x}-\sqrt{1-x}} + C$

39. $\displaystyle\int \frac{x^{1/2}}{x^{1/5}+1}\,dx = 10\left(\frac{1}{13}x^{13/10} - \frac{1}{11}x^{11/10} + \frac{1}{9}x^{9/10} - \frac{1}{7}x^{7/10} + \frac{1}{5}x^{5/10} - \frac{1}{3}x^{3/10} + x^{1/10} - \arctan x^{1/10}\right)$
$+ C$ (*Hint*: Let $u = x^{1/10}$.)

Chapter 36

Integration of Hyperbolic Functions

INTEGRATION FORMULAS. The following formulas are direct consequences of the differentiation formulas of Chapter 20.

28. $\int \sinh x\,dx = \cosh x + C$

29. $\int \cosh x\,dx = \sinh x + C$

30. $\int \tanh x\,dx = \ln \cosh x + C$

31. $\int \coth x\,dx = \ln |\sinh x| + C$

32. $\int \operatorname{sech}^2 x\,dx = \tanh x + C$

33. $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

34. $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$

35. $\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$

36. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C$

37. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C, \quad x > a > 0$

38. $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C, \quad x^2 < a^2$

39. $\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \coth^{-1} \frac{x}{a} + C, \quad x^2 > a^2$

Solved Problems

In Problems 1 to 13, evaluate the integral at the left.

1. $\int \sinh \tfrac{1}{2}x\,dx = 2\int \sinh \tfrac{1}{2}x\,d(\tfrac{1}{2}x) = 2\cosh \tfrac{1}{2}x + C$

2. $\int \cosh 2x\,dx = \tfrac{1}{2}\int \cosh 2x\,d(2x) = \tfrac{1}{2}\sinh 2x + C$

3. $\int \operatorname{sech}^2 (2x-1)\,dx = \tfrac{1}{2}\int \operatorname{sech}^2 (2x-1)\,d(2x-1) = \tfrac{1}{2}\tanh (2x-1) + C$

4. $\int \operatorname{csch} 3x \coth 3x\,dx = \tfrac{1}{3}\int \operatorname{csch} 3x \coth 3x\,d(3x) = -\tfrac{1}{3}\operatorname{csch} 3x + C$

5. $\int \operatorname{sech} x\,dx = \int \frac{1}{\cosh x}\,dx = \int \frac{\cosh x}{\cosh^2 x} = \int \frac{\cosh x}{1 + \sinh^2 x}\,dx = \arctan(\sinh x) + C$

6. $\int \sinh^2 x\,dx = \tfrac{1}{2}\int (\cosh 2x - 1)\,dx = \tfrac{1}{4}\sinh 2x - \tfrac{1}{2}x + C$

7. $\int \tanh^2 2x\,dx = \int (1 - \operatorname{sech}^2 2x)\,dx = x - \tfrac{1}{2}\tanh 2x + C$

8. $\int \cosh^3 \frac{1}{2}x\,dx = \int (1+\sinh^2 \frac{1}{2}x)\cosh \frac{1}{2}x\,dx = 2\sinh \frac{1}{2}x + \frac{2}{3}\sinh^3 \frac{1}{2}x + C$

9. $\int \text{sech}^4 x\,dx = \int (1-\tanh^2 x)\,\text{sech}^2 x\,dx = \tanh x - \frac{1}{3}\tanh^3 x + C$

10. $\int e^x \cosh x\,dx = \int e^x \frac{e^x + e^{-x}}{2}\,dx = \frac{1}{2}\int (e^{2x}+1)\,dx = \frac{1}{4}e^{2x} + \frac{1}{2}x + C$

11. $\int x \sinh x\,dx = \int x \frac{e^x - e^{-x}}{2}\,dx = \frac{1}{2}\int xe^x\,dx - \frac{1}{2}\int xe^{-x}\,dx$

$$= \frac{1}{2}(xe^x - e^x) - \frac{1}{2}(-xe^{-x} - e^{-x}) + C = x\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} + C$$
$$= x\cosh x - \sinh x + C$$

12. $\int \frac{dx}{\sqrt{4x^2 - 9}} = \frac{1}{2}\cosh^{-1}\frac{2x}{3} + C$

13. $\int \frac{dx}{9x^2 - 25} = -\frac{1}{15}\coth^{-1}\frac{3x}{5} + C$

14. Find $\int \sqrt{x^2+4}\,dx$.

Let $x = 2\sinh z$. Then $dx = 2\cosh z\,dz$, $\sqrt{x^2+4} = 2\cosh z$, and

$$\int \sqrt{x^2+4}\,dx = 4\int \cosh^2 z\,dz = 2\int (\cosh 2z + 1)\,dz = \sinh 2z + 2z + C$$
$$= 2\sinh z\cosh z + 2z + C = \tfrac{1}{2}x\sqrt{x^2+4} + 2\sinh^{-1}\tfrac{1}{2}x + C$$

15. Find $\int \frac{dx}{x\sqrt{1-x^2}}$.

Let $x = \text{sech}\,z$. Then $dx = -\text{sech}\,z\tanh z\,dz$, $1 - x^2 = \tanh z$, and

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\int \frac{\text{sech}\,z\tanh z}{\text{sech}\,z\tanh z}\,dz = -\int dz = -z + C = -\text{sech}^{-1}x + C$$

Supplementary Problems

In Problems 16 to 39, evaluate the integral at the left.

16. $\int \sinh 3x\,dx = \frac{1}{3}\cosh 3x + C$

17. $\int \cosh \frac{1}{4}x\,dx = 4\sinh \frac{1}{4}x + C$

18. $\int \coth \frac{3}{2}x\,dx = \frac{2}{3}\ln|\sinh \frac{3}{2}x| + C$

19. $\int \text{csch}^2(1+3x)\,dx = -\frac{1}{3}\coth(1+3x) + C$

20. $\int \text{sech}\,2x\tanh 2x\,dx = -\frac{1}{2}\text{sech}\,2x + C$

21. $\int \text{csch}\,x\,dx = \ln\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} + C$

22. $\int \cosh^2 \frac{1}{2}x\,dx = \frac{1}{2}(\sinh x + x) + C$

23. $\int \coth^2 3x\,dx = x - \frac{1}{3}\coth 3x + C$

24. $\int \sinh^3 x\,dx = \frac{1}{3}\cosh^3 x - \cosh x + C$

25. $\int e^x \sinh x\,dx = \frac{1}{4}e^{2x} - \frac{1}{2}x + C$

26. $\int e^{2x} \cosh x\, dx = \frac{1}{6}e^{3x} + \frac{1}{2}e^{x} + C$

27. $\int x \cosh x\, dx = x \sinh x - \cosh x + C$

28. $\int x^2 \sinh x\, dx = (x^2 + 2) \cosh x - 2x \sinh x + C$

29. $\int \sinh^3 x \cosh^2 x\, dx = \frac{1}{5} \cosh^5 x - \frac{1}{3} \cosh^3 x + C$

30. $\int \sinh x \ln \cosh^2 x\, dx = \cosh x\, (\ln \cosh^2 x - 2) + C$

31. $\int \frac{dx}{\sqrt{x^2 + 9}} = \sinh^{-1} \frac{x}{3} + C$

32. $\int \frac{dx}{\sqrt{x^2 - 25}} = \cosh^{-1} \frac{x}{5} + C$

33. $\int \frac{dx}{4 - 9x^2} = \frac{1}{6} \tanh^{-1} \frac{3}{2} x + C$

34. $\int \frac{dx}{16x^2 - 9} = -\frac{1}{12} \coth^{-1} \frac{4}{3} x + C$

35. $\int \sqrt{x^2 - 9}\, dx = \frac{x}{2} \sqrt{x^2 - 9} - \frac{9}{2} \cosh^{-1} \frac{x}{3} + C$

36. $\int \frac{dx}{\sqrt{x^2 - 2x + 17}} = \sinh^{-1} \frac{x - 1}{4} + C$

37. $\int \frac{dx}{4x^2 + 12x + 5} = -\frac{1}{4} \coth^{-1} \left(x + \frac{3}{2}\right) + C$

38. $\int \frac{x^2}{(x^2 + 4)^{3/2}}\, dx = \sinh^{-1} \frac{x}{2} - \frac{x}{\sqrt{x^2 + 4}} + C$

39. $\int \frac{\sqrt{x^2 + 1}}{x^2}\, dx = \sinh^{-1} x - \frac{\sqrt{1 + x^2}}{x} + C$