12/11/'19

DERIVATE DI FUNZIONI TRASCENDENTI

$$f(x) = x^{p/q} \qquad p \in \mathbb{Z}, \quad q \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$$

$$x \to \underset{\substack{\shortparallel \\ y}}{\left(x^{1/q}\right)} \xrightarrow{x^p} y^p = \left(x^{1/q}\right)^p = x^{p/q}$$

è l'inversa di $x \to x^q$

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f, g due funzioni derivabili.

$$(g \circ f)'(x)$$

$$\frac{(g \circ f)(x+h) - g \circ f(x)}{h} =$$

$\downarrow \pm f(x)$

$$= \frac{g(f(x+h)) - g(f(x))}{h} =$$

Ex.

$$f(x) = 3x^2 - 2 \rightsquigarrow f'(x) = 6x$$

$$g(x) = 2x^3 - 3 \cdot x \rightsquigarrow g'(x) = 6x^2 - 3$$

$$(g \circ f)'(x)$$

2 modi:

(I) $g \circ f(x) = g(3x^2 - 2) = 2(3x^2-2)^3 - 3(3x^2-2) = 2 \cdot (27x^6 \; 54x^4 + 36x^2 - 8) - 9x^2 + 6 = 54x^6 - 108x^4 + 63x^2 - 10$

$$(g \circ f)'(x) = \boxed{324 \cdot x^5 - 432 \cdot x^3 + 126 \cdot x}$$

(II) $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$

$$= g'(3x^2 - 2) \cdot 6x =$$

$$= \underbrace{\left[6(3x^2-2)^2 - 3\right]}_{+1+} \cdot 6x =$$

$$= \frac{g\left(\overbrace{f(x)}^{y} + \overbrace{f(x+h) - f(x)}^{h_1}\right) - g\left(\overbrace{f(x)}^{y}\right)}{[f(x+h) - f(x)]} \cdot \frac{[f(x+h) - f(x)]}{h}$$

$$= \frac{g(y + h_1) - g(y)}{h_1} \cdot \frac{f(x+h) - f(x)}{h}$$

$$h_1 = \overbrace{f(x+h)}^{\to f(x)} - f(x) \xrightarrow[h \to 0]{} 0$$

f deriv. in x $\Rightarrow$ f cont. in x

$$y = f(x)$$

$$\boxed{(g \circ f)'(x) = g'(f(x)) \cdot f'(x)}$$

$= 6x(\underbrace{54x^4 - 72x^2 + 21}) =$

$= \boxed{324x^5 - 432x^3 + 126 \cdot x}$

DERIVATA DI $f^{-1}(x)$?

$$f^{-1} \circ f(x) = f \circ f^{-1}(x) = x$$

FUNZIONE INVERSA

- f invertibile e derivabile
  $f'(y) \neq 0$, $x = f(y) \Rightarrow f^{-1}(x) = y$
- Rapp. increm. di f^{-1} in x:

$$\frac{f^{-1}(x+h) - f^{-1}(x)}{h} =$$

$$= \frac{f^{-1}(x+h) - y}{(x+h) - x} = (*)$$

chiamo $y_1 = f^{-1}(x+h)$

da cui $f(y_1) = x + h$.

sia $h_1 = (y_1 - y)$

Di conseguenza

se $h \to 0$, $h_1 \to 0$

$$h_1 = f^{-1}(x+h) - f^{-1}(x)$$

$$f^{-1}(x+h) \to f^{-1}(x)$$

$$\Delta f^{-1} \xrightarrow[h \to 0]{} 0$$

f^{-1} è cont.

(*) $$\frac{y_1 - y}{f(y_1) - f(y)} = \frac{1}{\frac{f(y+h_1) - f(y)}{h_1}}$$

$y = f^{-1}(x)$

$$\downarrow$$

$$f'$$

$$= \frac{f^{-1}(x+h) - f^{-1}(x)}{h}$$

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

$$\frac{d}{dx} x^{1/q} = \frac{1}{g_q'(x^{1/q})} =$$

$$g_q(x) = x^q \qquad g_q' = q x^{q-1}$$

$$= \frac{1}{q \cdot (x^{1/q})^{q-1}} = \frac{1}{q} x^{(1/q)-1}$$

$$d/dx \, x^p = p x^{p-1}$$

$$(x^{p/q})' = (x^{1/q})^p = g_p(x^{1/q})$$

$$= g_p'(x^{1/q}) \cdot (x^{1/q})'$$

$$= p x^{\frac{p-1}{q}} \cdot \frac{1}{q} x^{1/q - 1}$$

$\square$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Altro modo:

$$f \circ f^{-1}(x) = x$$

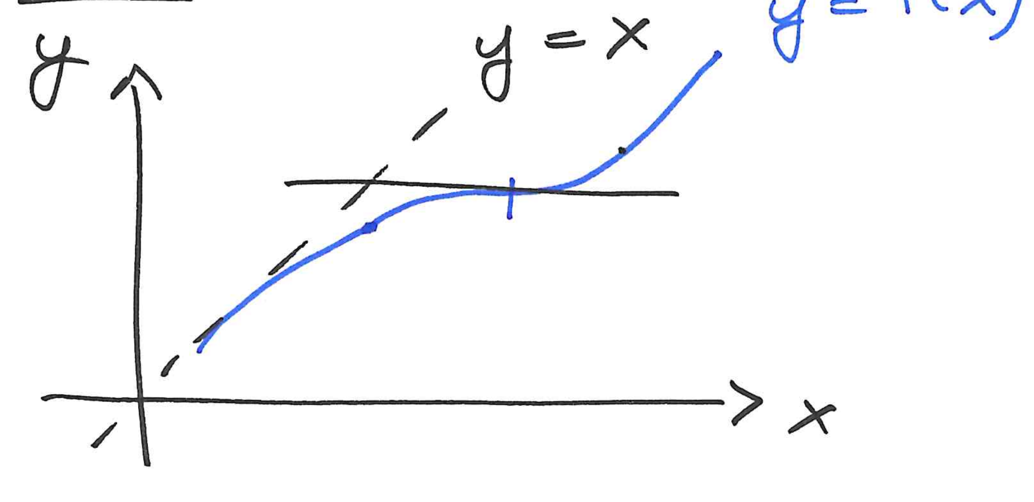
$$(f \circ f^{-1}(x))' = 1$$

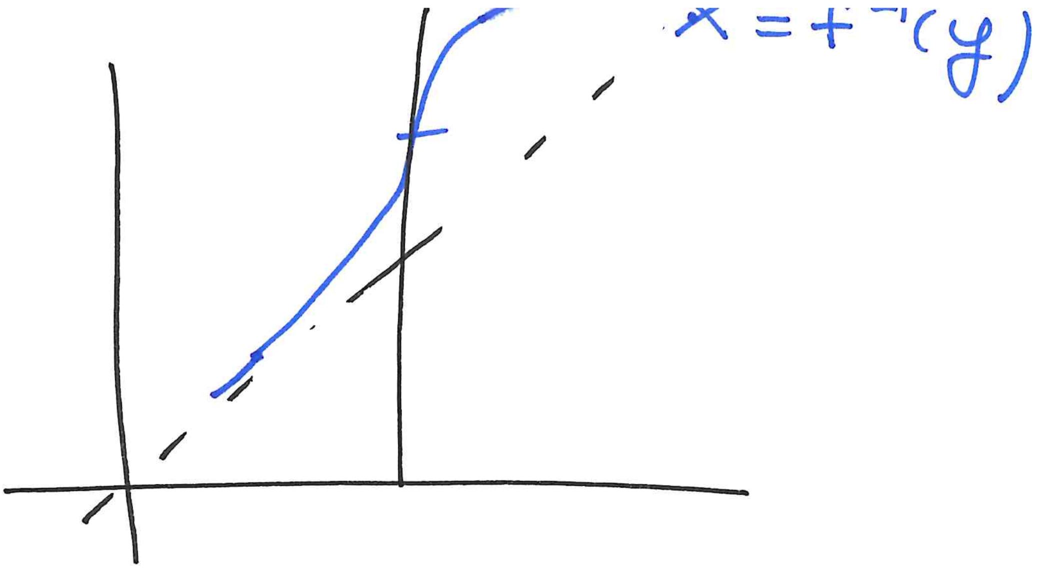
$$\parallel$$

$$\underbrace{f'(f^{-1}(x))} \cdot (f^{-1})'(x) = 1$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Oss.





DERIVATA DI LOG NATURALE

(log / ln)

Rapp. increm.

$$\frac{\log(x+h)-\log x}{h} =$$

$$= \frac{1}{h}\cdot\log\left(\frac{x+h}{x}\right) =$$

$$= \log\underbrace{\left(1+\frac{1}{\cancel{x}}\cdot h\right)}_{=}^{1/h}$$

$$\lim_{h\to 0}\frac{\log(x+h)-\log x}{h} = \log\Big($$

$$\lim_{h\to 0}\left\{\left(1+\frac{1}{x}\cdot h\right)^{1/h}\right) \to e^{1/x}$$

$$= \log e^{1/x} = 1/x$$

$$\boxed{\left[\log_e(x)\right]' = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}\log_p x = \frac{1}{(\log_e p)\cdot x}}$$

$$\boxed{\log(f(x))' = \frac{1}{f}\cdot f'}$$

$$x \longmapsto f(x) \xrightarrow{\log} \log(f(x))$$

$$(e^x)'$$

e^x è l'inversa di $\log_e x$

$$\boxed{(e^x)'} = \frac{1}{\frac{1}{e^x}} = \boxed{e^x}$$

uso la derivata di inversa

$$(a^x)' \qquad a \neq e$$

$$a^x = e^{\log_e(a^x)}$$

$$= e^{[x \log_e(a)]}$$

$$(a^x)' = \big(e^{x \cdot \underbrace{\log_e a}_{\in \mathbb{R}}}\big)' =$$

$$= \underbrace{e^{x \log_e a}}_{= a^x} \cdot (\log_e a) =$$

$$(e^{x \cdot b})' = b \cdot e^{xb}$$

$$\boxed{(a^x)'} = \boxed{a^x \cdot \log_e a}$$

se $a = e$

$\log_e e = 1$

$$(x^\alpha)'$$

$\alpha \in \mathbb{R}$

$$x^\alpha = e^{\log_e x^\alpha}$$

$$= e^{\alpha \cdot \log_e x}$$

$$\frac{d}{dx} x^{\alpha} = \frac{d}{dx}\left(e^{\alpha \cdot \log_e x}\right) =$$

$$= \underbrace{e^{\alpha \log_e x}}_{= x^{\alpha}} \cdot \frac{\alpha}{x} = \frac{x^{\alpha}}{x} \cdot \alpha = \alpha x^{\alpha - 1}$$

$$\boxed{(x^{\alpha})' = \alpha \cdot x^{\alpha - 1}}$$

$\forall \alpha \in \mathbb{R}$

$$\boxed{\left(e^{f(x)}\right)' = e^{f} \cdot f'}$$

$$\left(f^{g}\right)'$$

$$(f^{g})' = \left(e^{\log_e f^{g}}\right)' = \left(e^{g \cdot \log_e (f)}\right)'$$

$$\boxed{(f^{g})' = g f^{g-1} \cdot f' + g' \cdot f^{g} \log f}$$

DERIVATA DI $\sin(x)$:

scrivo rapp. increm.

$$\frac{\sin(x+h)-\sin x}{h} =$$

$$= \frac{\sin x \cos h + \sin h \cdot \cos x - \sin x}{h}$$

$$= \sin x \cdot \underbrace{\frac{(\cos h - 1)}{h}}_{\substack{h \to 0 \\ \downarrow \\ 0}} + \cos x \cdot \underbrace{\frac{\sin h}{h}}_{\substack{h \to 0 \\ \downarrow \\ 1}}$$

$\sim h^2$

$$\boxed{\sin(x)' = \cos x}$$

DERIVATA DI $\cos x$

$$\frac{\cos(x+h)-\cos x}{h} = \frac{\cos x \cdot \cos h -}{h}$$

$$\frac{-\sin h \cdot \sin x - \cos x}{h} =$$

$$= \cos x \cdot \underbrace{\frac{\cos h - 1}{h}}_{\downarrow 0,\ h \to 0} - \sin x \cdot \underbrace{\frac{\sin h}{h}}_{\downarrow 1,\ h \to 0}$$

$$\boxed{(\cos(x))' = - \sin x}$$

meno

DERIVATA di tan(x)

$$\boxed{\frac{d}{dx} \tan(x)} = \frac{d}{dx} \frac{\sin x}{\cos x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \boxed{\frac{1}{(\cos x)^2}}$$

$$\left(f/g\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$\boxed{\frac{d}{dx}\cot x} = \frac{d}{dx}\frac{\cos x}{\sin x}$$

$$= \frac{-(\sin x)^2 - \cos^2 x}{\sin^2 x} = \boxed{\frac{-1}{\sin^2 x}}$$

Da formula di derivata di f^{-1}:

$$\boxed{\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}}$$

$$\boxed{\frac{d}{dx}\operatorname{arctg} x = \frac{1}{1+x^2}}$$

Ex. (per casa)

Quant'è f' se f è:

(1) $e^{4 \cdot x}$

(2) e^{4x+1}

(3) $(x \cdot \log_e x - x)$

(4) $\sin x + \cos x$

(5) $2\sin x \cdot \cos x$

(6) $x \cdot \cos x$

(7) $\cos(x^3)$

(8) $\sin(2 \cdot x)$

(9) $\sin x + \tan x$

Ex. calcolare f'

$$f(x) = \frac{e^{\cos x} + \arctan(3x^4 + 7x^3)}{\log(1+\sin^2 x)}$$

Oss. 1

$$f = \frac{g_1(x)}{g_2(x)}$$

$$g_1(x) = e^{\cos x} + \arctan(3x^4 + 7x^3)$$

$$g_2(x) = \log(1 + \sin^2 x)$$

$$f' = \frac{g_1' \cdot g_2 - g_1 \cdot g_2'}{g_2^2}$$

$$g_1 = h_1 + h_2$$

$$g_1' = h_1' + h_2'$$

$$h_1(x) = e^{\cos x}$$

$$h_2(x) = \arctan(3x^4 + 7x^3)$$

$$h_1'(x) = e^{\cos x}(-\sin x)$$

$$h_2'(x) = \frac{1}{1+(3x^4+7x^3)^2} \cdot (12x^3 + 21x^2)$$

$$\left(\arctan(p(x))\right)' = \frac{1}{1+p^2(x)}\, p'(x)$$

$$g_2(x) = \log(1+\sin^2 x)$$

$$g_2'(x) = \frac{1}{1+\sin^2 x} \cdot \underbrace{2\sin x \cos x}_{=\ \sin(2x)}$$

$$(\log f)' = \frac{f'}{f}$$

$$g_2'(x) = \frac{\sin(2x)}{1+\sin^2 x}$$

Risultato:

$$f'(x) = \frac{1}{\log^2(1+\sin^2 x)} \cdot \left[\left(\frac{12x^4+21x^2}{1+(3x^4+7x^3)^2} - (\sin x) \cdot e^{\cos x} \right) \cdot \log(1+\sin^2 x) - \left(e^{\cos x} + \arctan(3x^4+7x^3) \right) \cdot \frac{\sin 2x}{1+\sin^2 x} \right]$$