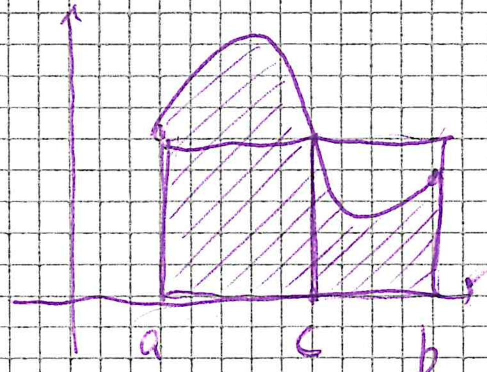
12 - 12 - 2018

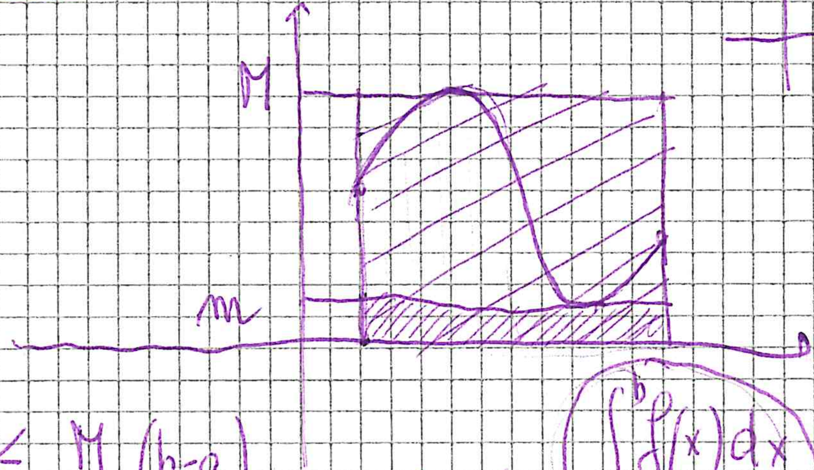
TEOREMA DELLA MEDIA INTEGRALE

Sia $f(x)$ continua in $[a,b]$. Allora $\exists\ c \in [a,b]$ tale che

$$\boxed{\int_a^b f(x)\,dx = f(c)\cdot(b-a)}$$

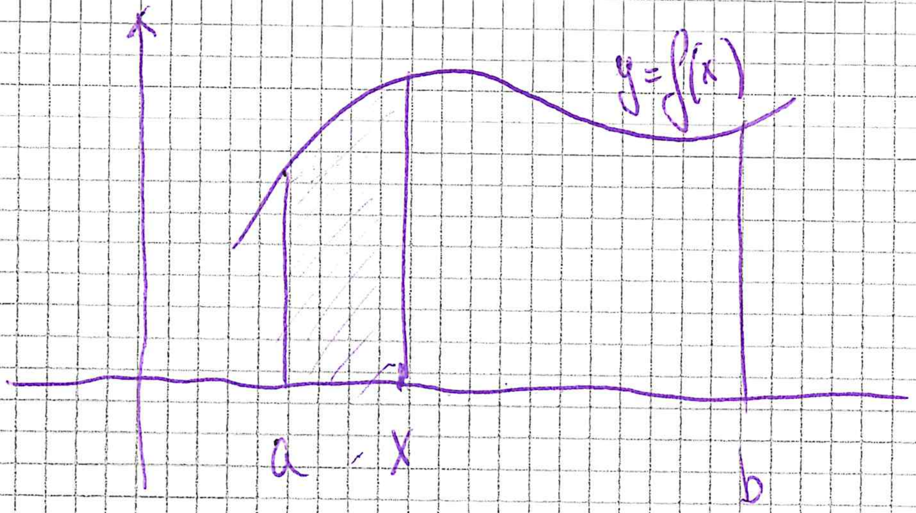


Dim.: Per W. $\exists$ M e m



$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \qquad m \le \frac{\int_a^b f(x)\,dx}{b-a} \le M$$

$$\exists\ c \in [a,b] \text{ t.c. } f(c) = \frac{\int_a^b f(x)\,dx}{b-a} \Rightarrow f(c)(b-a) = \int_a^b f(x)\,dx$$



$$F(x) \overset{def.}{=} \int_a^x f(t)\,dt$$

$F(x)$ si chiama "funzione integrale" della funzione $f(x)$.

$$F(a) = \int_a^a f(x)\,dx = 0$$

$$F(b) = \int_a^b f(x)\,dx$$

Teorema fondamentale del calcolo integrale (Torricelli - Barrow)

1) $F(x)$ è derivabile in $[a, b]$
2) $F'(x) = f(x)$

cioè $F(x)$ è una primitiva di $f(x)$

276

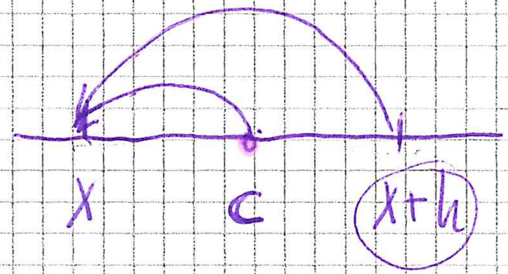
Dim.:

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt-\int_a^{x} f(t)\,dt}{h}=$$

$$=\lim_{h\to 0}\frac{\cancel{\int_a^{x} f(t)\,dt}+\int_x^{x+h} f(t)\,dt-\cancel{\int_a^{x} f(t)\,dt}}{h}=\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$\int_a^{c}+\int_c^{b}=\int_a^{b}$$

$\exists\, c\in[x,x+h]$ tale che $\displaystyle\int_x^{x+h} f(t)\,dt=f(c)(\cancel{x}+h-\cancel{x})=f(c)\,h$



$$\lim_{h\to 0}\frac{f(c)\cancel{h}}{\cancel{h}}=$$

$$=\lim_{c\to x} f(c)=f(x)$$ poiché $f(x)$ è continua!!!

Sia $f(x)$ continua in $[a,b]$ e sia $G(x)$ una primitiva in $[a,b]$ di $f(x)$. Allora

$$G(x) = F(x) + C \Rightarrow G(b) = F(b) + C$$

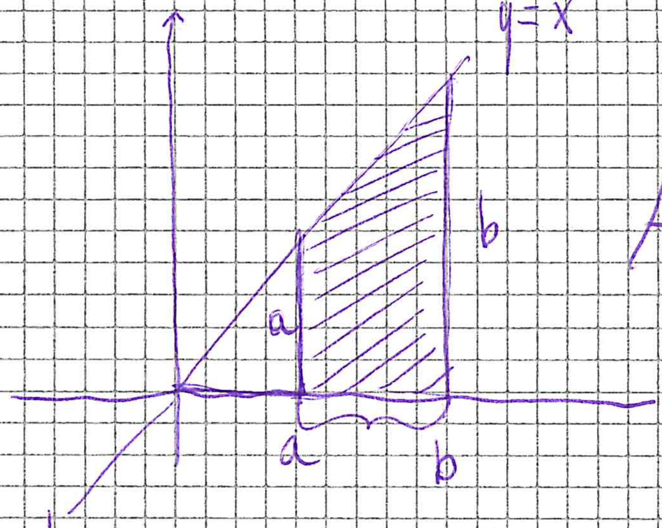
$$G(a) = F(a) + C$$

$$G(b) - G(a) = F(b) + \cancel{C} - F(a) - \cancel{C} = \int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = G(b) - G(a) = \Big[G(x)\Big]_a^b$$

278

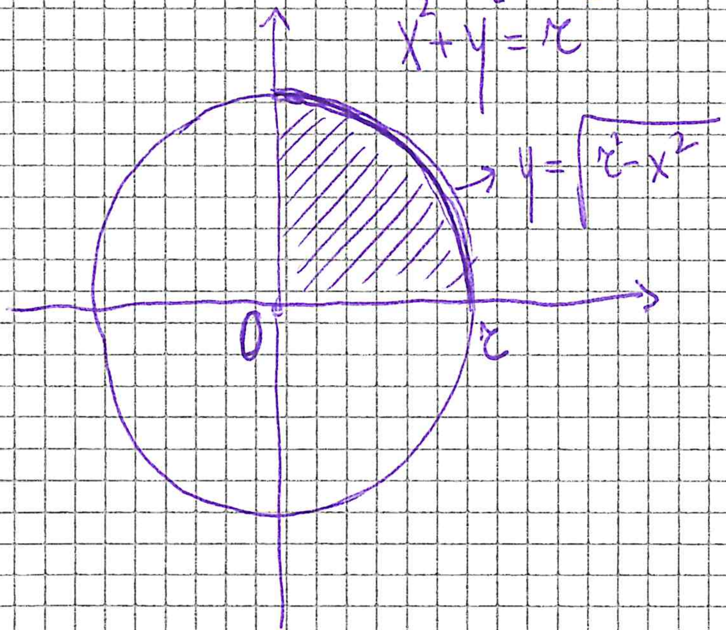
$$\int_a^b x\,dx$$



$$A = \frac{(b+a)(b-a)}{2} = \frac{b^2-a^2}{2}$$

$$\int_a^b x\,dx = \left[\frac{x^2}{2}\right]_a^b = \frac{b^2}{2} - \frac{a^2}{2} = \frac{b^2-a^2}{2}$$

$$x^2+y^2=r^2 \qquad y = \pm\sqrt{r^2-x^2}$$



$$4\int_0^r \sqrt{r^2-x^2}\,dx$$

279

$$4\int_0^r \sqrt{r^2-x^2}\,dx \qquad x = r\,\mathrm{sen}\,t \qquad dx = r\cos t\,dt$$

$x=0 \rightarrow r\,\mathrm{sen}\,t = 0 \rightarrow \mathrm{sen}\,t = 0 \rightarrow t = 0$

$x=r \rightarrow \cancel{r}\,\mathrm{sen}\,t = \cancel{r}^{\,1} \rightarrow t = \frac{\pi}{2}$

$$4\int_0^{\pi/2}\sqrt{r^2 - r^2\mathrm{sen}^2 t}\cdot r\cos t\,dt = 4\int_0^{\frac{\pi}{2}}\sqrt{r^2\cos^2 t}\cdot r\cos t\,dt$$

$r^2(1-\mathrm{sen}^2 t) = r^2\cos^2 t$

$$= 4\int_0^{\frac{\pi}{2}} r^2\cos^2 t\,dt = 4r^2\int_0^{\frac{\pi}{2}}\cos^2 t\,dt =$$

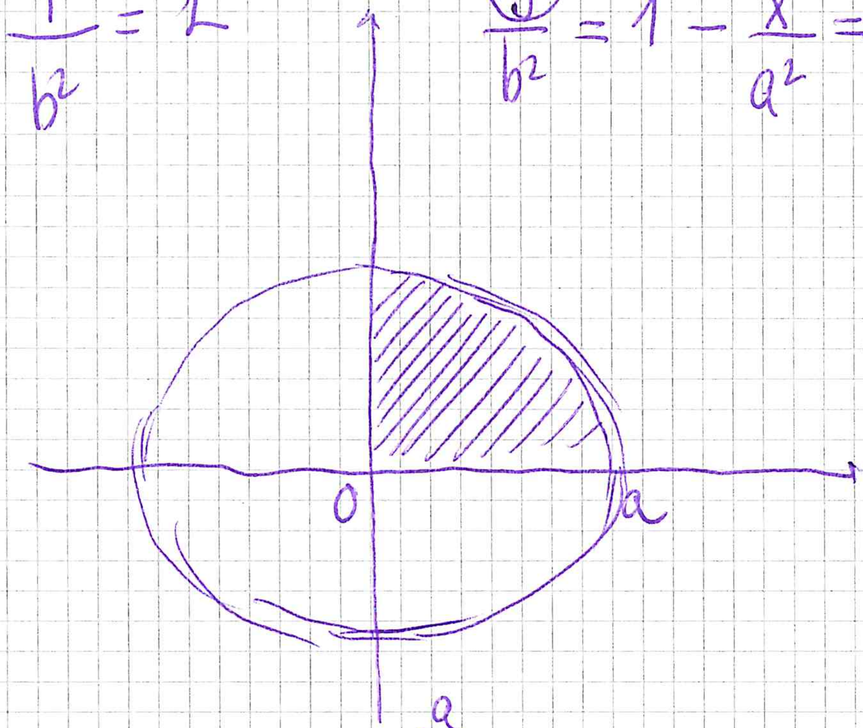
$$\int \mathrm{sen}^2 x\,dx = \frac{x - \mathrm{sen}\,x\cos x}{2} + C$$

$$\int \cos^2 x\,dx = \frac{x + \mathrm{sen}\,x\cos x}{2} + C$$

$$= 4r^2\left[\frac{t + \mathrm{sen}\,t\cos t}{2}\right]_0^{\frac{\pi}{2}} = 4r^2\left[\frac{\frac{\pi}{2}+0}{2} - \frac{0+0}{2}\right] = \cancel{4}r^2\cdot\frac{\pi}{\cancel{4}} = \pi r^2$$

280

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

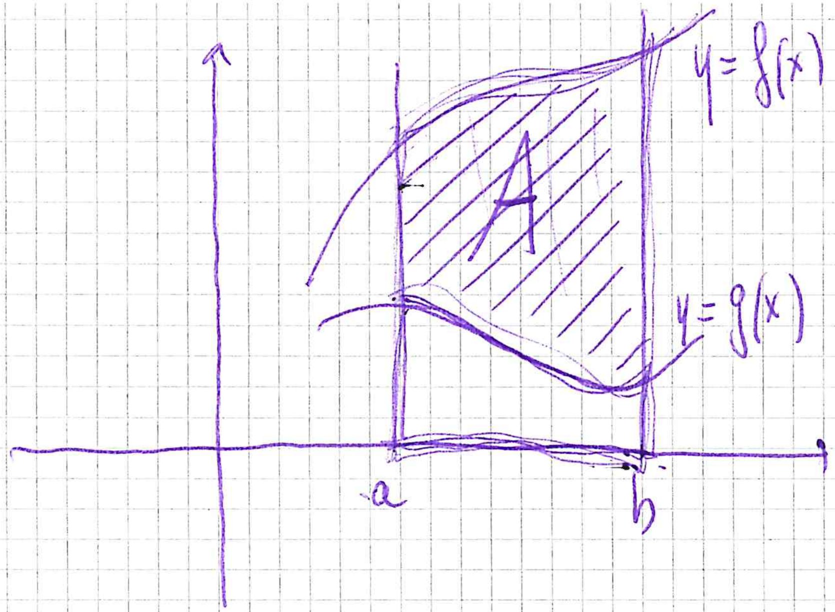


$$y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

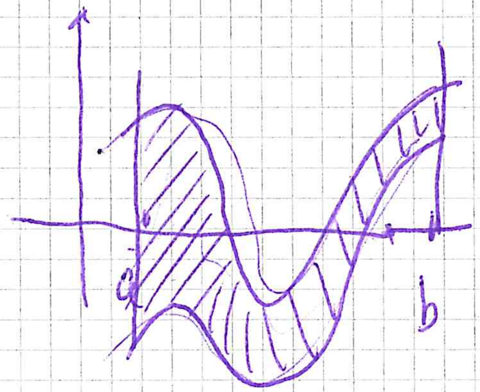
$$4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx = 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx = \frac{b}{\cancel{a}} \cdot \pi a^{\cancel{2}} = \pi a b$$

281



$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$A = \int_a^b \left(f(x)-g(x)\right)dx =$$



$$\int_1^e \lg x\, dx = \Big[x\lg x - x\Big]_1^e$$

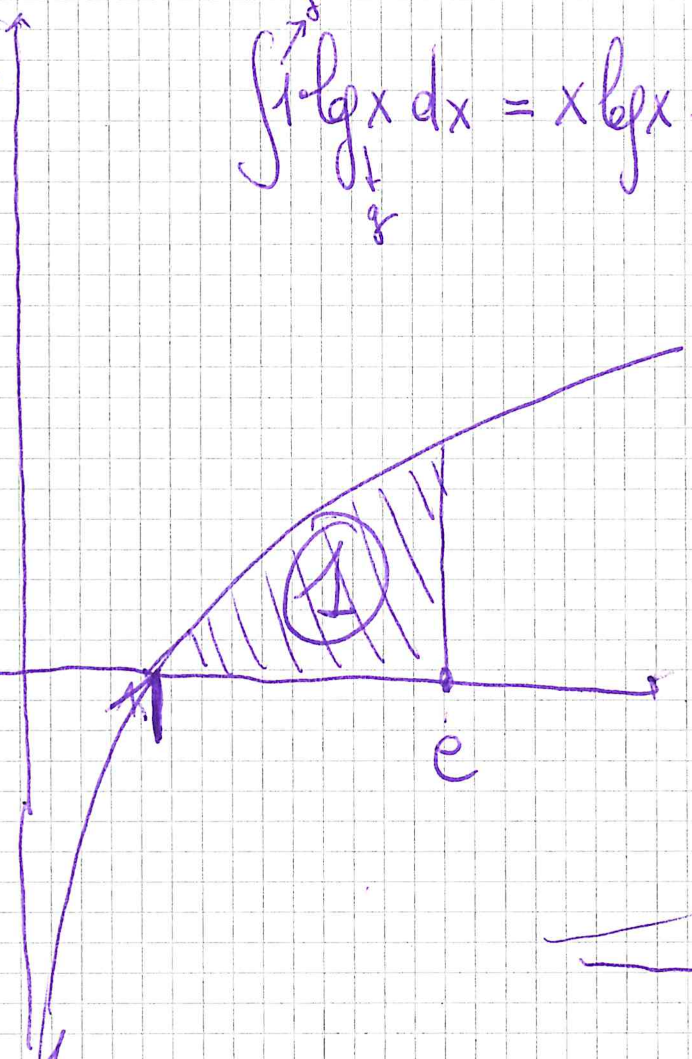
$$= e\underbrace{\lg e}_{1} - e - (\cancel{1\lg 1} - 1)$$

$$= \cancel{e} - \cancel{e} + 1 = 1$$

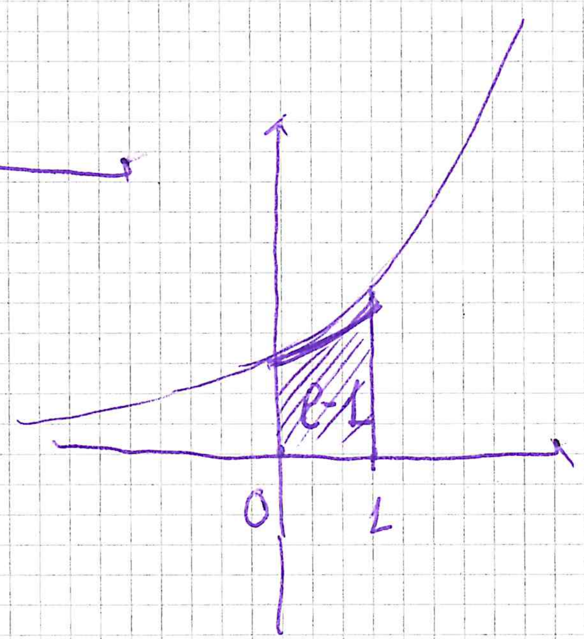
$$\int 1\cdot\lg x\, dx = x\lg x - \int x\cdot\frac{1}{x}\,dx$$

$$= x\lg x - x + C$$

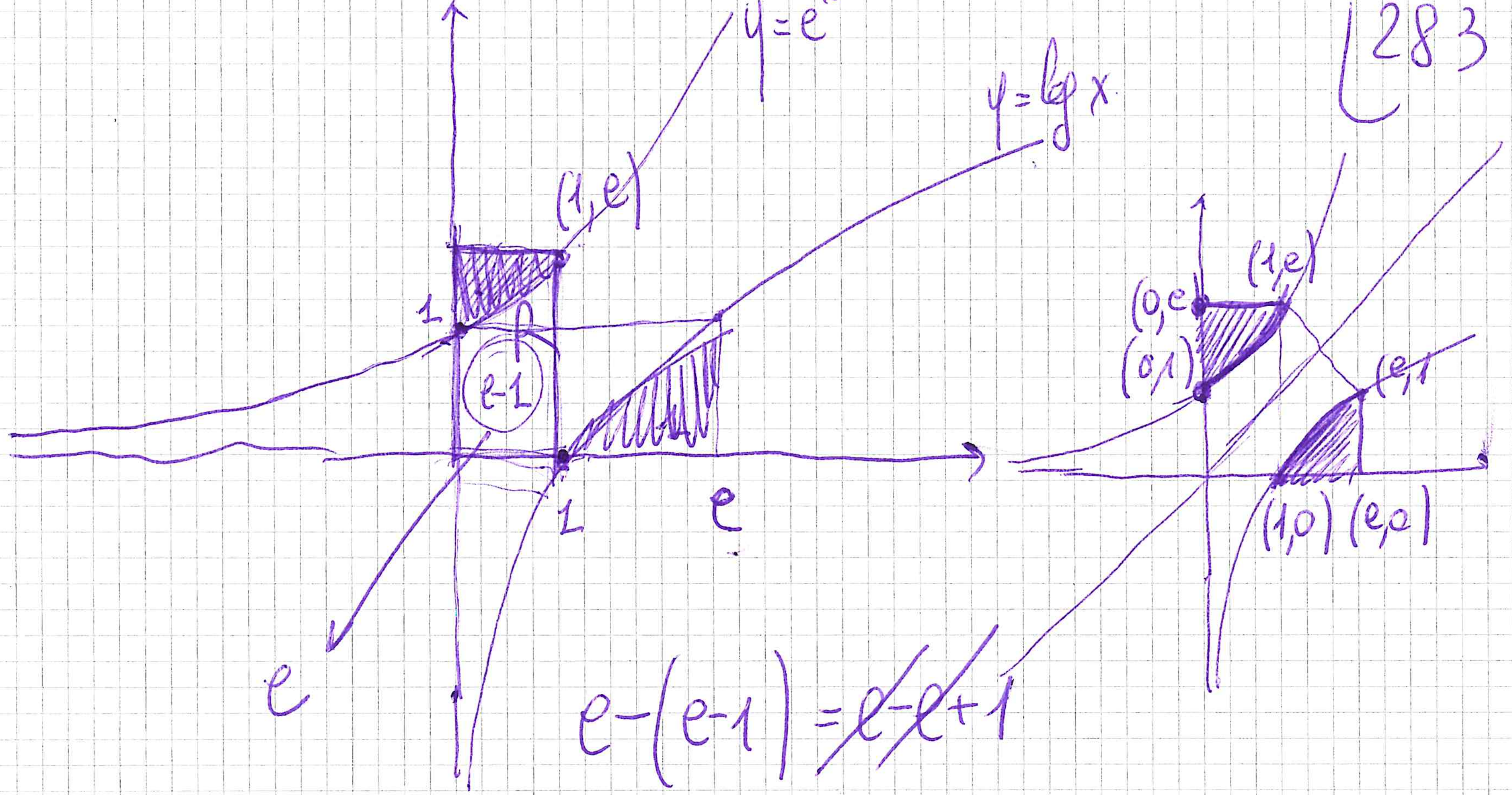
(282



$$\int_0^1 e^x\, dx = \Big[e^x\Big]_0^1 = e - 1$$



283



$$e-(e-1)=\cancel{e}-\cancel{e}+1$$

284

$$\int \frac{x^4+1}{x^2+25}\,dx$$

$$\begin{array}{r|l} x^4 \qquad\qquad +1 & x^2+25 \\ \hline -x^4 \ -25x^2 \qquad & x^2-25 = Q(x) \\ \hline // \quad -25x^2 \quad +1 & \\ +25x^2+625 & \\ \hline // \qquad 626 & = R \end{array}$$

$$\int (x^2-25)\,dx + \int \frac{626}{x^2+25}\,dx$$

$\Delta = 0-100=-100$

$$= \frac{x^3}{3} - 25x + 626\int \frac{1}{x^2+25}\,dx = \frac{x^3}{3} - 25x + 626\cdot\left(\frac{\cancel{2}^{\,1}}{\sqrt{\cancel{100}}^{\,5}}\cdot \operatorname{arctg}\frac{2x}{\sqrt{100}}\right)+C$$

$$= \frac{x^3}{3} - 25x + 626\left(\frac{1}{5}\operatorname{arctg}\frac{x}{5}\right)+C$$

285

$$\int \frac{1}{x^2+25}\,dx = \int \frac{1}{25\left(\frac{x^2}{25}+1\right)}\,dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx \qquad \text{arctg}\,\frac{x}{5}$$

$$= \frac{1}{25}\int \frac{1/5}{1+\left(\frac{x}{5}\right)^2}\,dx = \frac{1}{5}\,\text{arctg}\,\frac{x}{5}$$

$$\int \overset{g}{x^3}\,\overset{f'}{\sin x}\,dx = -\cos x \cdot x^3 - \int -\cos x \cdot 3x^2\,dx = -x^3\cos x + 3\int \overset{g}{x^2}\,\overset{f'}{\cos x}\,dx$$
$$= -x^3\cos x + 3\left(x^2\sin x - \int 2x\sin x\,dx\right) = -x^3\cos x + 3x^2\sin x - 6\int \overset{g}{x}\,\overset{f'}{\sin x}\,dx$$
$$= -x^3\cos x + 3x^2\sin x - 6\left(-x\cos x - \int -\cos x \cdot 1\,dx\right)$$

286

$$-x^3\,\text{συν}x+3x^2\,\text{ημ}x+6x\,\text{συν}x-6\int\text{συν}x\,dx$$

$$-x^3\,\text{συν}x+3x^2\,\text{ημ}x+6x\,\text{συν}x-6\,\text{ημ}x+C$$

$$\left(-x^3\,\text{συν}x+3x^2\,\text{ημ}x+6x\,\text{συν}x-6\,\text{ημ}x\right)'=\left(-x^3\,\text{συν}x\right)'+\left(3x^2\,\text{ημ}x\right)'+\left(6x\,\text{συν}x\right)'-\left(6\,\text{ημ}x\right)'=$$

$$=-3x^2\,\text{συν}x-x^3\cdot(-\text{ημ}x)+6x\,\text{ημ}x+3x^2\cdot\text{συν}x+6\,\text{συν}x+6x(-\text{ημ}x)-6\,\text{συν}x$$

$$=\cancel{-3x^2\,\text{συν}x}+x^3\,\text{ημ}x+\cancel{6x\,\text{ημ}x}+\cancel{3x^2\,\text{συν}x}+\cancel{6\,\text{συν}x}\cancel{-6x\,\text{ημ}x}\cancel{-6\,\text{συν}x}$$

$$=x^3\,\text{ημ}x$$