Lezione 2

Stern-Gerlach experiment

Performed in Frankfurt, Germany in 1922 and named after

Otto Stern  and Walther Gerlach 

1888 – 1969

1889 - 1979

At the time, Stern (34) and Gerlach (33) were assistants at the University of Frankfurt's Institute for Theoretical and Experimental Physics, respectively.

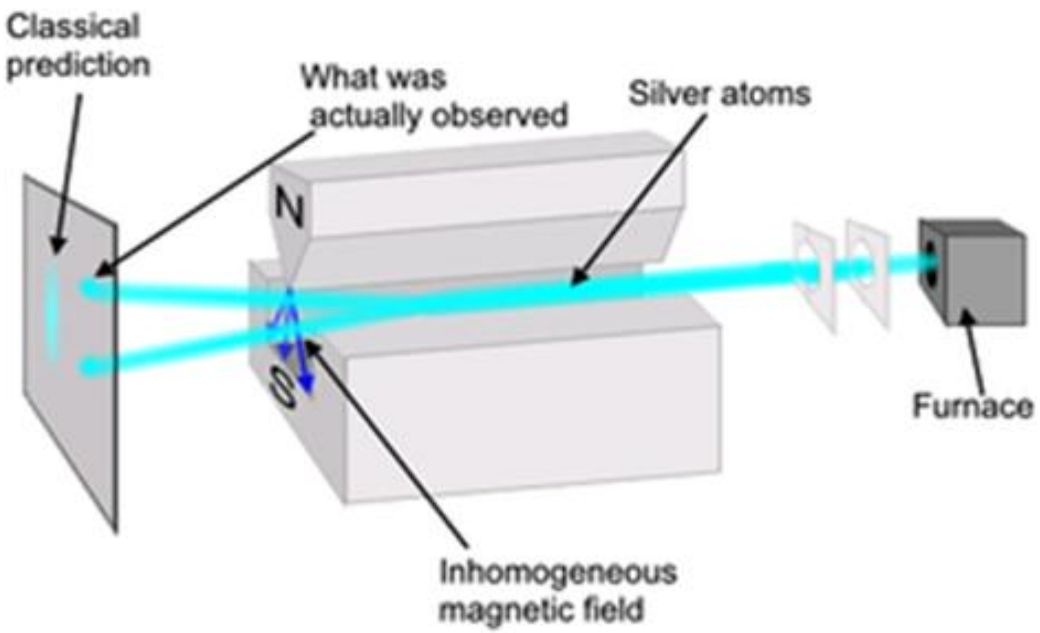
Often used to illustrate basic principles of quantum mechanics.

A simple(!) experiment on the deflection of particles demonstrating that:

1) microscopic systems (electrons and atoms) have intrinsically quantum properties;
2) a measurement in quantum mechanics affects the system being measured.

PROCEDURE: to send a (non polarized monocromatic) beam of silver (Ag) atoms (more generally **electrically neutral** particels or atoms) through an inhomogeneous magnetic field and to observe their deflection.

This avoids the large deflection to the orbit of a charged particle moving through a magnetic field (due to Lorentz force) and allows **magnetic dipole moment effects** to dominate.



RESULTS: show that particles possess an intrinsic angular momentum that is most closely analogous to the angular momentum of a classically spinning object, but that takes only certain quantized values.

A magnetic dipole **μ** in a magnetic field **B** has a potential energy

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}$$

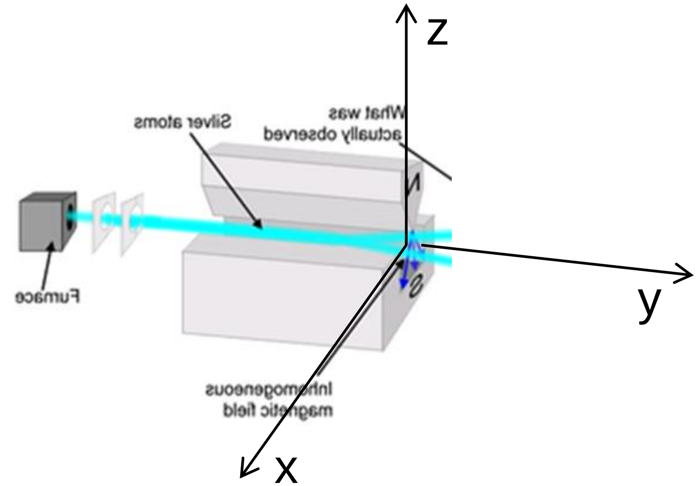
The configuration of minimum (maximum) potential energy is for **μ** parallel (antiparallel) to **B**.

If the particle moves through a **homogeneous** magnetic field its trajectory is unaffected (the forces exerted on opposite ends of the dipole cancel each other out – think at a wire of current)

$$\mathbf{F} = -\nabla U = 0$$

However, if the magnetic field is **inhomogeneous**, the force on one end of the dipole will be slightly greater than the opposing force on the other end, so that there is a net force which **deflects** the particle's trajectory.

The couple of magnets (one flat, the other sharp) is such that the induced **B** field is along **z** and inhomogeneous



$$\mathbf{B} = B_z \ \hat{\mathbf{z}} \qquad\qquad \frac{\partial B_z}{\partial z} \neq 0$$

While passing through the apparatus, the atom is then subject to the force $\mathbf{F} = F_z \ \hat{\mathbf{z}}$ where $F_z = \mu_z \dfrac{\partial B_z}{\partial z}$

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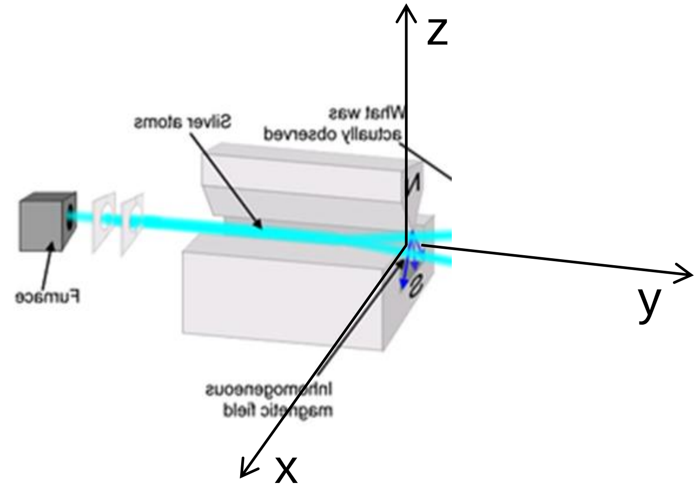

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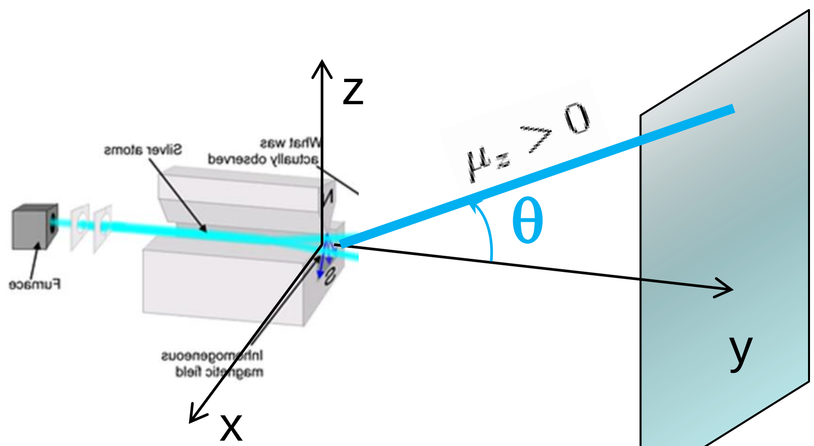
Take $\dfrac{\partial B_z}{\partial z} = \mathrm{cost} > 0$, the atom is deflected upwards if $\mu_z > 0$, downwards if $\mu_z < 0$

The deflection angle θ is proportional to μ_z



Atoms are collected on a screen (photographic film)

Note that if the particle is treated as a classical spinning magnetic dipole, the vector $\boldsymbol{\mu}$ will precess in a magnetic field **B**, because of the torque

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$$

Remember that the torque on a body determines the rate of change of the body's angular momentum **J**

$$\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt}$$

But the magnetic dipole moment $\boldsymbol{\mu}$ is aligned with the angular momentum **J** (think of a wire), and the proportionality coefficient is called **gyromagnetic ratio** γ

$$\boldsymbol{\mu} = \gamma\, \mathbf{J}$$

The time evolution of μ is in general described by the Landau-Lifshitz-Gilbert equation

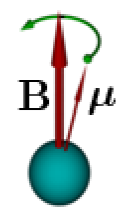
$$\frac{1}{\gamma}\frac{d\boldsymbol{\mu}}{dt} = \underbrace{\boldsymbol{\mu} \times \mathbf{B}}_{\text{precession term}} - \frac{\lambda}{\gamma}\hat{\boldsymbol{\mu}} \times \frac{d\boldsymbol{\mu}}{dt}$$

precession term

damping (dissipative) term
tends to align μ with **B**
NEGLIGIBLE for gases,
relevant for solids (and motors!)

In the Stern-Gerlach experiment, we have then pure precession

$$\frac{1}{\gamma}\frac{d\boldsymbol{\mu}}{dt} = \boldsymbol{\mu} \times \mathbf{B}$$



So that the component μ_z does not vary going through the apparatus.
The Larmor angular precession frequency is

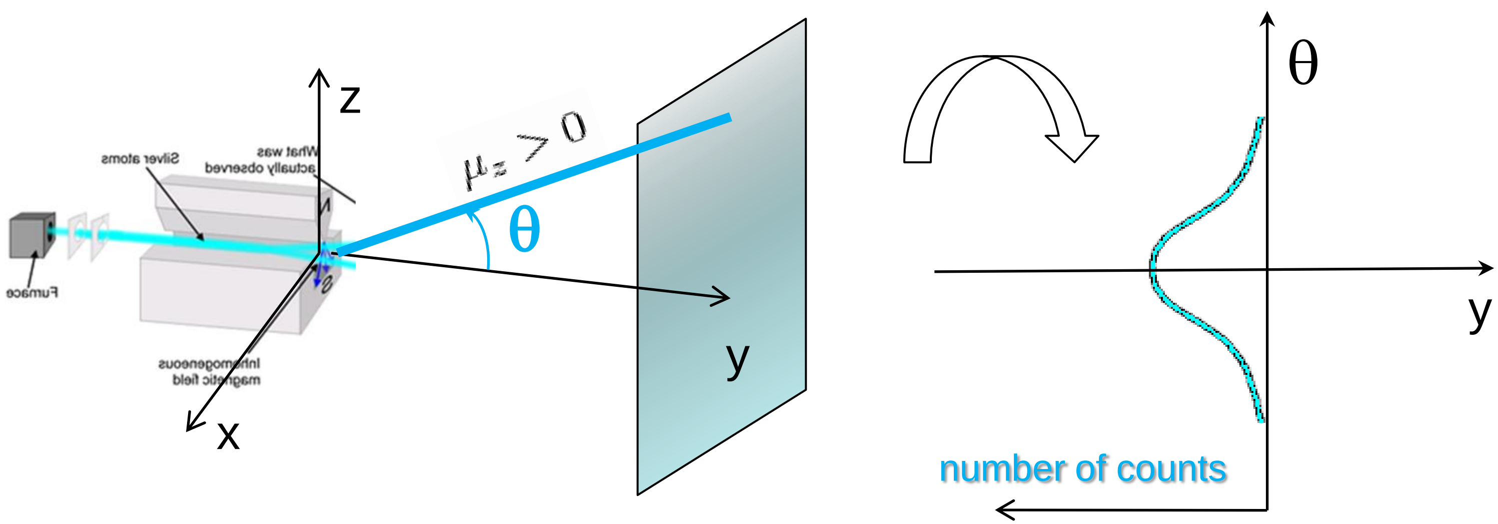
$$\omega_{\mathbf{L}} = -\gamma\mathbf{B}$$

CLASSICALLY

μ is a vector (whose 3 components can be simultaneously known at any time)

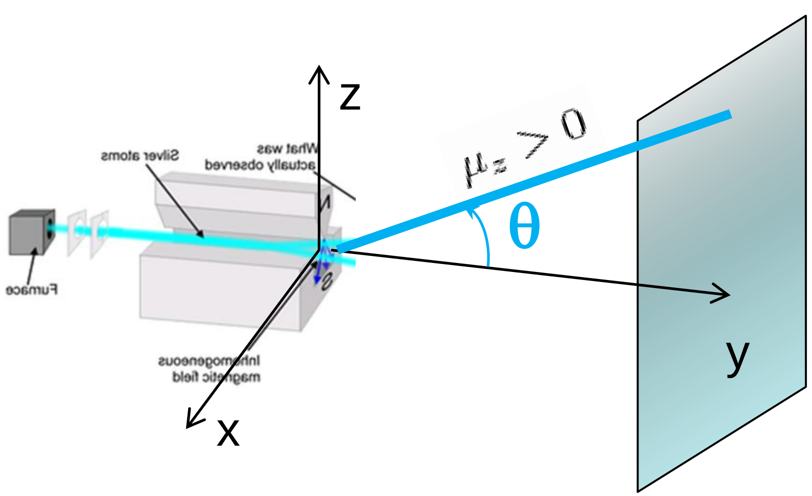
If the particles were classical spinning objects, one would expect the distribution of their magnetic dipole moment vectors to be random and continuous.

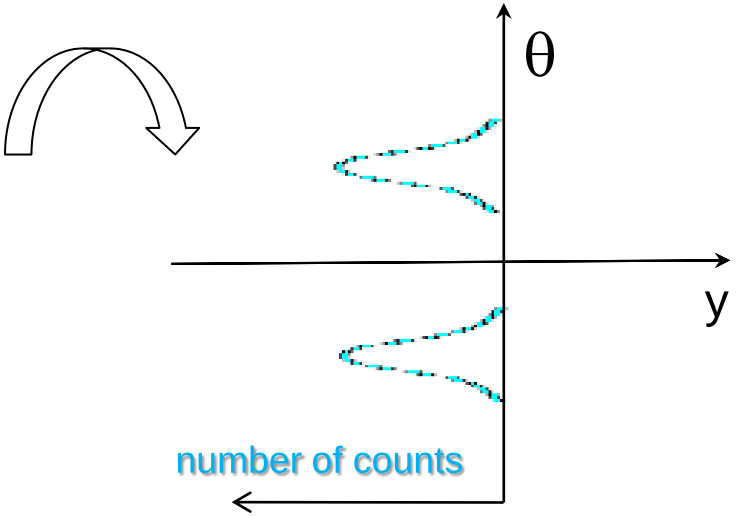
Each particle would be deflected by a different amount, producing a smooth distribution on the detector screen. There should be a maximum and a minimum deviation angle θ, corresponding respectively to μ parallel or antiparrallel with **B**.



EXPERIMENTALLY

Instead, the particles passing through the Stern-Gerlach apparatus are deflected either up or down by a specific amount.



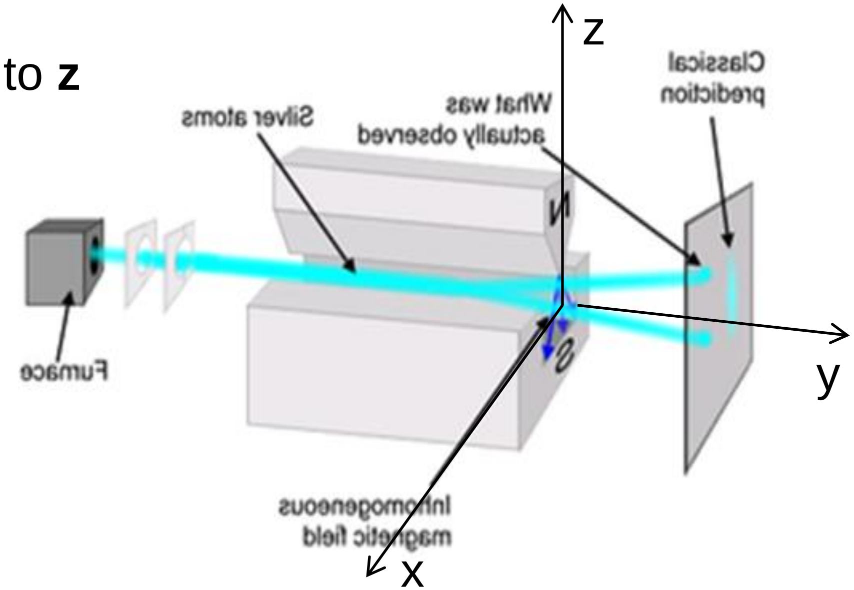


There are only 2 - equal and opposite - equally probable values for θ

This indicates that

μ_z , the magnetic dipole moment along the direction of the **B** field, is QUANTIZED, i.e. it can only take discrete values.

We took **B** parallel to **z**



but exactly the same pattern would have appeared on the screen if the inhomogeneous **B** field would have rather been oriented along any other axis.

Gerlach and Stern then concluded that

"Silver atoms in a magnetic field have only *two discrete* values of the component of the magnetic moment in the direction of the field strength; both have the same absolute value with each half of the atoms having a positive and a negative sign respectively"

Since $\boldsymbol{\mu} = \gamma\, \mathbf{J}$ where γ is a constant (specific for the particle or atom considered)

the Stern-Gerlach experiment actually shows that
the component of the angular momentum **J** in the direction of **B**
is quantized and assumes only two equal and opposite values

However, it is known that atoms in their fundamental state (as were the Ag atoms) have null ORBITAL ANGULAR MOMENTUM **L** (the one associated with the motion of electrons around the nucleus - which is much heavier than electrons).

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Suppose any atom to be also endowed with an INTRISIC ANGULAR MOMENTUM, called SPIN **S**, so that the TOTAL ANGULAR MOMENTUM **J** is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

$\Rightarrow$ The SG experiment measures (the component of) the magnetic moment (in the direction of the field) associated with the SPIN of the atom

$$\boldsymbol{\mu} = \gamma\,\mathbf{S}$$

What's S for Ag?

We know that the nucleons (protons and neutrons) contribution to the atom magnetic moment of can be neglected (in first approximation),being much smaller than the electrons magnetic moment contribution.

The Ag atom has Z=47. We know that spins of the inner 46 electrons sum up to zero, hence the same happens for their total magnetic moment.

THE COMPONENT OF THE MAGNETIC MOMENT IN THE DIRECTION OF THE FIELD MEASURED BY THE SG EXPERIMENT IS THE ONE OF THE 47TH OUTER ELECTRON

$$\boldsymbol{\mu} = \boldsymbol{\mu_e} = \gamma_e \, \mathbf{S_e}$$

AND IS THUS PROPORTIONAL TO
THE COMPONENT OF THE ELECTON SPIN IN THE DIRECTION OF THE FIELD

1. The electron gyromagnetic ratio can be determined by measuring (via another experimental technique) its Larmor precession frequency $\nu_L = \frac{\omega_L}{2\pi} = -\frac{\gamma_e B}{2\pi}$

obtaining $\gamma_e = -\frac{e}{m_e}$

2. Consider then a SG experiment and take for definitness **B** parallel to **z**.

One measures $\mu_z = \pm \mu_B$ where $\mu_B = -\frac{e\hbar}{2m_e}$ is called Bohr magneton

The result is: $S_{ez} = \pm \frac{\mu_B}{\gamma_e} = \pm \frac{-\frac{e\hbar}{2m_e}}{-\frac{e}{m_e}} = \pm \frac{\hbar}{2}$ with equal probabilities

Clearly, for any other orientation of **B** we would have obtained the same results for the values of the component of **S** in the direction of the field

⇨ Electrons are **spin-½** particles:
they have only two possible spin values
measured along any axis, +ħ/2 or −ħ/2.

If this value arises as a result of the particles rotating the way a planet rotates, then the individual particles would have to be spinning impossibly fast.

Indeed, hbar has dimensions of

angular momentum = lenght x (mass x velocity)

hence, on dimensional grounds

$$\hbar \approx r_e \; m_e \; v_e$$

Planck constant $\hbar \approx 10^{-34}\mathrm{J\ s} = 10^{-34}\ \mathrm{kg\ m^2/s}$

Classical electron radius $r_e \approx 3 \times 10^{-15}\mathrm{m} = 3\ \mathrm{fm}$

Electron rest mass $m_e \approx 10^{-30}\mathrm{kg} = 0.5\ \mathrm{MeV/c^2}$

and the electron surface would have to be rotating at

$$v_e = \frac{\hbar}{r_e m_e} \approx \frac{10^{-34}\ \mathrm{kg\ m^2/s}}{.3 \times 10^{-14}\mathrm{m}\ \ 10^{-30}\mathrm{kg}} \approx 3 \times 10^{10}\mathrm{m/s}$$

while the speed of light c is two orders of magnitude smaller!

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… instead of crazy numbers try this: $\boxed{\hbar c \approx 200\ \mathrm{MeV\ \ fm}}$

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$$\frac{v_e}{c} = \frac{\hbar c}{r_e \ m_e c^2} \approx \frac{200\ \mathrm{MeV\ \ fm}}{3\ \mathrm{fm}\ \ 0.5\ \mathrm{MeV}} \approx 10^2$$

The spin angular momentum is a purely quantum mechanical phenomenon

(forget about spinning tops!)

Spin is not a prediction of non-relativistic quantum mechanics:
it is introduced as a postulate.

It is instead a prediction of relativistic quantum mechanics (Dirac equation).

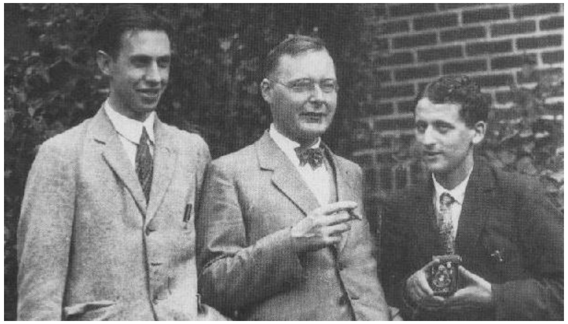
History

In **1922** Stern (34) and Gerlach (33) carry out their experiment in Frankfurt.

Spin was first discovered in the context of the emission spectrum of alkali metals: in **1924** Wolfgang Pauli (24) introduced what he called a "two-valued quantum degree of freedom" associated with the electron in the outermost shell. This allowed him to formulate the Pauli exclusion principle, stating that no two electrons can share the same quantum state at the same time.

The physical interpretation of Pauli's "degree of freedom" was initially unknown.

Ralph Kronig (21), one of Landé’s assistants, suggested in early **1925** that it was produced by the self-rotation of the electron. When Pauli heard about the idea, he criticized it severely, noting that the electron’s hypothetical surface would have to be moving faster than the speed of light in order for it to rotate quickly enough to produce the necessary angular momentum. This would violate the theory of relativity. Largely due to Pauli’s criticism, Kronig decided not to publish his idea.1


Uhlenbeck, Kramers and Goudsmit, circa 1928

In the **autumn of 1925**, the same thought came to two Dutch physicists, George Uhlenbeck (25) and Samuel Goudsmit (23). Under the advice of Paul Ehrenfest, they published their results. It met a favorable response, especially after Llewellyn Thomas managed to resolve a factor-of-two discrepancy between experimental results and Uhlenbeck and Goudsmit's calculations (and Kronig's unpublished ones). This discrepancy was due to the orientation of the electron's tangent frame, in addition to its position.

Despite his initial objections, Pauli formalized the theory of spin in **1927**, using the modern theory of quantum mechanics discovered by Schrödinger and Heisenberg. He pioneered the use of Pauli matrices as a representation of the spin operators, and introduced a two-component spinor wave-function. Pauli's theory of spin was non-relativistic.

In retrospect, the first direct experimental evidence of the electron spin was the Stern-Gerlach experiment of 1922. However, the correct explanation of this experiment was only given in 1927!

In **1928,** Paul Dirac published the Dirac equation, which described the relativistic electron. In the Dirac equation, a four-component spinor (known as a "Dirac spinor") was used for the electron wave-function.

In **1940**, Pauli proved the *spin-statistics theorem*, which states that fermions have half-integer spin and bosons integer spin.

Particle's Spins

For **electrons** there are two possible values for spin angular momentum measured along an axis.

The same is true for the **proton** and the **neutron**, which are composite particles made up of three quarks each (which are themselves spin-$\frac{1}{2}$ particles).

Other particles have a different number of possible spin values.

Delta baryons (Δ^{++}, Δ^{+}, Δ^{0}, Δ^{-}), for example, are spin $+\frac{3}{2}$ particles and have four possible values for spin angular momentum.

Elementary particles

Integer spin = bosons
Half-integer spin = fermions

spin	Known particles	hypothetical particles
0		Higgs squarks&sleptons
½	quarks&leptons	Higgsino photino gluino Zino Wino
1	photon, gluon, Z,W	
3/2		gravitino
2		graviton

supersymmetric particles

Particle's magnetic moments

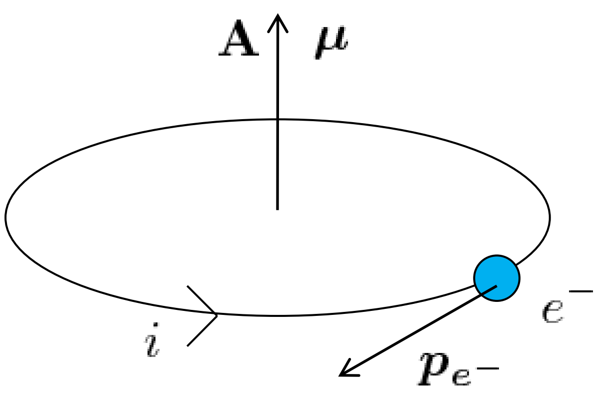
….WAIT… first come back to

the classic magnetic dipole moment of a planar loop of an electric current

Consider an electron (whose charge is -e with e>0) orbiting in a circular loop of radius R

the direction of *conventional current* is defined
to be the direction of the flow of positive charges

$$i = \frac{e}{T} = \frac{e\, p_{e^-}}{2\pi R m_e}$$



the induced magnetic moment is

$$\boldsymbol{\mu} = i\mathbf{A} = \frac{e}{2m_e}\, R\, p_{e^-}\, \hat{\mathbf{A}}$$

where **A** is the vector area of the current loop

(by convention, the direction of **A** is given by the right hand grip rule: curling the fingers of one's right hand in the direction of the current around the loop, when the palm of the hand is "touching" the loop's outer edge, and the straight thumb indicates the direction of the vector area and thus of the magnetic moment)

the orbital angular momentum is $\mathbf{L}_{e^-} = -R p_{e^-} \mathbf{A}$ so that $\boldsymbol{\mu} = -\frac{e}{2m_e}\mathbf{L}_{e^-}$

The classic gyromagnetic ratio is then

$$\gamma_{\rm cl} \equiv -\frac{e}{2m_e} = \frac{\mu_B}{\hbar}$$

For a particle X one defines the **Lande' g-factor** as

$$g_X \equiv \frac{\gamma_X}{\gamma_{\rm cl}} = \frac{\gamma_X \hbar}{\mu_B}$$

where

$$\mu_B = -\frac{e\hbar}{2m_e}$$

$$\gamma_{\rm cl} \equiv -\frac{e}{2m_e} = \frac{\mu_B}{\hbar}$$

its magnetic moment is then

$$\boldsymbol{\mu}_X = g_X \frac{\mu_B}{\hbar} \mathbf{S}_X$$

For the **electron** we have $\gamma_e = -\frac{e}{m_e}$ so that

$$g_e = 2$$

the electron is not classic at all!

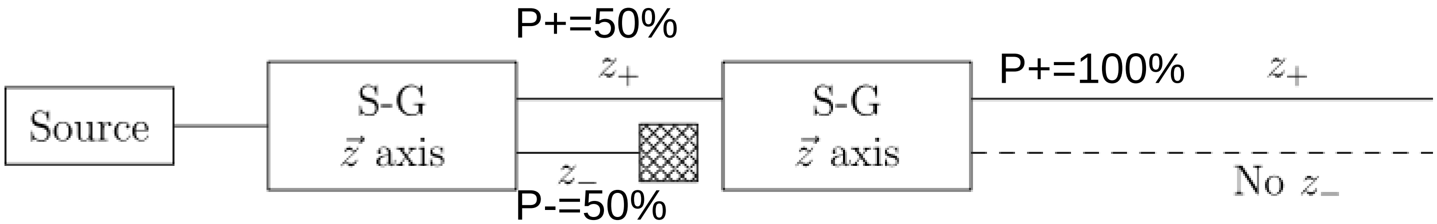
The prediction that g_e=2 (plus permille radiative corrections) is a major success of the Dirac equation, in the context of relativistic quantum mechanics (called field theory).

Some particle's g-factors

	g-factor	uncertainty
e	2.002319…	10^-12
μ	2.002331…	10^-9
p	- 5.585694…	10^-7
n	3.826085	10^-6

Sequences of SG experiments

1. First is SG(**z**), beam is 50-50 split and we take just Sz=+1/2 (unit of hbar understood here and in the following); second is another SG(**z**):



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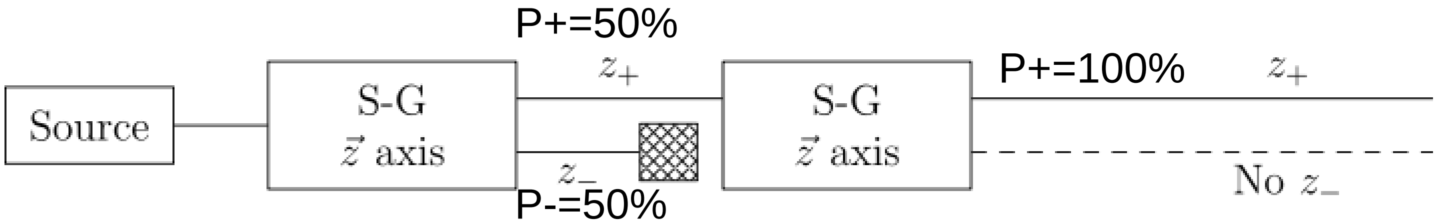
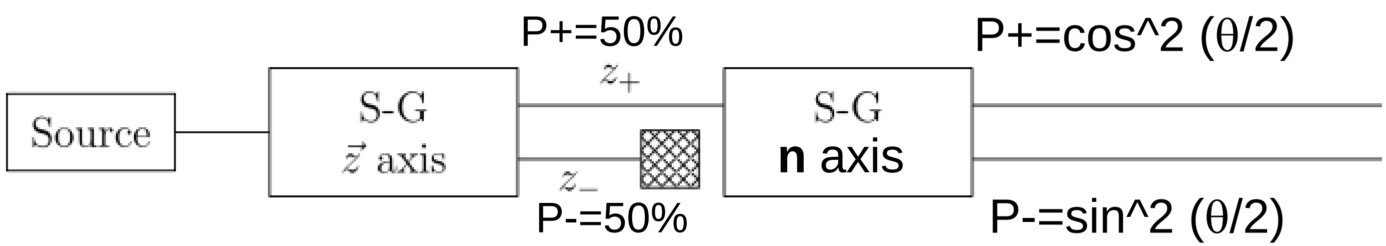


That's reasonable!

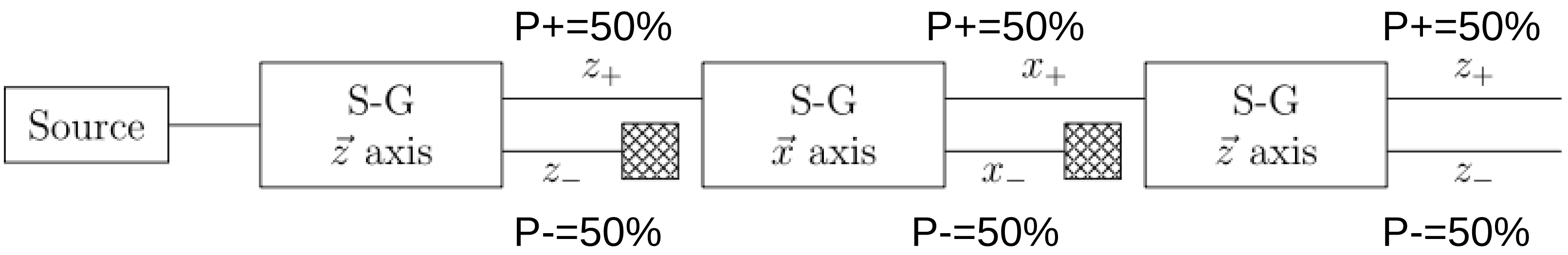
2. First is SG(**z**), beam is 50-50 split and we take +1/2; second SG is now rotated around y by an angle θ, call it SG(**n**); you obtain +1/2 and -1/2 with probabilities



Well, that's very reasonable!

Consider e.g. θ=0, π/2, π so that P+= 100%,50%, 0.

3. First is SG(**z**), beam is 50-50 split and we take +1/2;
second is SG(**x**) (namely $\theta=\pi/2$), beam is 50-50 split and we take +1/2;
third is SG(**z**) and… again 50-50!!!



That's NOT reasonable…

…those coming out from the first SG had already said Sz=+1/2, and now after the third SG also Sz=-1/2 comes out!

⇨ The measure of Sx has modified the system!!!
Classically this is not the case…

One realises that, at the contrary of classical physics:

1) there exists quantised observable quantities (like spin)
2) measurements alter the system under investigation

In quantum mechanics a measurement on a system affects it and produces a system which is different from the original one.

It is not possible to disentangle the system from the experimental apparatus measuring it.

To measure a system = To affect a system

Let's then associate to a measurement of an observable quantity an operator, namely a mathematical object acting on a vectorial space.

observable quantity $\Longleftrightarrow$ operator

physical states on which the measurement is done $\Longleftrightarrow$ elements of a vectorial space

Watch out the unit system

We worked in the SI (m, kg, s, etc)

where $[B] = [E]/c$ = charge force/c

But in the Gaussian CGS (cm, g, s, …)

one has $[B] = [E]$ = charge force

So that $[\mu]_{G\text{-}CGS} = [\mu]_{SI} /c$

Virtual SG experiment

http://phet.colorado.edu/en/simulation/stern-gerlach