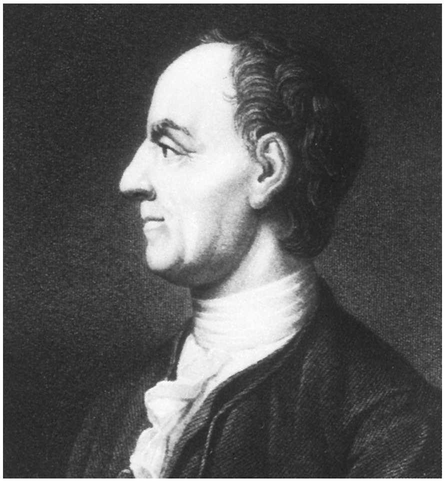


Leonhard Euler (1707–1783) was a Swiss mathematician who became a court mathematician and later a professor of mathematics in Saint Petersburg, Russia. He produced many works in algebra and geometry and was interested in the geometrical form of deflection curves in strength of materials. Euler's column buckling load is quite familiar to mechanical and civil engineers, and Euler's constant and Euler's coordinate system are well known to mathematicians. He derived the equation of motion for the bending vibrations of a rod (Euler-Bernoulli theory) and presented a series form of solution, as well as studying the dynamics of a vibrating ring. (Courtesy of Dirk J. Struik, *A Concise History of Mathematics,* 2nd ed., Dover Publications, New York, 1948.)

CHAPTER 9

Vibration Control

Chapter Outline

We studied all the aspects of modeling and analysis of vibrating systems in the previous chapters. We will now consider methods of eliminating or reducing unwanted vibration. The acceptable levels of vibration must be known before we can quantify the levels to be eliminated or reduced. The vibration nomograph and vibration criteria which indicate acceptable levels of vibration are outlined at the beginning. The vibration to be eliminated or reduced can be in the form of one or more forms of disturbance—displacement, velocity, acceleration, and transmitted force. The following methods are discussed to eliminate/reduce vibration at the source:

- Balancing of rotating machines—single- and two-plane balancing.
- Controlling the response and stability of rotating shafts.
- Balancing of reciprocating engines.
- Reducing vibration caused by impacts due to clearances in the joints of machines and mechanisms.

The following methods are discussed to reduce transmission of vibration from the source:

- Changing the natural frequency of the system when the forcing frequency cannot be altered.
- Introducing a power-dissipation mechanism by adding dashpots or viscoelastic materials.
- Designing an isolator which changes the stiffness/damping of the system.
- Using an active control technique.
- Designing a vibration absorber by adding an auxiliary mass to absorb the vibration energy of the original mass.

Finally, the solution of various vibration-control problems using MATLAB is presented with numerical examples.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Use vibration nomographs and vibration criteria to determine the levels of vibration to be controlled or reduced.
- Apply one- and two-plane balancing techniques for eliminating vibration (unbalance).
- Control the vibration caused by the unbalance in rotating shafts.
- Reduce the unbalance in reciprocating engines.
- Design vibration and shock isolations for systems with fixed base as well as vibrating base.
- Design active vibration-control systems.
- Design undamped and damped vibration absorbers.
- Use MATLAB for solving vibration-control problems.

9.1 Introduction

There are numerous sources of vibration in an industrial environment: impact processes such as pile driving and blasting; rotating or reciprocating machinery such as engines, compressors, and motors; transportation vehicles such as trucks, trains, and aircraft; the flow of fluids; and many others. The presence of vibration often leads to excessive wear of bearings, formation of cracks, loosening of fasteners, structural and mechanical failures, frequent and costly maintenance of machines, electronic malfunctions through fracture of solder joints, and abrasion of insulation around electric conductors causing shorts. The occupational exposure of humans to vibration leads to pain, discomfort, and reduced efficiency. Vibration can sometimes be eliminated on the basis of theoretical analysis. However, the manufacturing costs involved in eliminating the vibration may be too high; a designer must compromise between an acceptable amount of vibration and a reasonable manufacturing cost. In some cases the excitation or shaking force is inherent in the machine. As seen earlier, even a relatively small excitation force can cause an undesirably large response near resonance, especially in lightly damped systems. In these cases, the magnitude of the response can be significantly reduced by the use of isolators and auxiliary mass absorbers [9.1]. In this chapter, we shall consider various techniques of vibration control—that is, methods involving the elimination or reduction of vibration.

9.2 Vibration Nomograph and Vibration Criteria

The acceptable levels of vibration are often specified in terms of the response of an undamped single-degree-of-freedom system undergoing harmonic vibration. The bounds are shown in a graph, called the *vibration nomograph,* which displays the variations of displacement, velocity, and acceleration amplitudes with respect to the frequency of vibration. For the harmonic motion

$$x(t) = X \sin \omega t \tag{9.1}$$

the velocity and accelerations are given by

$$v(t) = \dot{x}(t) = \omega X \cos \omega t = 2\pi f X \cos \omega t \tag{9.2}$$

$$a(t) = \ddot{x}(t) = -\omega^2 X \sin \omega t = -4\pi^2 f^2 X \sin \omega t \tag{9.3}$$

where ω is the circular frequency (rad/s), f is the linear frequency (Hz), and X is the amplitude of displacement. The amplitudes of displacement (X), velocity (v_{max}) and acceleration (a_{max}) are related as

$$v_{max} = 2\pi f X \tag{9.4}$$

$$a_{max} = -4\pi^2 f^2 X = -2\pi f v_{max} \tag{9.5}$$

By taking logarithms of Eqs. (9.4) and (9.5), we obtain the following linear relations:

$$\ln v_{max} = \ln(2\pi f) + \ln X \tag{9.6}$$

$$\ln v_{max} = -\ln a_{max} - \ln(2\pi f) \tag{9.7}$$

It can be seen that for a constant value of the displacement amplitude (X), Eq. (9.6) shows that $\ln v_{max}$ varies with $\ln(2\pi f)$ as a straight line with slope +1. Similarly, for a constant value of the acceleration amplitude (a_{max}), Eq. (9.7) indicates that $\ln v_{max}$ varies with $\ln(2\pi f)$ as a straight line with slope −1. These variations are shown as a nomograph in Fig. 9.1. Thus every point on the nomograph denotes a specific sinusoidal (harmonic) vibration.

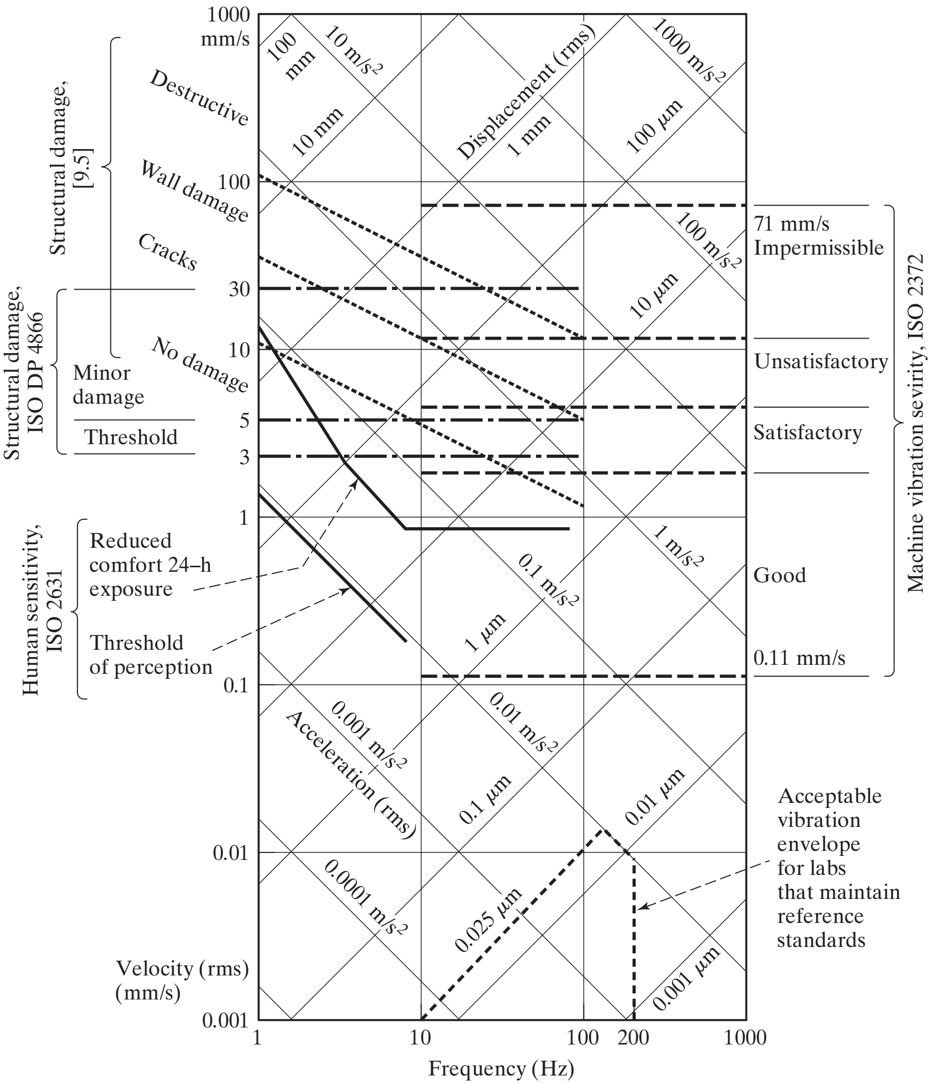


FIGURE 9.1 Vibration nomograph and vibration criteria [9.2].

Since the vibration imparted to a human or machine is composed of many frequencies—rarely of just one frequency—the root mean square values of $x(t)$, $v(t)$, and $a(t)$ are used in the specification of vibration levels.

The usual ranges of vibration encountered in different scientific and engineering applications are given below [9.2]:

1. Atomic vibrations: Frequency $= 10^{12}$ Hz, displacement amplitude $= 10^{-8}$ to 10^{-6} mm.
2. Microseisms or minor tremors of earth's crust: Frequency = 0.1 to 1 Hz, displacement amplitude $= 10^{-5}$ to 10^{-3} mm. This vibration also denotes the threshold of disturbance of optical, electronic, and computer equipment.
3. Machinery and building vibration: Frequency = 10 to 100 Hz, displacement amplitude = 0.01 to 1 mm. The threshold of human perception falls in the frequency range 1 to 8 Hz.
4. Swaying of tall buildings: Frequency range = 0.1 to 5 Hz, displacement amplitude = 10 to 1000 mm.

Vibration severity of machinery is defined in terms of the rms value of the vibration velocity in ISO 2372 [9.3]. The ISO definition identifies 15 vibration severity ranges in the velocity range 0.11–71 mm/s for four classes of machines: (1) small, (2) medium, (3) large, and (4) turbomachine. The vibration severity of class 3 machines, including large prime movers, is shown in Fig. 9.1. In order to apply these criteria, the vibration is to be measured on machine surfaces such as bearing caps in the frequency range 10–1000 Hz.

ISO DP 4866 [9.4] gives the vibration severity for whole-building vibration under blasting and steady-state vibration in the frequency range 1–100 Hz. For the vibration from blasting, the velocity is to be measured at the building foundation nearest the blast, and for the steady-state vibration, the peak velocity is to be measured on the top floor. The limits given are 3–5 mm/s for threshold of damage and 5–30 mm/s for minor damage. The vibration results reported by Steffens [9.5] on structural damage are also shown in Fig. 9.1.

The vibration limits recommended in ISO 2631 [9.6] on human sensitivity to vibration are also shown in Fig. 9.1. In the United States an estimated 8 million workers are exposed to either whole-body vibration or segmented vibration to specific body parts. The whole-body vibration may be due to transmission through a supporting structure such as the seat of a helicopter, and the vibration to specific body parts may be due to work processes such as compacting, drilling, and chain-saw operations. Human tolerance of whole-body vibration is found to be lowest in the 4–8 Hz frequency range. The segmental vibration is found to cause localized stress injuries to different body parts at different frequencies, as indicated in Fig. 9.2. In addition, the following effects have been observed at different frequencies [9.7]: motion sickness (0.1–1 Hz), blurring vision (2–20 Hz), speech disturbance (1–20 Hz), interference with tasks (0.5–20 Hz), and after-fatigue (0.2–15 Hz).

The acceptable vibration levels for laboratories that maintain reference standards are also shown in Fig. 9.1.

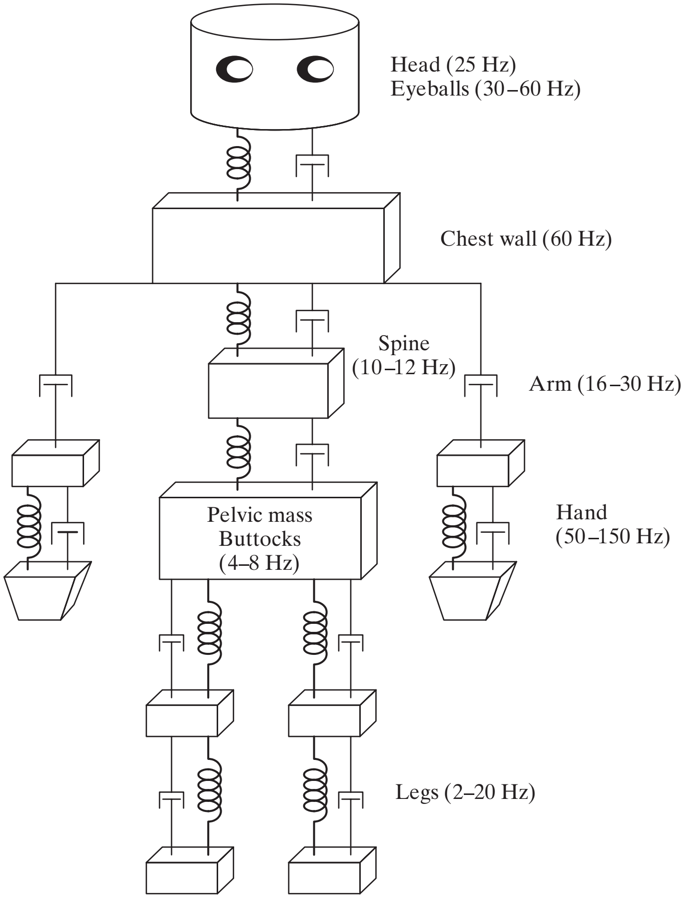


FIGURE 9.2 Vibration frequency sensitivity of different parts of human body.

EXAMPLE 9.1 Helicopter Seat Vibration Reduction

The seat of a helicopter, with the pilot, weighs 1000 N and is found to have a static deflection of 10 mm under self weight. The vibration of the rotor is transmitted to the base of the seat as harmonic motion with frequency 4 Hz and amplitude 0.2 mm.

a. What is the level of vibration felt by the pilot?

b. How can the seat be redesigned to reduce the effect of vibration?

Solution:

a. By modeling the seat as an undamped single-degree-of-freedom system, we can compute the following:

$$\text{Mass} = m = 1000/9.81 = 101.9368 \text{ kg}$$

$$\text{Stiffness} = k = \frac{W}{\delta_{st}} = \frac{1000}{0.01} = 10^5 \text{ N/m}$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^5}{101.9368}} = 31.3209 \text{ rad/s} = 4.9849 \text{ Hz}$$

$$\text{Frequency ratio} = r = \frac{\omega}{\omega_n} = \frac{4.9849}{4.0} = 1.2462$$

Since the seat is subject to harmonic base excitation, the amplitude of vibration felt by the pilot (mass of the seat) is given by Eq. (3.68) with $\zeta = 0$:

$$X = \pm \frac{Y}{1 - r^2} \tag{E.1}$$

where Y is the amplitude of base displacement. Equation (E.1) yields

$$X = \frac{0.2}{1 - 1.2462^2} = 0.3616 \text{ mm}$$

The amplitudes of velocity and acceleration felt by the pilot are given by $\omega X = 2\pi f X = 2(\pi)(5)(0.3616) = 9.0887$ mm/s, and $\omega^2 X = (2\pi f)^2 X = 228.4074$ mm/s$^2 = 0.2284$ m/s^2. Corresponding to the frequency 4 Hz, Fig. 9.1 shows that the amplitude of motion of 0.3616 mm may not cause much discomfort. However, the velocity and acceleration levels at the same frequency (4 Hz) are not acceptable for a comfortable ride.

b. To bring the vibration level to an acceptable level, let us try to bring the acceleration felt by the pilot from the level 0.2284 m/s^2 to 0.01 m/s^2. Using $a_{\max} = 10$ mm/s$^2 = -(2\pi f)^2 X = -(8\pi)^2 X$, we obtain $X = 0.01583$ mm. This leads to

$$\frac{X}{Y} = \frac{0.01583}{0.2} = \pm \frac{1}{1 - r^2} \quad \text{or} \quad r = 3.6923$$

This gives the new natural frequency of the seat as

$$\omega_n = \frac{\omega}{3.6923} = \frac{8\pi}{3.6923} = 6.8068 \text{ rad/s}$$

Using the relation $\omega_n = \sqrt{k/m}$ with $m = 101.9368$ kg, the new stiffness is given by $k = 4722.9837$ N/m. This implies that the stiffness of the seat is to be reduced from 10^5 N/m to 4722.9837 N/m. This can be accomplished by using a softer material for the seat or by using a different spring design. Alternatively, the desired acceleration level can be achieved by increasing the mass of the seat. However, this solution is not usually acceptable, as it increases the weight of the helicopter.

■

9.3 Reduction of Vibration at the Source

The first thing to be explored to control vibrations is to try to alter the source so that it produces less vibration. This method may not always be feasible. Some examples of the sources of vibration that cannot be altered are earthquake excitation, atmospheric turbulence, road roughness, and engine combustion instability. On the other hand, certain sources such as unbalance in rotating or reciprocating machines can be altered to reduce the vibrations. This can be achieved, usually, by using either internal balancing or an increase in the precision of machine elements. The use of close tolerances and better surface finish for machine parts (which have relative motion with respect to one another) make the machine less susceptible to vibration. Of course, there may be economic and manufacturing constraints on the degree of balancing that can be achieved or the precision with which the machine parts can be made. We shall consider the analysis of rotating and reciprocating machines in the presence of unbalance as well as the means of controlling the vibrations that result from unbalanced forces.

9.4 Balancing of Rotating Machines

The presence of an eccentric or unbalanced mass in a rotating disc causes vibration, which may be acceptable up to a certain level. If the vibration caused by an unbalanced mass is not acceptable, it can be eliminated either by removing the eccentric mass or by adding an equal mass in such a position that it cancels the effect of the unbalance. In order to use this procedure, we need to determine the amount and location of the eccentric mass experimentally. The unbalance in practical machines can be attributed to such irregularities as machining errors and variations in sizes of bolts, nuts, rivets, and welds. In this section, we shall consider two types of balancing: *single-plane* or *static balancing* and *two-plane* or *dynamic balancing* [9.9, 9.10].

9.4.1 Single-Plane Balancing

Consider a machine element in the form of a thin circular disc, such as a fan, flywheel, gear, and a grinding wheel mounted on a shaft. When the center of mass is displaced from the axis of rotation due to manufacturing errors, the machine element is said to be statically unbalanced. To determine whether a disc is balanced or not, mount the shaft on two low-friction bearings, as shown in Fig. 9.3(a). Rotate the disc and permit it to come to rest. Mark the lowest point on the circumference of the disc with chalk. Repeat the process several times, each time marking the lowest point on the disc with chalk. If the disc is balanced, the chalk marks will be scattered randomly all over the circumference. On the other hand, if the disc is unbalanced, all the chalk marks will coincide.

The unbalance detected by this procedure is known as *static unbalance*. The static unbalance can be corrected by removing (drilling) metal at the chalk mark or by adding a weight at 180° from the chalk mark. Since the magnitude of unbalance is not known, the amount of material to be removed or added must be determined by trial and error. This procedure is called "single-plane balancing," since all the mass lies practically in a single plane. The amount of unbalance can be found by rotating the disc at a known speed ω and measuring the reactions at the two bearings (see Fig. 9.3(b)). If an unbalanced mass m is located at a radius r of the disc, the centrifugal force will be $mr\omega^2$. Thus the measured bearing reactions F_1 and F_2 give m and r:

$$F_1 = \frac{a_2}{l} mr\omega^2, \qquad F_2 = \frac{a_1}{l} mr\omega^2 \tag{9.8}$$

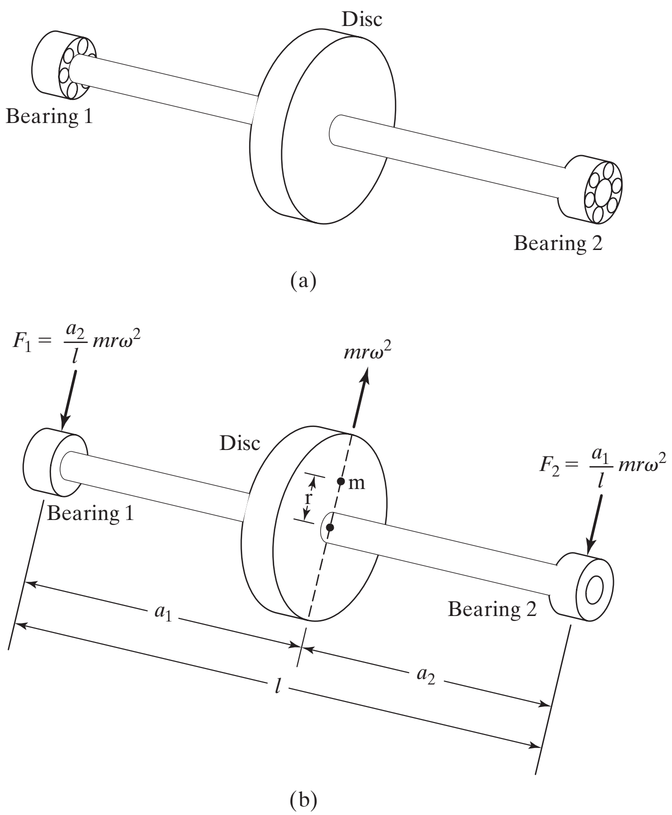


FIGURE 9.3 Single-plane balancing of a disc.

Another procedure for single-plane balancing, using a vibration analyzer, is illustrated in Fig. 9.4. Here, a grinding wheel (disc) is attached to a rotating shaft that has bearing at A and is driven by an electric motor rotating at an angular velocity ω.

Before starting the procedure, *reference marks,* also known as *phase marks,* are placed both on the rotor (wheel) and the stator, as shown in Fig. 9.5(a). A vibration pickup is placed in contact with the bearing, as shown in Fig. 9.4, and the vibration analyzer is set to a frequency corresponding to the angular velocity of the grinding wheel. The vibration signal (the displacement amplitude) produced by the unbalance can be read from the indicating meter of the vibration analyzer. A stroboscopic light is fired by the vibration analyzer at the frequency of the rotating wheel. When the rotor rotates at speed ω, the phase mark on the rotor appears stationary under the stroboscopic light but is positioned at an angle θ from the mark on the stator, as shown in Fig. 9.5(b), due to phase lag in the response. Both the angle θ and the amplitude A_u (read from the vibration analyzer) caused by the original unbalance are noted. The rotor is then stopped, and a known trial weight W is attached to the rotor, as shown in Fig. 9.5(b). When the rotor runs at speed ω, the new angular position

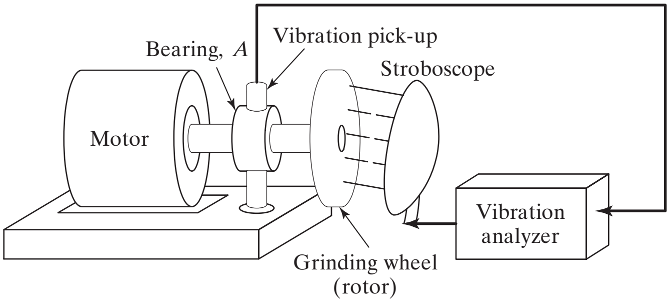


FIGURE 9.4 Single-plane balancing using vibration analyzer.

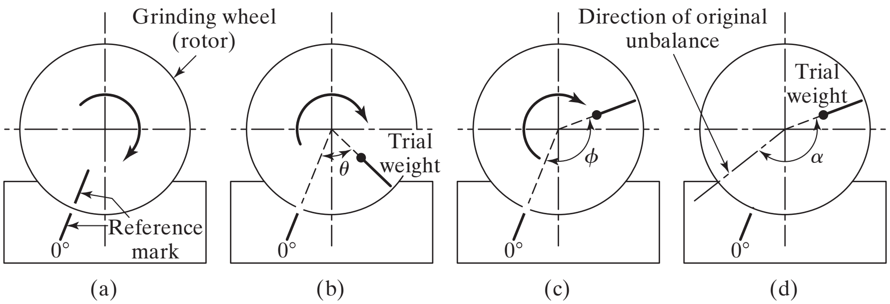


FIGURE 9.5 Use of phase marks.

of the rotor phase mark ϕ and the vibration amplitude A_{u+w}, caused by the combined unbalance of rotor and trial weight, are noted (see Fig. 9.5(c)).[1]

Now we construct a vector diagram to find the magnitude and location of the correction mass for balancing the wheel. The original unbalance vector $\vec{A}_u$ is drawn in an arbitrary direction, with its length equal to A_u, as shown in Fig. 9.6. Then the combined unbalance vector is drawn as $\vec{A}_{u+w}$ at an angle $\phi - \theta$ from the direction of $\vec{A}_u$ with a length of A_{u+w}. The difference vector $\vec{A}_w = \vec{A}_{u+w} - \vec{A}_u$ in Fig. 9.6 then represents the unbalance vector due to the trial weight W. The magnitude of $\vec{A}_w$ can be computed using the law of cosines:

$$A_w = [A_u^2 + A_{u+w}^2 - 2A_u A_{u+w} \cos(\phi - \theta)]^{1/2} \tag{9.9}$$

Since the magnitude of the trial weight W and its direction relative to the original unbalance (α in Fig. 9.6) are known, the original unbalance itself must be at an angle α away

[1]Note that if the trial weight is placed in a position that shifts the net unbalance in a clockwise direction, the stationary position of the phase mark will be shifted by exactly the same amount in the counterclockwise direction, and vice versa.

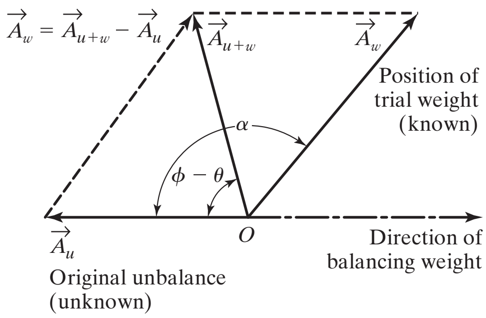


FIGURE 9.6 Unbalance due to trial weight W.

from the position of the trial weight, as shown in Fig. 9.5(d). The angle α can be obtained from the law of cosines:

$$\alpha = \cos^{-1}\left[\frac{A_u^2 + A_w^2 - A_{u+w}^2}{2A_uA_w}\right] \tag{9.10}$$

The magnitude of the original unbalance is $W_0 = (A_u/A_w) \cdot W$, located at the same radial distance from the rotation axis of the rotor as the weight W. Once the location and magnitude of the original unbalance are known, correction weight can be added to balance the wheel properly.

9.4.2 Two-Plane Balancing

The single-plane balancing procedure can be used for balancing in one plane—that is, for rotors of the rigid disc type. If the rotor is an elongated rigid body, as shown in Fig. 9.7, the unbalance can be anywhere along the length of the rotor. In this case, the rotor can be balanced by adding balancing weights in any two planes [9.10, 9.11]. For convenience, the two planes are usually chosen as the end planes of the rotor (shown by dashed lines in Fig. 9.7).

To see that any unbalanced mass in the rotor can be replaced by two equivalent unbalanced masses (in any two planes), consider a rotor with an unbalanced mass m at a distance

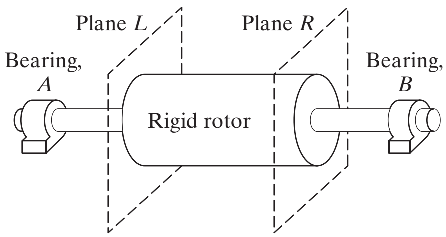


FIGURE 9.7 Two-plane balancing of a rotor.

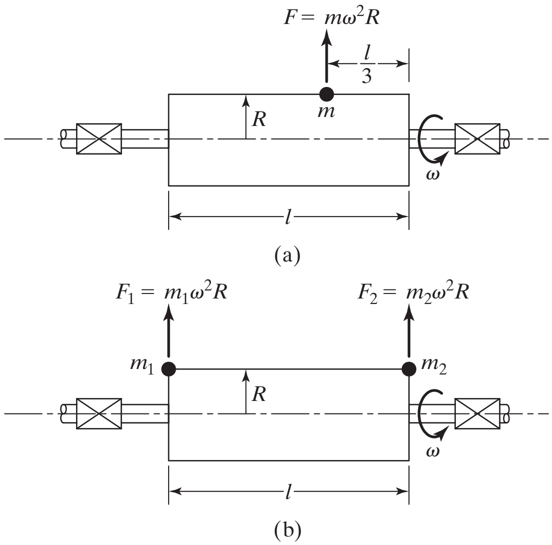


FIGURE 9.8 Representation of an unbalanced mass as two equivalent unbalanced masses.

$l/3$ from the right end, as shown in Fig. 9.8(a). When the rotor rotates at a speed of ω, the force due to the unbalance will be $F = m\omega^2R$, where R is the radius of the rotor. The unbalanced mass m can be replaced by two masses m_1 and m_2, located at the ends of the rotor, as shown in Fig. 9.8(b). The forces exerted on the rotor by these masses are $F_1 = m_1\omega^2R$ and $F_2 = m_2\omega^2R$. For the equivalence of force in Figs. 9.8(a) and (b), we have

$$m\omega^2R = m_1\omega^2R + m_2\omega^2R \qquad \text{or} \qquad m = m_1 + m_2 \tag{9.11}$$

For the equivalence of moments in the two cases, we consider moments about the right end so that

$$m\omega^2R\frac{l}{3} = m_1\omega^2Rl \qquad \text{or} \qquad m = 3m_1 \tag{9.12}$$

Equations (9.11) and (9.12) give $m_1 = m/3$ and $m_2 = 2m/3$. Thus any unbalanced mass can be replaced by two equivalent unbalanced masses in the end planes of the rotor.

We now consider the two-plane balancing procedure using a vibration analyzer. In Fig. 9.9, the total unbalance in the rotor is replaced by two unbalanced weights U_L and U_R in the left and the right planes, respectively. At the rotor's operating speed ω, the vibration amplitude and phase due to the original unbalance are measured at the two bearings A and B, and the results are recorded as vectors $\vec{V}_A$ and $\vec{V}_B$. The magnitude of the vibration vector is taken as the vibration amplitude, while the direction of the vector is taken as the negative

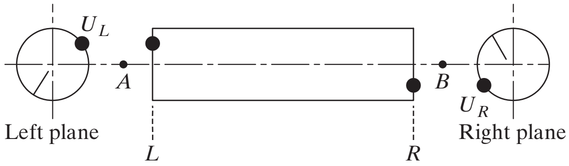


FIGURE 9.9 Two-plane balancing.

of the phase angle observed under stroboscopic light with reference to the stator reference line. The measured vectors $\vec{V}_A$ and $\vec{V}_B$ can be expressed as

$$\vec{V}_A = \vec{A}_{AL}\vec{U}_L + \vec{A}_{AR}\vec{U}_R \tag{9.13}$$

$$\vec{V}_B = \vec{A}_{BL}\vec{U}_L + \vec{A}_{BR}\vec{U}_R \tag{9.14}$$

where $\vec{A}_{ij}$ can be considered as a vector, reflecting the effect of the unbalance in plane j $(j = L, R)$ on the vibration at bearing i $(i = A, B)$. Note that $\vec{U}_L$, $\vec{U}_R$, and all the vectors $\vec{A}_{ij}$ are unknown in Eqs. (9.13) and (9.14).

As in the case of single-plane balancing, we add known trial weights and take measurements to obtain information about the unbalanced masses. We first add a known weight $\vec{W}_L$ in the left plane at a known angular position and measure the displacement and phase of vibration at the two bearings while the rotor is rotating at speed ω. We denote these measured vibrations as vectors as

$$\vec{V}_A' = \vec{A}_{AL}(\vec{U}_L + \vec{W}_L) + \vec{A}_{AR}\vec{U}_R \tag{9.15}$$

$$\vec{V}_B' = \vec{A}_{BL}(\vec{U}_L + \vec{W}_L) + \vec{A}_{BR}\vec{U}_R \tag{9.16}$$

By subtracting Eqs. (9.13) and (9.14) from Eqs. (9.15) and (9.16), respectively, and solving, we obtain[2]

$$\vec{A}_{AL} = \frac{\vec{V}_A' - \vec{V}_A}{\vec{W}_L} \tag{9.17}$$

[2]It can be seen that complex subtraction, division, and multiplication are often used in the computation of the balancing weights. If

$$\vec{A} = a\angle\theta_A \qquad \text{and} \qquad \vec{B} = b\angle\theta_B$$

we can rewrite $\vec{A}$ and $\vec{B}$ as $\vec{A} = a_1 + ia_2$ and $\vec{B} = b_1 + ib_2$, where $a_1 = a\cos\theta_A$, $a_2 = a\sin\theta_A$, $b_1 = b\cos\theta_B$, and $b_2 = b\sin\theta_B$. Then the formulas for complex subtraction, division, and multiplication are [9.12]:

$$\vec{A} - \vec{B} = (a_1 - b_1) + i(a_2 - b_2)$$

$$\frac{\vec{A}}{\vec{B}} = \frac{(a_1b_1 + a_2b_2) + i(a_2b_1 - a_1b_2)}{(b_1^2 + b_2^2)}$$

$$\vec{A}\cdot\vec{B} = (a_1b_1 - a_2b_2) + i(a_2b_1 + a_1b_2)$$

$$\vec{A}_{BL} = \frac{\vec{V}_B' - \vec{V}_B}{\vec{W}_L} \tag{9.18}$$

We then remove $\vec{W}_L$ and add a known weight $\vec{W}_R$ in the right plane at a known angular position and measure the resulting vibrations while the rotor is running at speed ω. The measured vibrations can be denoted as vectors:

$$\vec{V}_A'' = \vec{A}_{AR}(\vec{U}_R + \vec{W}_R) + \vec{A}_{AL}\vec{U}_L \tag{9.19}$$

$$\vec{V}_B'' = \vec{A}_{BR}(\vec{U}_R + \vec{W}_R) + \vec{A}_{BL}\vec{U}_L \tag{9.20}$$

As before, we subtract Eqs. (9.13) and (9.14) from Eqs. (9.19) and (9.20), respectively, to find

$$\vec{A}_{AR} = \frac{\vec{V}_A'' - \vec{V}_A}{\vec{W}_R} \tag{9.21}$$

$$\vec{A}_{BR} = \frac{\vec{V}_B'' - \vec{V}_B}{\vec{W}_R} \tag{9.22}$$

Once the vector operators $\vec{A}_{ij}$ are known, Eqs. (9.13) and (9.14) can be solved to find the unbalance vectors $\vec{U}_L$ and $\vec{U}_R$:

$$\vec{U}_L = \frac{\vec{A}_{BR}\vec{V}_A - \vec{A}_{AR}\vec{V}_B}{\vec{A}_{BR}\vec{A}_{AL} - \vec{A}_{AR}\vec{A}_{BL}} \tag{9.23}$$

$$\vec{U}_R = \frac{\vec{A}_{BL}\vec{V}_A - \vec{A}_{AL}\vec{V}_B}{\vec{A}_{BL}\vec{A}_{AR} - \vec{A}_{AL}\vec{A}_{BR}} \tag{9.24}$$

The rotor can now be balanced by adding equal and opposite balancing weights in each plane. The balancing weights in the left and right planes can be denoted vectorially as $\vec{B}_L = -\vec{U}_L$ and $\vec{B}_R = -\vec{U}_R$. It can be seen that the two-plane balancing procedure is a straightforward extension of the single-plane balancing procedure. Although high-speed rotors are balanced during manufacture, usually it becomes necessary to rebalance them in the field due to slight unbalances introduced due to creep, high-temperature operation, and the like. Figure 9.10 shows a practical example of two-plane balancing.

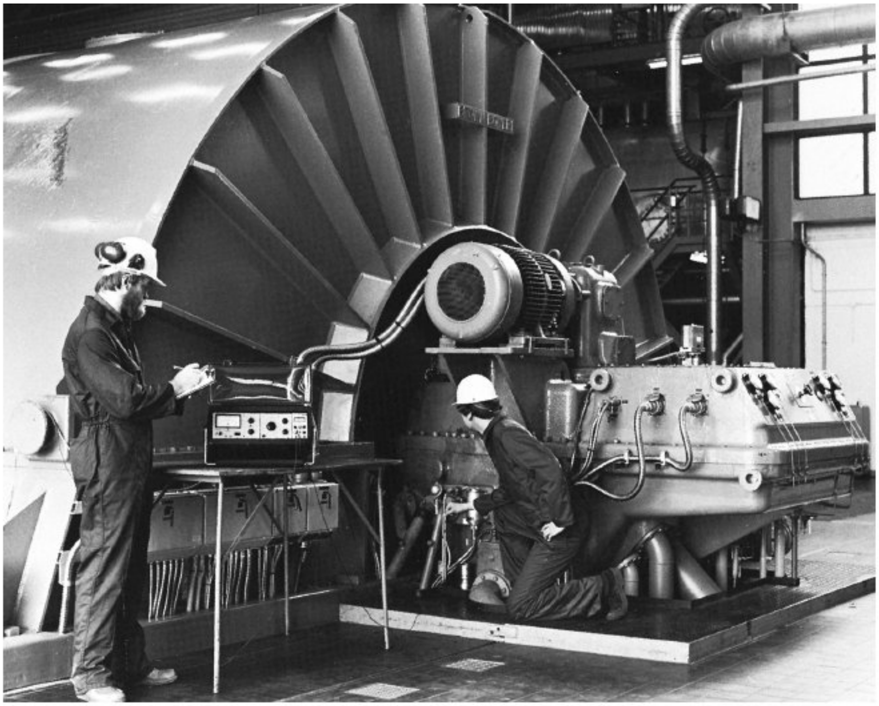


FIGURE 9.10 Two-plane balancing. (Courtesy of Bruel and Kjaer Instruments, Inc., Marlborough, MA.)

EXAMPLE 9.2

Two-Plane Balancing of Turbine Rotor

In the two-plane balancing of a turbine rotor, the data obtained from measurement of the original unbalance, the right-plane trial weight, and the left-plane trial weight are shown below. The displacement amplitudes are in mils (1/1000 inch.) Determine the size and location of the balance weights required.

	Vibration (Displacement) Amplitude		Phase Angle	
Condition	**At Bearing *A***	**At Bearing *B***	**At Bearing *A***	**At Bearing *B***
Original unbalance	8.5	6.5	60°	205°
W_L = 10.0 oz added at 270° from reference mark	6.0	4.5	125°	230°
W_R = 12.0 oz added at 180° from reference mark	6.0	10.5	35°	160°

Solution: The given data can be expressed in vector notation as

$$\vec{V}_A = 8.5\angle 60^\circ = 4.2500 + i7.3612$$
$$\vec{V}_B = 6.5\angle 205^\circ = -5.8910 - i2.7470$$
$$\vec{V}_A' = 6.0\angle 125^\circ = -3.4415 + i4.9149$$
$$\vec{V}_B' = 4.5\angle 230^\circ = -2.8926 - i3.4472$$
$$\vec{V}_A'' = 6.0\angle 35^\circ = 4.9149 + i3.4472$$
$$\vec{V}_B'' = 10.5\angle 160^\circ = -9.8668 + i3.5912$$
$$\vec{W}_L = 10.0\angle 270^\circ = 0.0000 - i10.0000$$
$$\vec{W}_R = 12.0\angle 180^\circ = -12.0000 + i0.0000$$

Equations (9.17) and (9.18) give

$$\vec{A}_{AL} = \frac{\vec{V}_A' - \vec{V}_A}{\vec{W}_L} = \frac{-7.6915 - i2.4463}{0.0000 - i10.0000} = 0.2446 - i0.7691$$

$$\vec{A}_{BL} = \frac{\vec{V}_B' - \vec{V}_B}{\vec{W}_L} = \frac{2.9985 - i0.7002}{0.0000 - i10.0000} = 0.0700 + i0.2998$$

The use of Eqs. (9.21) and (9.22) leads to

$$\vec{A}_{AR} = \frac{\vec{V}_A'' - \vec{V}_A}{\vec{W}_R} = \frac{0.6649 - i3.9198}{-12.0000 + i0.0000} = -0.0554 + i0.3266$$

$$\vec{A}_{BR} = \frac{\vec{V}_B'' - \vec{V}_B}{\vec{W}_R} = \frac{-3.9758 + i6.3382}{-12.0000 + i0.0000} = 0.3313 - i0.5282$$

The unbalance weights can be determined from Eqs. (9.23) and (9.24):

$$\vec{U}_L = \frac{(5.2962 + i0.1941) - (1.2237 - i1.7721)}{(-0.3252 - i0.3840) - (-0.1018 + i0.0063)} = \frac{(4.0725 + i1.9661)}{(-0.2234 - i0.3903)}$$
$$= -8.2930 + i5.6879$$

$$\vec{U}_R = \frac{(-1.9096 + i1.7898) - (3.5540 + i3.8590)}{(-0.1018 + i0.0063) - (-0.3252 - i0.3840)} = \frac{(1.6443 - i2.0693)}{(0.2234 + i0.3903)}$$
$$= -2.1773 - i5.4592$$

Thus the required balance weights are given by

$$\vec{B}_L = -\vec{U}_L = (8.2930 - i5.6879) = 10.0561\angle 145.5548^\circ$$
$$\vec{B}_R = -\vec{U}_R = (2.1773 + i5.4592) = 5.8774\angle 248.2559^\circ$$

This shows that the addition of a 10.0561-oz weight in the left plane at 145.5548° and a 5.8774-oz weight in the right plane at 248.2559° from the reference position will balance the turbine rotor. It is implied that the balance weights are added at the same radial distance as the trial weights. If a balance weight is to be located at a different radial position, the required balance weight is to be modified in inverse proportion to the radial distance from the axis of rotation.

■

9.5 Whirling of Rotating Shafts

In the previous section, the rotor system—the shaft as well as the rotating body—was assumed to be rigid. However, in many practical applications, such as turbines, compressors, electric motors, and pumps, a heavy rotor is mounted on a lightweight, flexible shaft that is supported in bearings. There will be unbalance in all rotors due to manufacturing errors. These unbalances as well as other effects, such as the stiffness and damping of the shaft, gyroscopic effects, and fluid friction in bearings, will cause a shaft to bend in a complicated manner at certain rotational speeds, known as the whirling, whipping, or critical speeds. Whirling is defined as the rotation of the plane made by the line of centers of the bearings and the bent shaft. We consider the aspects of modeling the rotor system, critical speeds, response of the system, and stability in this section [9.13–9.14].

9.5.1 Equations of Motion

Consider a shaft supported by two bearings and carrying a rotor or disc of mass m at the middle, as shown in Fig. 9.11. We shall assume that the rotor is subjected to a steady-state excitation due to mass unbalance. The forces acting on the rotor are the inertia force due to the acceleration of the mass center, the spring force due to the elasticity of the shaft, and the external and internal damping forces.[3]

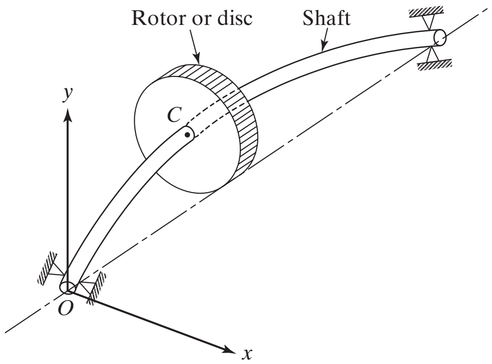


FIGURE 9.11 Shaft carrying a rotor.

[3]Any rotating system responds in two different ways to damping or friction forces, depending upon whether the forces rotate with the shaft or not. When the positions at which the forces act remain fixed in space, as in the case of damping forces (which cause energy losses) in the bearing support structure, the damping is called *stationary* or *external damping.* On the other hand, if the positions at which they act rotate with the shaft in space, as in the case of internal friction of the shaft material, the damping is called *rotary* or *internal damping.*

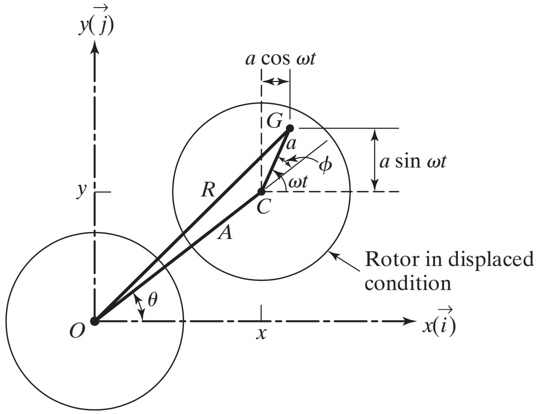


FIGURE 9.12 Rotor with eccentricity.

Let O denote the equilibrium position of the shaft when balanced perfectly, as shown in Fig. 9.12. The shaft (line CG) is assumed to rotate with a constant angular velocity ω. During rotation, the rotor deflects radially by a distance $A = OC$ (in steady state). The rotor (disc) is assumed to have an eccentricity a so that its mass center (center of gravity) G is at a distance a from the geometric center, C. We use a fixed coordinate system (x and y fixed to the earth) with O as the origin for describing the motion of the system. The angular velocity of the line OC, $\dot{\theta} = d\theta/dt$, is known as the whirling speed and, in general, is not equal to ω. The equations of motion of the rotor (mass m) can be written as

$$\begin{aligned}\text{Inertia force } (\vec{F}_i) &= \text{Elastic force } (\vec{F}_e) \\ &\quad + \text{Internal damping force } (\vec{F}_{di}) \\ &\quad + \text{External damping force } (\vec{F}_{de})\end{aligned} \tag{9.25}$$

The various forces in Eq. (9.25) can be expressed as follows:

$$\text{Inertia force: } \vec{F}_i = m\ddot{\vec{R}} \tag{9.26}$$

where $\vec{R}$ denotes the radius vector of the mass center G given by

$$\vec{R} = (x + a\cos\omega t)\vec{i} + (y + a\sin\omega t)\vec{j} \tag{9.27}$$

with x and y representing the coordinates of the geometric center C and $\vec{i}$ and $\vec{j}$ denoting the unit vectors along the x and y coordinates, respectively. Equations (9.26) and (9.27) lead to

$$\vec{F}_i = m[(\ddot{x} - a\omega^2\cos\omega t)\vec{i} + (\ddot{y} - a\omega^2\sin\omega t)\vec{j}] \tag{9.28}$$

$$\text{Elastic force: } \vec{F}_e = -k(x\vec{i} + y\vec{j}) \tag{9.29}$$

where k is the stiffness of the shaft.

$$\text{Internal damping force: } \vec{F}_{di} = -c_i[(\dot{x} + \omega y)\vec{i} + (\dot{y} + \omega x)\vec{j}] \tag{9.30}$$

where c_i is the internal or rotary damping coefficient:

$$\text{External damping force: } \vec{F}_{de} = -c(\dot{x}\vec{i} + \dot{y}\vec{j}) \tag{9.31}$$

where c is the external damping coefficient. By substituting Eqs. (9.28) to (9.31) into Eq. (9.25), we obtain the equations of motion in scalar form:

$$m\ddot{x} + (c_i + c)\dot{x} + kx - c_i\omega y = m\omega^2 a \cos \omega t \tag{9.32}$$

$$m\ddot{y} + (c_i + c)\dot{y} + ky - c_i\omega x = m\omega^2 a \sin \omega t \tag{9.33}$$

These equations of motion, which describe the lateral vibration of the rotor, are coupled and are dependent on the speed of the steady-state rotation of the shaft, ω. By defining a complex quantity w as

$$w = x + iy \tag{9.34}$$

where $i = (-1)^{1/2}$, and by adding Eq. (9.32) to Eq. (9.33) multiplied by i, we obtain a single equation of motion:

$$m\ddot{w} + (c_i + c)\dot{w} + kw - i\omega c_i w = m\omega^2 a e^{i\omega t} \tag{9.35}$$

9.5.2 Critical Speeds

A critical speed is said to exist when the frequency of the rotation of a shaft equals one of the natural frequencies of the shaft. The undamped natural frequency of the rotor system can be obtained by solving Eqs. (9.32), (9.33), or (9.35), retaining only the homogeneous part with $c_i = c = 0$. This gives the natural frequency of the system (or critical speed of the undamped system):

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{9.36}$$

When the rotational speed is equal to this critical speed, the rotor undergoes large deflections, and the force transmitted to the bearings can cause bearing failures. A rapid transition of the rotating shaft through a critical speed is expected to limit the whirl amplitudes, while a slow transition through the critical speed aids the development of large amplitudes. Reference [9.15] investigates the behavior of the rotor during acceleration and deceleration through critical speeds. A FORTRAN computer program for calculating the critical speeds of rotating shafts is given in reference [9.16].

9.5.3 Response of the System

To determine the response of the rotor, we assume the excitation to be a harmonic force due to the unbalance of the rotor. In addition, we assume the internal damping to be negligible $(c_i = 0)$. Then, we can solve Eqs. (9.32) and (9.33) (or equivalently, Eq. (9.35)) and find the rotor's dynamic whirl amplitudes resulting from the mass unbalance. With $c_i = 0$, Eq. (9.35) reduces to

$$m\ddot{w} + c\dot{w} + kw = m\omega^2 a e^{i\omega t} \tag{9.37}$$

The solution of Eq. (9.37) can be expressed as

$$w(t) = Ce^{-(\alpha t + \beta)} + Ae^{i(\omega t - \phi)} \tag{9.38}$$

where C, β, A, and ϕ are constants. Note that the first term on the right-hand side of Eq. (9.38) contains a decaying exponential term representing a transient solution and the second term denotes a steady-state circular motion (whirl). By substituting the steady-state part of Eq. (9.38) into Eq. (9.37), we can find the amplitude of the circular motion (whirl) as

$$A = \frac{m\omega^2 a}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}} = \frac{ar^2}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \tag{9.39}$$

and the phase angle as

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{9.40}$$

where

$$r = \frac{\omega}{\omega_n}, \qquad \omega_n = \sqrt{\frac{k}{m}}, \qquad \text{and} \qquad \zeta = \frac{c}{2\sqrt{km}}.$$

By differentiating Eq. (9.39) with respect to ω and setting the result equal to zero, we can find the rotational speed ω at which the whirl amplitude becomes a maximum:

$$\omega \approx \frac{\omega_n}{\sqrt{1 - 2\zeta^2}} \tag{9.41}$$

where ω_n is given by Eq. (9.36). It can be seen that the critical speed corresponds exactly to the natural frequency ω_n only when the damping (c) is zero. Furthermore, Eq. (9.41) shows that the presence of damping, in general, increases the value of the critical speed compared to ω_n. A plot of Eqs. (9.39) and (9.40) is shown in Fig. 9.13 [9.14]. Since the forcing function is proportional to ω^2, we normally expect the vibration amplitude to increase with the speed ω. However, the actual amplitude appears as shown in Fig. 9.13. From Eq. (9.39), we note that the amplitude of circular whirl A at low speeds is determined

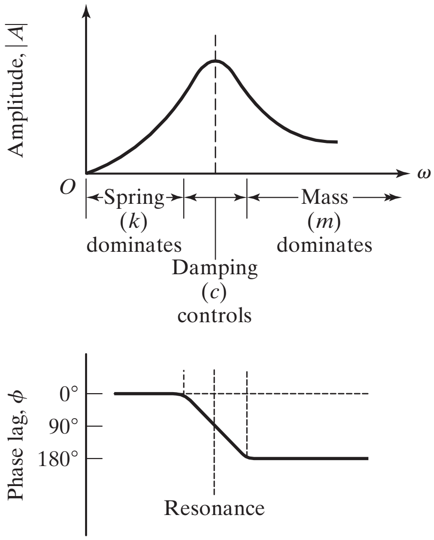


FIGURE 9.13 Plots of Eqs. (9.39) and (9.40).

by the spring constant k, since the other two terms, $m\omega^2$ and $c^2\omega^2$, are small. Also, the value of the phase angle ϕ can be seen to be 0° from Eq. (9.40) for small values of ω. As ω increases, the amplitude of the response reaches a peak, since resonance occurs at $k - m\omega^2 = 0$. Around resonance, the response is essentially limited by the damping term. The phase lag is 90° at resonance. As the speed ω increases beyond ω_n, the response is dominated by the mass term $m^2\omega^4$ in Eq. (9.39). Since this term is 180° out of phase with the unbalanced force, the shaft rotates in a direction opposite to that of the unbalanced force, hence the response of the shaft will be limited.

Notes

1. Equation (9.38) implicitly assumes a condition of forward synchronous whirl under steady state (that is, $\dot{\theta} = \omega$). As a general case, if the steady-state solution of Eq. (9.37) is assumed as $w(t) = Ae^{i(\gamma t - \phi)}$, the solution can be obtained as $\gamma = \pm\omega$, with $\gamma = +\omega$ representing the forward synchronous whirl and $\gamma = -\omega$ denoting a backward synchronous whirl. For simple rotors, such as the one shown in Fig. 9.11, only forward synchronous whirl occurs in practice.
2. To determine the bearing reactions, we first find the deflection of the mass center of the disc from the bearing axis, R in Fig. 9.12, as

$$R^2 = A^2 + a^2 + 2Aa\cos\phi \tag{9.42}$$

In view of Eqs. (9.39) and (9.40), Eq. (9.42) can be rewritten as

$$R = a\left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2} \tag{9.43}$$

The bearing reactions can then be determined from the centrifugal force, $m\omega^2 R$. The vibration and balancing of unbalanced flexible rotors are presented in references [9.17, 9.18].

9.5.4 Stability Analysis

Instability in a flexible rotor system can occur due to several factors like internal friction, eccentricity of the rotor, and the oil whip in the bearings. As seen earlier, the stability of the system can be investigated by considering the equation governing the dynamics of the system. Assuming $w(t) = e^{st}$, the characteristic equation corresponding to the homogeneous part of Eq. (9.35) can be written as

$$ms^2 + (c_i + c)s + k - i\omega c_i = 0 \tag{9.44}$$

With $s = i\lambda$, Eq. (9.44) becomes

$$-m\lambda^2 + (c_i + c)i\lambda + k - i\omega c_i = 0 \tag{9.45}$$

This equation is a particular case of the more general equation

$$(p_2 + iq_2)\lambda^2 + (p_1 + iq_1)\lambda + (p_0 + iq_0) = 0 \tag{9.46}$$

A necessary and sufficient condition for the system governed by Eq. (9.46) to be stable, according to Routh-Hurwitz criterion, is that the following inequalities are satisfied:

$$-\begin{vmatrix} p_2 & p_1 \\ q_2 & q_1 \end{vmatrix} > 0 \tag{9.47}$$

and

$$\begin{vmatrix} p_2 & p_1 & p_0 & 0 \\ q_2 & q_1 & q_0 & 0 \\ 0 & p_2 & p_1 & p_0 \\ 0 & q_2 & q_1 & q_0 \end{vmatrix} > 0 \tag{9.48}$$

Noting that $p_2 = -m$, $q_2 = 0$, $p_1 = 0$, $q_1 = c_i + c$, $p_0 = k$, and $q_0 = -\omega c_i$, from Eq. (9.45), the application of Eqs. (9.47) and (9.48) leads to

$$m(c_i + c) > 0 \tag{9.49}$$

and

$$km(c_i + c)^2 - m^2(\omega^2 c_i^2) > 0 \tag{9.50}$$

Equation (9.49) is automatically satisfied, while Eq. (9.50) yields the condition

$$\sqrt{\frac{k}{m}}\left(1 + \frac{c}{c_i}\right) - \omega > 0 \tag{9.51}$$

This equation also shows that internal and external friction can cause instability at rotating speeds above the first critical speed of $\omega = \sqrt{\frac{k}{m}}$.

EXAMPLE 9.3 Whirl Amplitude of a Shaft Carrying an Unbalanced Rotor

A shaft, carrying a rotor of weight 100 lb and eccentricity 0.1 in., rotates at 1200 rpm. Determine (a) the steady-state whirl amplitude and (b) the maximum whirl amplitude during start-up conditions of the system. Assume the stiffness of the shaft as 2×10^5 lb/in. and the external damping ratio as 0.1.

Solution: The forcing frequency of the rotor (rotational speed of the shaft) is given by

$$\omega = \frac{1200 \times 2\pi}{60} = 40\pi = 125.6640 \text{ rad/s}$$

The natural frequency of the system can be determined as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.0 \times 10^5}{(100/386.4)}} = 87.9090 \text{ rad/s}$$

and the frequency ratio as

$$r = \frac{\omega}{\omega_n} = \frac{125.6640}{87.9090} = 1.4295$$

(a) The steady-state amplitude is given by Eq. (9.39):

$$A = \frac{ar^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{E.1}$$

$$= \frac{(0.1)(1.4295)^2}{\sqrt{(1 - 1.4295^2)^2 + (2 \times 0.1 \times 1.4295)^2}} = 0.18887 \text{ in.} \tag{E.2}$$

(b) During start-up conditions, the frequency (speed) of the rotor, ω, passes through the natural frequency of the system. Thus, using $r = 1$ in Eq. (E.1), we obtain the whirl amplitude as

$$A\big|_{r=1} = \frac{a}{2\zeta} = \frac{0.1}{2(0.1)} = 0.5 \text{ in.}$$

■

9.6 Balancing of Reciprocating Engines

The essential moving elements of a reciprocating engine are the piston, the crank, and the connecting rod. Vibrations in reciprocating engines arise due to (1) periodic variations of the gas pressure in the cylinder and (2) inertia forces associated with the moving parts [9.19]. We shall now analyze a reciprocating engine and find the unbalanced forces caused by these factors.

9.6.1 Unbalanced Forces Due to Fluctuations in Gas Pressure

Figure 9.14(a) is a schematic diagram of a cylinder of a reciprocating engine. The engine is driven by the expanding gas in the cylinder. The expanding gas exerts on the piston a pressure force F, which is transmitted to the crankshaft through the connecting rod. The reaction to the force F can be resolved into two components: one of magnitude $F/\cos\phi$, acting along the connecting rod, and the other of magnitude $F\tan\phi$, acting in a horizontal direction. The force $F/\cos\phi$ induces a torque M_t, which tends to rotate the crankshaft. (In Fig. 9.14(b), M_t acts about an axis perpendicular to the plane of the paper and passes through point Q.)

$$M_t = \left(\frac{F}{\cos\phi}\right) r \cos\theta \tag{9.52}$$

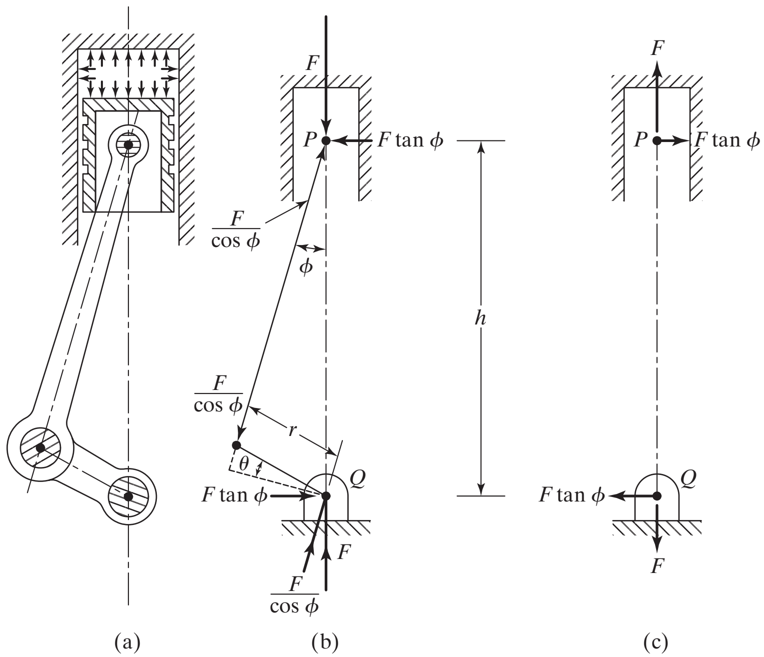


FIGURE 9.14 Forces in a reciprocating engine.

For force equilibrium of the overall system, the forces at the bearings of the crankshaft will be F in the vertical direction and $F \tan \phi$ in the horizontal direction.

Thus the forces transmitted to the stationary parts of the engine are as follows:

1. Force F acting upward at the cylinder head
2. Force $F \tan \phi$ acting toward the right at the cylinder head
3. Force F acting downward at the crankshaft bearing Q
4. Force $F \tan \phi$ acting toward the left at the crankshaft bearing

These forces are shown in Fig. 9.14(c). Although the total resultant force is zero, there is a resultant torque $M_Q = Fh \tan \phi$ on the body of the engine, where h can be found from the geometry of the system:

$$h = \frac{r \cos \theta}{\sin \phi} \tag{9.53}$$

Thus the resultant torque is given by

$$M_Q = \frac{Fr \cos \theta}{\cos \phi} \tag{9.54}$$

As expected, M_t and M_Q given by Eqs. (9.52) and (9.54) can be seen to be identical, which indicates that the torque induced on the crankshaft due to the gas pressure on the piston is felt at the support of the engine. Since the magnitude of the gas force F varies with time, the torque M_Q also varies with time. The magnitude of force F changes from a maximum to a minimum at a frequency governed by the number of cylinders in the engine, the type of the operating cycle, and the rotating speed of the engine.

9.6.2 Unbalanced Forces Due to Inertia of the Moving Parts

Acceleration of the Piston. Figure 9.15 shows the crank (of length r), the connecting rod (of length l), and the piston of a reciprocating engine. The crank is assumed to rotate in an anticlockwise direction at a constant angular speed of ω, as shown in Fig. 9.15. If we consider the origin of the x-axis (O) as the uppermost position of the piston, the displacement of the piston P corresponding to an angular displacement of the crank of $\theta = \omega t$ can be expressed as in Fig. 9.15. The displacement of the piston P corresponding to an angular displacement of the crank $\theta = \omega t$ from its topmost position (origin O) can be expressed as

$$\begin{aligned} x_p &= r + l - r \cos \theta - l \cos \phi \\ &= r + l - r \cos \omega t - l\sqrt{1 - \sin^2 \phi} \end{aligned} \tag{9.55}$$

But

$$l \sin \phi = r \sin \theta = r \sin \omega t \tag{9.56}$$

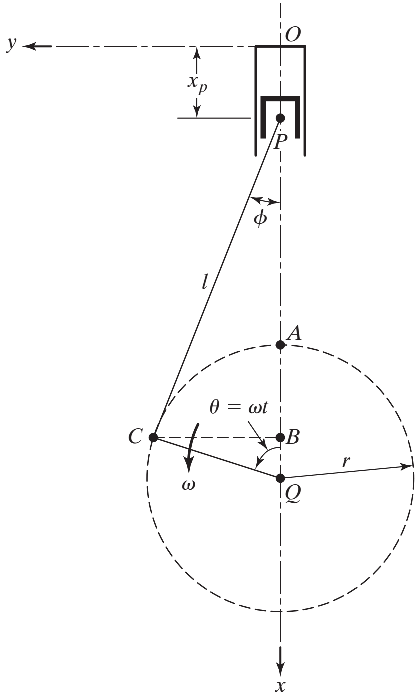


FIGURE 9.15 Motions of crank, connecting rod, and piston.

and hence

$$\cos\phi = \left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{1/2} \tag{9.57}$$

By substituting Eq. (9.57) into Eq. (9.55), we obtain

$$x_p = r + l - r\cos\omega t - l\sqrt{1 - \frac{r^2}{l^2}\sin^2\omega t} \tag{9.58}$$

Due to the presence of the term involving the square root, Eq. (9.58) is not very convenient in further calculation. Equation (9.58) can be simplified by noting that, in general, $r/l < \frac{1}{4}$ and by using the expansion relation

$$\sqrt{1 - \varepsilon} \simeq 1 - \frac{\varepsilon}{2} \tag{9.59}$$

Hence Eq. (9.58) can be approximated as

$$x_p \simeq r(1 - \cos \omega t) + \frac{r^2}{2l} \sin^2 \omega t \tag{9.60}$$

or, equivalently,

$$x_p = r\left(1 + \frac{r}{2l}\right) - r\left(\cos \omega t + \frac{r}{4l} \cos 2\omega t\right) \tag{9.61}$$

Equation (9.61) can be differentiated with respect to time to obtain expressions for the velocity and the acceleration of the piston:

$$\dot{x}_p = r\omega\left(\sin \omega t + \frac{r}{2l} \sin 2\omega t\right) \tag{9.62}$$

$$\ddot{x}_p = r\omega^2\left(\cos \omega t + \frac{r}{l} \cos 2\omega t\right) \tag{9.63}$$

Acceleration of the Crankpin. With respect to the xy coordinate axes shown in Fig. 9.15, the vertical and horizontal displacements of the crankpin C are given by

$$x_c = OA + AB = l + r(1 - \cos \omega t) \tag{9.64}$$

$$y_c = CB = r \sin \omega t \tag{9.65}$$

Differentiation of Eqs. (9.64) and (9.65) with respect to time gives the velocity and acceleration components of the crankpin as

$$\dot{x}_c = r\omega \sin \omega t \tag{9.66}$$

$$\dot{y}_c = r\omega \cos \omega t \tag{9.67}$$

$$\ddot{x}_c = r\omega^2 \cos \omega t \tag{9.68}$$

$$\ddot{y}_c = -r\omega^2 \sin \omega t \tag{9.69}$$

Inertia Forces. Although the mass of the connecting rod is distributed throughout its length, it is generally idealized as a massless link with two masses concentrated at its ends—the piston end and the crankpin end. If m_p and m_c denote the total mass of the piston and of the crankpin (including the concentrated mass of the connecting rod) respectively, the vertical component of the inertia force (F_x) for one cylinder is given by

$$F_x = m_p\ddot{x}_p + m_c\ddot{x}_c \tag{9.70}$$

By substituting Eqs. (9.63) and (9.68) for the accelerations of P and C, Eq. (9.70) becomes

$$F_x = (m_p + m_c)r\omega^2 \cos \omega t + m_p \frac{r^2\omega^2}{l} \cos 2\omega t \tag{9.71}$$

It can be observed that the vertical component of the inertia force consists of two parts. One part, known as the *primary part,* has a frequency equal to the rotational frequency of the crank ω. The other part, known as the *secondary part,* has a frequency equal to twice the rotational frequency of the crank.

Similarly, the horizontal component of inertia force for a cylinder can be obtained

$$F_y = m_p \ddot{y}_p + m_c \ddot{y}_c \tag{9.72}$$

where $\ddot{y}_p = 0$ and $\ddot{y}_c$ is given by Eq. (9.69). Thus

$$F_y = -m_c r\omega^2 \sin \omega t \tag{9.73}$$

The horizontal component of the inertia force can be observed to have only a primary part.

9.6.3 Balancing of Reciprocating Engines

The unbalanced or inertia forces on a single cylinder are given by Eqs. (9.71) and (9.73). In these equations, m_p and m_c represent the equivalent reciprocating and rotating masses, respectively. The mass m_p is always positive, but m_c can be made zero by counterbalancing the crank. It is therefore possible to reduce the horizontal inertia force F_y to zero, but the vertical unbalanced force always exists. Thus a single-cylinder engine is inherently unbalanced.

In a multicylinder engine, it is possible to balance some or all of the inertia forces and torques by proper arrangement of the cranks. Figure 9.16(a) shows the general arrangement of an N-cylinder engine (only six cylinders, $N = 6$, are shown in the figure). The lengths of all the cranks and connecting rods are assumed to be r and l, respectively, and the angular velocity of all the cranks is taken to be a constant, ω. The axial displacement and angular orientation of ith cylinder from those of the first cylinder are assumed to be α_i and l_i, respectively; $i = 2, 3, \ldots, N$. For force balance, the total inertia force in the x and y directions must be zero. Thus

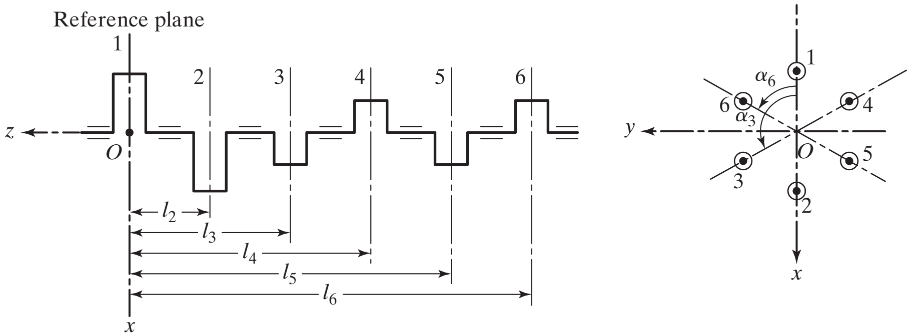
$$(F_x)_{\text{total}} = \sum_{i=1}^{N} (F_x)_i = 0 \tag{9.74}$$

$$(F_y)_{\text{total}} = \sum_{i=1}^{N} (F_y)_i = 0 \tag{9.75}$$

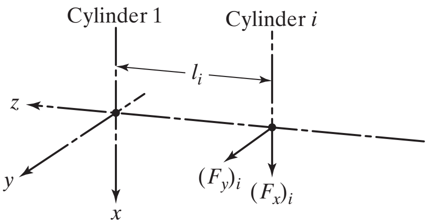
where $(F_x)_i$ and $(F_y)_i$ are the vertical and horizontal components of inertia force of cylinder i given by (see Eqs. (9.71) and (9.73)):

$$(F_x)_i = (m_p + m_c)_i r\omega^2 \cos(\omega t + \alpha_i) + (m_p)_i \frac{r^2\omega^2}{l} \cos(2\omega t + 2\alpha_i) \tag{9.76}$$

$$(F_y)_i = -(m_c)_i r\omega^2 \sin(\omega t + \alpha_i) \tag{9.77}$$



(a)



(b)

FIGURE 9.16 Arrangement of an N-cylinder engine.

For simplicity, we assume the reciprocating and rotating masses for each cylinder to be same—that is, $(m_p)_i = m_p$ and $(m_c)_i = m_c$ for $i = 1, 2, \ldots, N$. Without loss of generality, Eqs. (9.74) and (9.75) can be applied at time $t = 0$. Thus the conditions necessary for the total force balance are given by

$$\sum_{i=1}^{N} \cos \alpha_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{N} \cos 2\alpha_i = 0 \tag{9.78}$$

$$\sum_{i=1}^{N} \sin \alpha_i = 0 \tag{9.79}$$

The inertia forces $(F_x)_i$ and $(F_y)_i$ of the ith cylinder induce moments about the y- and x-axes, respectively, as shown in Fig. 9.16(b). The moments about the z- and x-axes are given by

$$M_z = \sum_{i=2}^{N} (F_x)_i l_i = 0 \tag{9.80}$$

$$M_x = \sum_{i=2}^{N} (F_y)_i l_i = 0 \tag{9.81}$$

By substituting Eqs. (9.76) and (9.77) into Eqs. (9.80) and (9.81) and assuming $t = 0$, we obtain the necessary conditions to be satisfied for the balancing of moments about the z- and x-axes as

$$\sum_{i=2}^{N} l_i \cos \alpha_i = 0 \quad \text{and} \quad \sum_{i=2}^{N} l_i \cos 2\alpha_i = 0 \tag{9.82}$$

$$\sum_{i=2}^{N} l_i \sin \alpha_i = 0 \tag{9.83}$$

Thus we can arrange the cylinders of a multicylinder reciprocating engine so as to satisfy Eqs. (9.78), (9.79), (9.82), and (9.83); it will be completely balanced against the inertia forces and moments.

9.7 Control of Vibration

In many practical situations, it is possible to reduce but not eliminate the dynamic forces that cause vibrations. Several methods can be used to control vibrations. Among them, the following are important:

1. Controlling the natural frequencies of the system and avoiding resonance under external excitations.
2. Preventing excessive response of the system, even at resonance, by introducing a damping or energy-dissipating mechanism.
3. Reducing the transmission of the excitation forces from one part of the machine to another by the use of vibration isolators.
4. Reducing the response of the system by the addition of an auxiliary mass neutralizer or vibration absorber.

We shall now consider the details of these methods.

9.8 Control of Natural Frequencies

It is well known that whenever the frequency of excitation coincides with one of the natural frequencies of the system, resonance occurs. The most prominent feature of resonance is a large displacement. In most mechanical and structural systems, large displacements indicate undesirably large strains and stresses, which can lead to the failure of the system. Hence in any system resonance conditions must be avoided. In most cases, the excitation frequency cannot be controlled, because it is imposed by the functional requirements of the system or machine. We must concentrate on controlling the natural frequencies of the system to avoid resonance.

As indicated by Eq. (2.14), the natural frequency of a system can be changed by varying either the mass m or the stiffness k.[4] In many practical cases, however, the mass cannot be changed easily, since its value is determined by the functional requirements of the system. For example, the mass of a flywheel on a shaft is determined by the amount of energy it must store in one cycle. Therefore, the stiffness of the system is the factor that is most often changed to alter its natural frequencies. For example, the stiffness of a rotating shaft can be altered by varying one or more of its parameters, such as materials or the number and location of support points (bearings).

9.9 Introduction of Damping

Although damping is disregarded so as to simplify the analysis, especially in finding the natural frequencies, most systems possess damping to some extent. The presence of damping is helpful in many cases. In systems such as automobile shock absorbers and many vibration-measuring instruments, damping must be introduced to fulfill the functional requirements [9.20–9.21].

If the system undergoes forced vibration, its response or amplitude of vibration tends to become large near resonance if there is no damping. The presence of damping always limits the amplitude of vibration. If the forcing frequency is known, it may be possible to avoid resonance by changing the natural frequency of the system. However, the system or the machine may be required to operate over a range of speeds, as in the case of a variable-speed electric motor or an internal combustion engine. It may not be possible to avoid resonance under all operating conditions. In such cases, we can introduce damping into the system to control its response, by the use of structural materials having high internal damping, such as cast iron or laminated or sandwich materials.

In some structural applications, damping is introduced through joints. For example, bolted and riveted joints, which permit slip between surfaces, dissipate more energy compared to welded joints, which do not permit slip. Hence a bolted or riveted joint is desirable to increase the damping of the structure. However, bolted and riveted joints reduce the stiffness of the structure, produce debris due to joint slip, and cause fretting corrosion. In spite of this, if a highly damped structure is desired, bolted or riveted joints should not be ignored.

Use of Viscoelastic Materials. The equation of motion of a single-degree-of-freedom system with internal damping, under harmonic excitation $F(t) = F_0 e^{i\omega t}$, can be expressed as

$$m\ddot{x} + k(1 + i\eta)x = F_0 e^{i\omega t} \tag{9.84}$$

where η is called the *loss factor* (or *loss coefficient*), which is defined as (see Section 2.6.4)

$$\eta = \frac{(\Delta W/2\pi)}{W} = \left(\frac{\text{Energy dissipated during 1 cycle of harmonic displacement/radian}}{\text{Maximum strain energy in cycle}}\right) \tag{9.85}$$

[4]Although this statement is made with reference to a single-degree-of-freedom system, it is generally true even for multidegree-of-freedom and continuous systems.

The amplitude of the response of the system at resonance $(\omega = \omega_n)$ is given by

$$\frac{F_0}{k\eta} = \frac{F_0}{aE\eta} \tag{9.86}$$

since the stiffness is proportional to the Young's modulus $(k = aE;\ a = \text{constant})$.

The viscoelastic materials have larger values of the loss factor and hence are used to provide internal damping. When viscoelastic materials are used for vibration control, they are subjected to shear or direct strains. In the simplest arrangement, a layer of viscoelastic material is attached to an elastic one. In another arrangement, a viscoelastic layer is sandwiched between the elastic layers. This arrangement is known as constrained layer damping.[5] Damping tapes, consisting of thin metal foil covered with a viscoelastic adhesive, are used on existing vibrating structures. A disadvantage with the use of viscoelastic materials is that their properties change with temperature, frequency, and strain. Equation (9.86) shows that a material with the highest value of $(E\eta)$ gives the smallest resonance amplitude. Since the strain is proportional to the displacement x and the stress is proportional to Ex, the material with the largest value of the loss factor will be subjected to the smallest stresses. The values of loss coefficient for some materials are given below:

Material	**Loss Factor (η)**
Polystyrene	2.0
Hard rubber	1.0
Fiber mats with matrix	0.1
Cork	0.13–0.17
Aluminum	1×10^{-4}
Iron and steel	$2-6 \times 10^{-4}$

The damping ratios obtainable with different types of construction/arrangement are indicated below:

Type of Construction/Arrangement	**Equivalent Viscous Damping Ratio (%)**
Welded construction	1–4
Bolted construction	3–10
Steel frame	5–6
Unconstrained viscoelastic layer on steel-concrete girder	4–5
Constrained viscoelastic layer on steel-concrete girder	5–8

[5]It appears that constrained layer damping was used, possibly unknowingly, as far back as the seventeenth century, in the manufacture of violins [9.22]. Antonio Stradivari (1644–1737), the renowned Italian violin manufacturer, bought the wood necessary for the manufacture of violins from Venice. The varnish used for sealing the wood was made from a mixture of resin and ground gem stones. This varnish—stone particles in resin matrix—acted as a form of constrained layer (friction mechanism) that created enough damping to explain why many of his violins had a rich, full tone.

9.10 Vibration Isolation

Vibration isolation is a procedure by which the undesirable effects of vibration are reduced [9.22–9.24]. Basically, it involves the insertion of a resilient member (or isolator) between the vibrating mass (or equipment or payload) and the source of vibration so that a reduction in the dynamic response of the system is achieved under specified conditions of vibration excitation. An isolation system is said to be active or passive depending on whether or not external power is required for the isolator to perform its function. A passive isolator consists of a resilient member (stiffness) and an energy dissipator (damping). Examples of passive isolators include metal springs, cork, felt, pneumatic springs, and elastomer (rubber) springs. Figure 9.17 shows typical spring and pneumatic mounts that can be used as passive isolators, and Fig. 9.18 illustrates the use of passive isolators in the mounting of a high-speed punch press [9.25]. The optimal synthesis of vibration isolators is presented in references [9.26–9.30]. An active isolator is comprised of a servomechanism with a sensor, signal processor, and actuator.

FIGURE 9.17 (a) Undamped spring mount; (b) damped spring mount; (c) pneumatic rubber mount. (Courtesy of *Sound and Vibration.*)

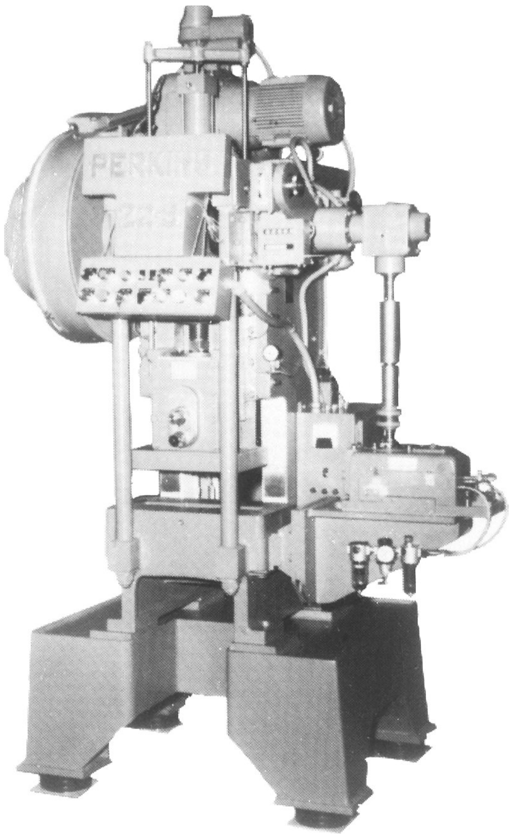


FIGURE 9.18 High-speed punch press mounted on pneumatic rubber mounts. (Courtesy of *Sound and Vibration.*)

Vibration isolation can be used in two types of situations. In the first type, the foundation or base of a vibrating machine is protected against large unbalanced forces. In the second type, the system is protected against the motion of its foundation or base.

The first type of isolation is used when a mass (or a machine) is subjected to a force or excitation. For example, in forging and stamping presses, large impulsive forces act on the object to be formed or stamped. These impacts are transmitted to the base or foundation of the forging or stamping machine, which can damage not only the base or foundation but also the surrounding or nearby structures and machines. They can also cause discomfort to operators of these machines. Similarly, in the case of reciprocating and rotating machines, the inherent unbalanced forces are transmitted to the base or foundation of the machine. In such cases, the force transmitted to the base, $F_t(t)$, varies harmonically, and the resulting stresses in the foundation bolts also vary harmonically, which might lead to fatigue

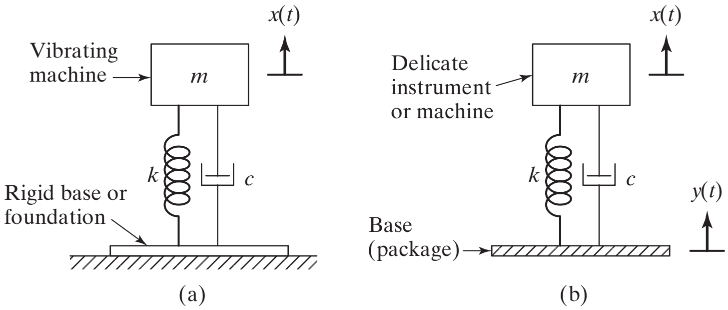


FIGURE 9.19 Vibration isolation.

failure. Even if the force transmitted is not harmonic, its magnitude is to be limited to safe permissible values. In these applications, we can insert an isolator (in the form of stiffness and/or damping) between the mass being subjected to force or excitation and the base or foundation to reduce the force transmitted to the base or foundation. This is called *force isolation*. In many applications, the isolator is also intended to reduce the vibratory motion of the mass under the applied force (as in the case of forging or stamping machines). Thus both force and displacement transmissibilities become important for this of isolators.

The second type of isolation is used when a mass to be protected against the motion or excitation of its base or foundation. When the base is subjected to vibration, the mass m will experience not only a displacement $x(t)$ but also a force $F_t(t)$. The displacement of the mass $x(t)$ is expected to be smaller than the displacement of the base $y(t)$. For example, a delicate instrument or equipment is to be protected from the motion of its container or package (as when the vehicle carrying the package experiences vibration while moving on a rough road). The force transmitted to the mass also needs to be reduced. For example, the package or container is to be designed properly to avoid transmission of large forces to the delicate instrument inside to avoid damage. The force experienced by the instrument or mass m (same as the force transmitted to mass m) is given by

$$F_t(t) = m\ddot{x}(t) = k\{x(t) - y(t)\} + c\{\dot{x}(t) - \dot{y}(t)\} \tag{9.87}$$

where $y(t)$ is the displacement of the base, $x(t) - y(t)$ is the relative displacement of the spring, and $\dot{x}(t) - \dot{y}(t)$ is the relative velocity of the damper. In such cases, we can insert an isolator (which provides stiffness and /or damping) between the base being subjected to force or excitation and the mass to reduce the motion and/or force transmitted to the mass. Thus both displacement isolation and force isolation become important in this case also.

It is to be noted that the effectiveness of an isolator depends on the nature of the force or excitation. For example, an isolator designed to reduce the force transmitted to the base or foundation due to impact forces of forging or stamping may not be effective if the disturbance is a harmonic unbalanced force. Similarly, an isolator designed to handle harmonic excitation at a particular frequency may not be effective for other frequencies or other types of excitation such as step-type excitation.

9.10.1 Vibration Isolation System with Rigid Foundation

Reduction of the Force Transmitted to Foundation. When a machine is bolted directly to a rigid foundation or floor, the foundation will be subjected to a harmonic load due to the unbalance in the machine in addition to the static load due to the weight of the machine. Hence an elastic or resilient member is placed between the machine and the rigid foundation to reduce the force transmitted to the foundation. The system can then be idealized as a single-degree-of-freedom system, as shown in Fig. 9.20(a). The resilient member is assumed to have both elasticity and damping and is modeled as a spring k and a dashpot c, as shown in Fig. 9.20(b). It is assumed that the operation of the machine gives rise to a harmonically varying force $F(t) = F_0 \cos \omega t$. The equation of motion of the machine (of mass m) is given by

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \tag{9.88}$$

Since the transient solution dies out after some time, only the steady-state solution will be left. The steady-state solution of Eq. (9.88) is given by (see Eq. (3.25))

$$x(t) = X \cos(\omega t - \phi) \tag{9.89}$$

where

$$X = \frac{F_0}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}} \tag{9.90}$$

and

$$\phi = \tan^{-1}\left(\frac{\omega c}{k - m\omega^2}\right) \tag{9.91}$$

The force transmitted to the foundation through the spring and the dashpot, $F_t(t)$, is given by

$$F_t(t) = kx(t) + c\dot{x}(t) = kX \cos(\omega t - \phi) - c\omega X \sin(\omega t - \phi) \tag{9.92}$$

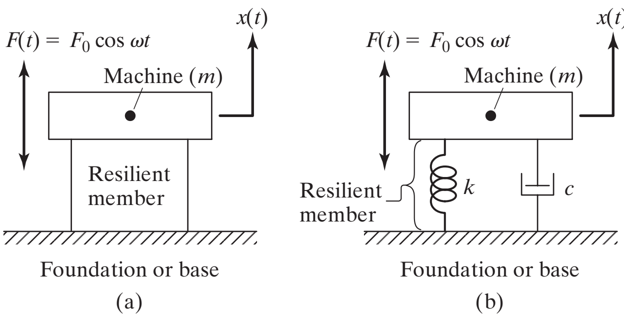


FIGURE 9.20 Machine and resilient member on rigid foundation.

The magnitude of the total transmitted force (F_T) is given by

$$F_T = [(kx)^2 + (c\dot{x})^2]^{1/2} = X\sqrt{k^2 + \omega^2 c^2}$$
$$= \frac{F_0(k^2 + \omega^2 c^2)^{1/2}}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}} \tag{9.93}$$

The transmissibility or transmission ratio of the isolator (T_f) is defined as the ratio of the magnitude of the force transmitted to that of the exciting force:

$$T_f = \frac{F_T}{F_0} = \left\{\frac{k^2 + \omega^2 c^2}{(k - m\omega^2)^2 + \omega^2 c^2}\right\}^{1/2}$$
$$= \left\{\frac{1 + (2\zeta r)^2}{[1 - r^2]^2 + (2\zeta r)^2}\right\}^{1/2} \tag{9.94}$$

where $r = \frac{\omega}{\omega_n}$ is the frequency ratio. The variation of T_f with the frequency ratio $r = \frac{\omega}{\omega_n}$ is shown in Fig. 9.21. In order to achieve isolation, the force transmitted to the foundation needs to be less than the excitation force. It can be seen, from Fig. 9.21, that the forcing frequency has to be greater than $\sqrt{2}$ times the natural frequency of the system in order to achieve isolation of vibration.

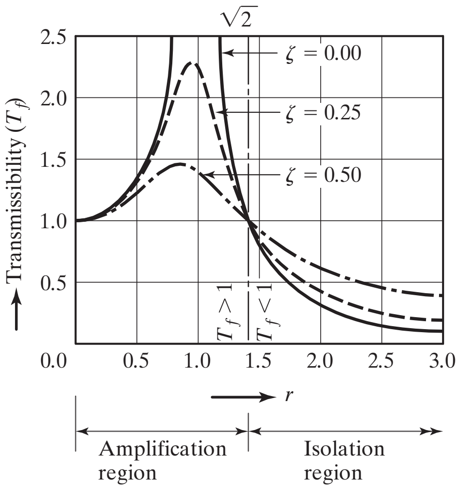


FIGURE 9.21 Variation of transmission ratio (T_f) with r.

For small values of damping ratio ζ and for frequency ratio $r > 1$, the force transmissibility, given by Eq. (9.94), can be approximated as

$$T_f = \frac{F_t}{F} \approx \frac{1}{r^2 - 1} \quad \text{or} \quad r^2 \approx \frac{1 + T_f}{T_f} \tag{9.95}$$

Notes

1. The magnitude of the force transmitted to the foundation can be reduced by decreasing the natural frequency of the system (ω_n).
2. The force transmitted to the foundation can also be reduced by decreasing the damping ratio. However, since vibration isolation requires $r > \sqrt{2}$, the machine should pass through resonance during start-up and stopping. Hence, some damping is essential to avoid infinitely large amplitudes at resonance.
3. Although damping reduces the amplitude of the mass (X) for all frequencies, it reduces the maximum force transmitted to the foundation (F_t) only if $r < \sqrt{2}$. Above that value, the addition of damping increases the force transmitted.
4. If the speed of the machine (forcing frequency) varies, we must compromise in choosing the amount of damping to minimize the force transmitted. The amount of damping should be sufficient to limit the amplitude X and the force transmitted F_t while passing through the resonance, but not so much to increase unnecessarily the force transmitted at the operating speed.

Reduction of the Vibratory Motion of the Mass. In many applications, the isolation is required to reduce the motion of the mass (machine) under the applied force. The displacement amplitude of the mass m due to the force $F(t)$, given by Eq. (9.90), can be expressed as:

$$T_d = \frac{X}{\delta_{st}} = \frac{kX}{F_0} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{9.96}$$

where $\frac{X}{\delta_{st}}$ is called, in the present context, the *displacement transmissibility* or *amplitude ratio* and indicates the ratio of the amplitude of the mass, X, to the static deflection under the constant force F_0, $\delta_{st} = \frac{F_0}{k}$. The variation of the displacement transmissibility with the frequency ratio r for several values of the damping ratio ζ is shown in Fig. 9.22. The following observations can be made from Fig. 9.22:

1. The displacement transmissibility increases to a maximum value at (Eq. (3.33)):

$$r = \sqrt{1 - 2\zeta^2} \tag{9.97}$$

Equation (9.97) shows that, for small values of damping ratio ζ, the displacement transmissibility (or the amplitude of the mass) will be maximum at $r \approx 1$ or $\omega \approx \omega_n$.

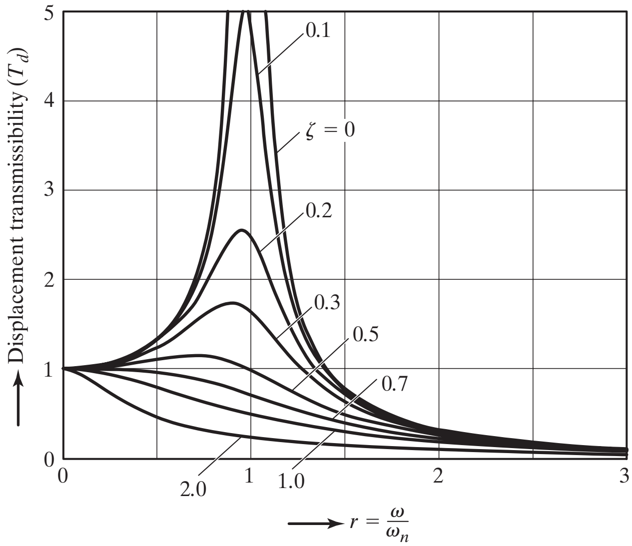


FIGURE 9.22 Variation of displacement transmissibility (T_d) with r.

Thus the value of $r \approx 1$ is to be avoided in practice. In most cases, the excitation frequency ω is fixed and hence we can avoid $r \approx 1$ by altering the value of the natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ which can be accomplished by changing the value of either or both of m and k.

2. The amplitude of the mass, X, approaches zero as r increases to a large value. The reason is that at large values of r, the applied force $F(t)$ varies very rapidly and the inertia of the mass prevents it from following the fluctuating force.

EXAMPLE 9.4 Spring Support for Exhaust Fan

An exhaust fan, rotating at 1000 rpm, is to be supported by four springs, each having a stiffness of K. If only 10 percent of the unbalanced force of the fan is to be transmitted to the base, what should be the value of K? Assume the mass of the exhaust fan to be 40 kg.

Solution: Since the transmissibility has to be 0.1, we have, from Eq. (9.94),

$$0.1 = \left[\frac{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2} \right]^{1/2} \tag{E.1}$$

where the forcing frequency is given by

$$\omega = \frac{1000 \times 2\pi}{60} = 104.72 \text{ rad/s} \tag{E.2}$$

and the natural frequency of the system by

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} = \left(\frac{4K}{40}\right)^{1/2} = \frac{\sqrt{K}}{3.1623} \tag{E.3}$$

By assuming the damping ratio to be $\zeta = 0$, we obtain from Eq. (E.1),

$$0.1 = \frac{\pm 1}{\left\{1 - \left(\frac{104.72 \times 3.1623}{\sqrt{K}}\right)^2\right\}} \tag{E.4}$$

To avoid imaginary values, we need to consider the negative sign on the right-hand side of Eq. (E.4). This leads to

$$\frac{331.1561}{\sqrt{K}} = 3.3166$$

or

$$K = 9969.6365 \text{ N/m}$$

■

EXAMPLE 9.5 Design of an Undamped Isolator

A 50-kg mass is subjected to the harmonic force $F(t) = 1000 \cos 120t$ N. Design an undamped isolator so that the force transmitted to the base does not exceed 5% of the applied force. Also, find the displacement amplitude of the mass of the system with isolation.

Solution: By setting the value of force transmissibility as 0.05 and using $\zeta = 0$, Eq. (9.95) gives

$$r^2 \approx \frac{1 + T_f}{T_f} = \frac{1 + 0.05}{0.05} = 21 \tag{E.1}$$

Using the definition of r, along with the values of $m = 50$ kg and $\omega = 120$ rad/s, Eq. (E.1) yields

$$r^2 = \frac{\omega^2}{\omega_n^2} = \frac{\omega^2 m}{k}$$

or

$$k = \frac{\omega^2 m}{r^2} = \frac{(120^2)(50)}{21} = 34.2857 \times 10^3 \text{ N/m} \tag{E.2}$$

The displacement amplitude of the mass of the system with isolation can be found from Eq. (9.96), with $\zeta = 0$:

$$X = \frac{F_0}{k}\frac{1}{(r^2 - 1)} = \frac{1000}{34.2857 \times 10^3}\frac{1}{(21 - 1)} = 1.4583 \times 10^{-3}\ \text{m} \tag{E.3}$$

■

Design Chart for Isolation:

The force transmitted to the base or ground by a source of vibration (vibrating mass) is given by Eq. (9.94) and is shown in Fig. 9.21 as a graph between $T_f = F_T/F_0$ and $r = \omega/\omega_n$. As noted earlier, vibration isolation—reduction of the force transmitted to the ground—can be achieved for $r > \sqrt{2}$. In the region $r > \sqrt{2}$, low values of damping are desired for more effective isolation. For large values of r and low values of ζ, the term $(2\zeta r)^2$ becomes very small and can be neglected in Eq. (9.94) for simplicity. Thus Eq. (9.94) can be approximated as shown in Eq. (9.95) for $r > \sqrt{2}$ and ζ small.

By defining the natural frequency of vibration of the undamped system as

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}} \tag{9.98}$$

and the exciting frequency ω as

$$\omega = \frac{2\pi N}{60} \tag{9.99}$$

where δ_{st} is the static deflection of the spring and N is the frequency in cycles per minute or revolutions per minute (rpm) of rotating machines such as electric motors and turbines, Eqs. (9.95) to (9.99) can be combined to obtain

$$r = \frac{\omega}{\omega_n} = \frac{2\pi N}{60}\sqrt{\frac{\delta_{st}}{g}} = \sqrt{\frac{2 - R}{1 - R}} \tag{9.100}$$

where $R = 1 - T_f$ is used to indicate the quality of the isolator and denotes the percent reduction achieved in the transmitted force. Equation (9.100) can be rewritten as

$$N = \frac{30}{\pi}\sqrt{\frac{g}{\delta_{st}}\left(\frac{2 - R}{1 - R}\right)} = 29.9092\sqrt{\frac{2 - R}{\delta_{st}(1 - R)}} \tag{9.101}$$

Equation (9.101) can be used to generate a graph between log N and log δ_{st} as a series of straight lines for different values of R, as shown in Fig. 9.23. This graph serves as a design chart for selecting a suitable spring for the isolation.

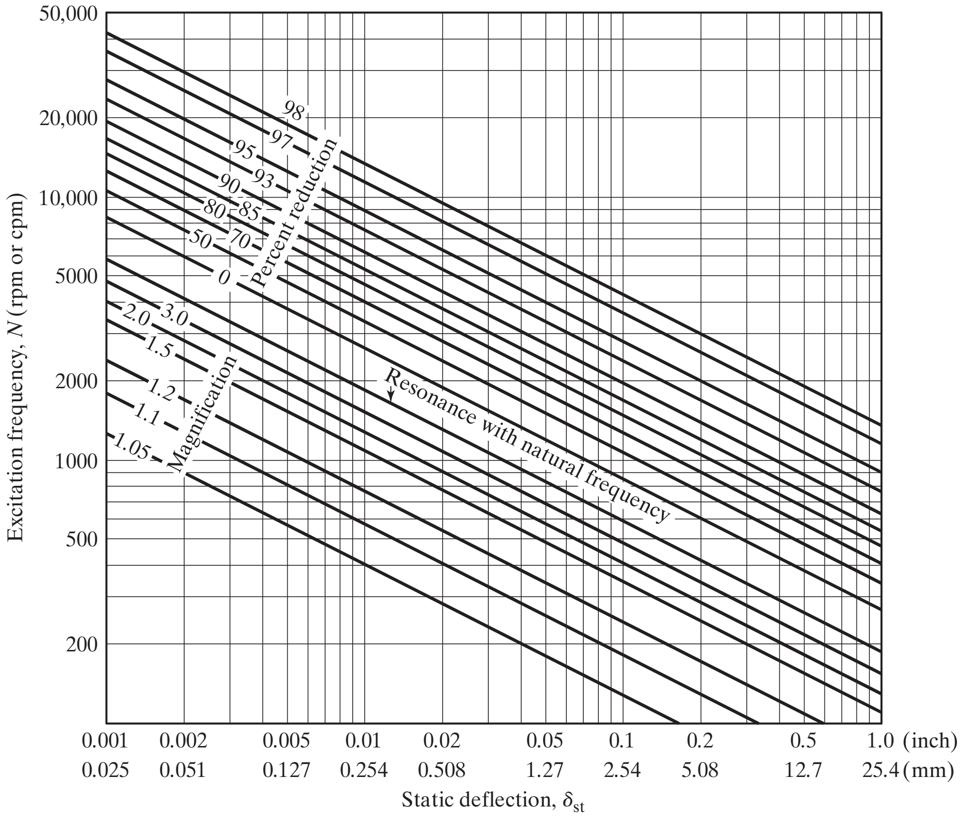


FIGURE 9.23 Isolation efficiency.

EXAMPLE 9.6

Isolator for Stereo Turntable

A stereo turntable, of mass 1 kg, generates an excitation force at a frequency of 3 Hz. If it is supported on a base through a rubber mount, determine the stiffness of the rubber mount to reduce the vibration transmitted to the base by 80 percent.

Solution: Using $N = 3 \times 60 = 180$ cpm and $R = 0.80$, Eq. (9.105) gives

$$180 = 29.9092\sqrt{\frac{2 - 0.80}{\delta_{st}(1 - 0.80)}}$$

or

$$\delta_{st} = 0.1657 \text{ m}$$

The static deflection of the rubber mount can be expressed, in terms of its stiffness (k), as

$$\delta_{st} = \frac{mg}{k}$$

which gives the stiffness of the rubber mount as

$$0.1657 = \frac{1(9.81)}{k} \qquad \text{or} \qquad k = 59.2179 \text{ N/m}$$

■

Isolation of systems with rotating unbalance:

A common source of forced harmonic force is imbalance in rotating machines such as turbines, centrifugal pumps, and turbogenerators. Imbalance in a rotating machine implies that the axis of rotation does not coincide with the center of mass of the whole system. Even a very small eccentricity can cause a large unbalanced force in high-speed machines such as turbines. A typical rotating system with an unbalance is shown in Fig. 9.24. Here the total mass of the system is assumed to be M and the unbalanced mass is considered as a point mass m located at the center of mass of the system (which has an eccentricity of e from the center of rotation) as shown in Fig. 9.24. If the unbalanced mass rotates at an angular velocity ω and the system is constrained to move in the vertical direction, the equation of motion of the system is given by

$$M\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \equiv me\omega^2 \sin \omega t \tag{9.102}$$

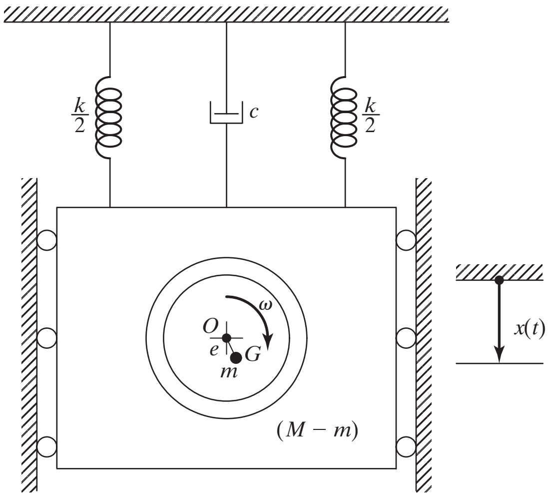


FIGURE 9.24 A system with rotating unbalance.

Using $F_0 = me\omega^2$, the force transmissibility of the system can be found from Eq. (9.88).

However, the presence of ω^2 in F_0 results in the following equation for the force transmissibility (T_f) due to rotating unbalance:

$$T_f = \frac{F_t}{F_0} = \frac{F_t}{me\omega^2} = \frac{F_t}{mer^2\omega_n^2}$$

or

$$\frac{F_t}{me\omega_n^2} = r^2\left\{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right\}^{\frac{1}{2}} \tag{9.103}$$

EXAMPLE 9.7 Centrifugal Pump with Rotating Unbalance—Rattle Space

A centrifugal pump, with a mass of 50 kg and rotational speed of 3000 rpm, is mounted in the middle of a simply supported beam of length 100 cm, width 20 cm, and thickness 0.5 cm. The damping ratio of the system (beam) can be assumed as $\zeta = 0.05$. The impeller (rotating part) of the pump has a mass of 5 kg with an eccentricity of 1 mm. If the maximum deflection of the beam is is constrained to be less than the available rattle space[6] of 3 mm. Determine whether the support system of the pump is adequate.

Solution: The bending stiffness or spring constant of the simply supported beam is given by

$$k = \frac{48EI}{l^3}$$

where the moment of inertia of the beam cross section can be computed as

$$I = \frac{1}{12}wt^3 = \frac{(20)(0.05^3)}{12} = 0.208333 \text{ cm}^4 = 20.8333 \times 10^{-10} \text{ m}^4$$

Using $E = 207 \times 10^9$ Pa, the spring constant of the beam can be found as

$$k = \frac{48(207 \times 10^9)(20.8333 \times 10^{-10})}{(1.0^3)} = 206{,}999.6688 \text{ N/m}$$

Using the density of steel as 7.85 gram/cm^3, the mass of the beam (m_b) can be determined as

$$m_b = 7.85(100)(20)(0.5) = 7850 \text{ gram} = 7.85 \text{ kg}$$

[6]The available clearance space that permits the system to undergo the induced deflection freely during vibration is called the *rattle space* or *clearance*. If the rattle space is too small to accommodate the deflection of the system, the system will undergo impacts (as it hits the surrounding or nearby surface or object) in each cycle of vibration.

The total mass of the system (M) is equal to the mass of the pump plus the effective mass of the beam at its center (equal to $\frac{17}{35}m_b$; see Problem 2.86):

$$M = m_{\text{pump}} + \frac{17}{35}m_b = 50 + \frac{17}{35}(7.85) = 53.8128 \text{ kg}$$

The natural frequency of the system is given by

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{206999.6688}{53.8128}} = 62.0215 \text{ rad/s}$$

The impeller (rotor) speed of 3000 rpm gives $\omega = 2\pi(3000)/60 = 314.16$ rad/s. Thus the frequency ratio (r) becomes

$$r = \frac{\omega}{\omega_n} = \frac{314.16}{62.0215} = 5.0653; \qquad r^2 = 25.6577$$

The amplitude of the forcing function is

$$me\omega^2 = 5(10^{-3})(314.16^2) = 493.4825 \text{ N}$$

Using $\zeta = 0.05$, the steady-state amplitude of the pump can be found from Eq. (9.96), using $me\omega^2$ for F_0, as

$$X = \frac{me\omega^2}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{493.4825}{206999.6688}\frac{1}{\sqrt{(1-25.6577)^2 + \{2(0.05)(5.0653)\}^2}}$$

$$= \frac{493.4825}{206999.6688}\frac{1}{24.6629} = 9.6662 \times 10^{-5}\ m$$

The static deflection of the beam under the weight of the pump can be determined as

$$\delta_{\text{pump}} = \frac{W_{\text{pump}}}{k} = \frac{(50)(9.81)}{206999.6688} = 236.9569 \times 10^{-5} \text{ m}$$

Thus the total deflection of the system is

$$\delta_{\text{total}} = X + \delta_{\text{pump}} = 9.662 \times 10^{-5} + 236.9569 \times 10^{-5} = 246.6231 \times 10^{-5} \text{ m} = 2.4662 \text{ mm}$$

This deflection is less than the rattle space of 3 mm. As such the support system of the pump is adequate. In case the value of δ_{total} exceeds the rattle space, we need to redesign (modify) the support system. This can be achieved by changing the spring constant (dimensions) of the beam and/or by introducing a damper.

■

9.10.2 Vibration Isolation System with Base Motion

In some applications, the base of the system is subjected to a vibratory motion. For example, the base or foundation of a machine such as a turbine in a power plant may be subjected to ground motion during an earthquake. In the absence of a suitably designed isolation system, the motion of the base transmitted to the mass (turbine) might cause damage and power failure. Similarly, a delicate instrument (mass) may have to be protected from a force or shock when the package containing the instrument is dropped from a height accidentally. Also, if the instrument is to be transported, the vehicle carrying it may experience vibration as it travels on a rough road with potholes. In this case, also, proper isolation is to be used to protect the instrument against excessive displacement or force transmitted from the base motion.

For a single-degree-of-freedom system with base excitation, shown in Fig. 9.19(b), the analysis was presented in Section 3.6. When the base of the system is subjected to a harmonic motion, $y(t) = Y \sin \omega t$, the equation of motion is given by Eq. (3.75):

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{9.104}$$

where $z = x - y$ denotes the displacement of the mass relative to the base. If the base motion is harmonic, then the motion of the mass will also be harmonic. Hence the displacement transmissibility, $T_d = \frac{X}{Y}$, is given by Eq. (3.68):

$$T_d = \frac{X}{Y} = \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{1/2} \tag{9.105}$$

where X and Y denote the displacement amplitudes of the mass and the base, respectively, and the right-hand-side expression can be identified to be the same as that in Eq. (9.94). Note that Eq. (9.105) is also equal to the ratio of the maximum steady-state accelerations of the mass and the base. The variation of the displacement transmissibility with the frequency ratio (r) for different values of the damping ratio (ζ) is shown in Fig. 9.25. The following observations can be made from Fig. 9.25:

1. For an undamped system, the displacement transmissibility approaches infinity at resonance ($r = 1$). Thus the undamped isolator (stiffness) is to be designed to ensure that the natural frequency of the system (ω_n) is away from the excitation frequency (ω).
2. For a damped system, the displacement transmissibility (and hence the displacement amplitude) attains a maximum for frequency ratios close to 1. The maximum displacement amplitude of the mass can be larger than the amplitude of base motion—that is, the base motion can get amplified by a large factor.
3. The displacement transmissibility is close to 1 for small values of the frequency ratio (r) and is exactly equal to 1 at $r = \sqrt{2}$.
4. The displacement amplitude is larger than 1 for $r < \sqrt{2}$ and smaller than 1 for $r > \sqrt{2}$. Note that a smaller damping ratio corresponds to a larger T_d for $r < \sqrt{2}$ and a smaller T_d for $r > \sqrt{2}$. Thus, if the damping of the system cannot be altered, the natural frequency of the system (stiffness) can be changed to achieve a value of $r > \sqrt{2}$.

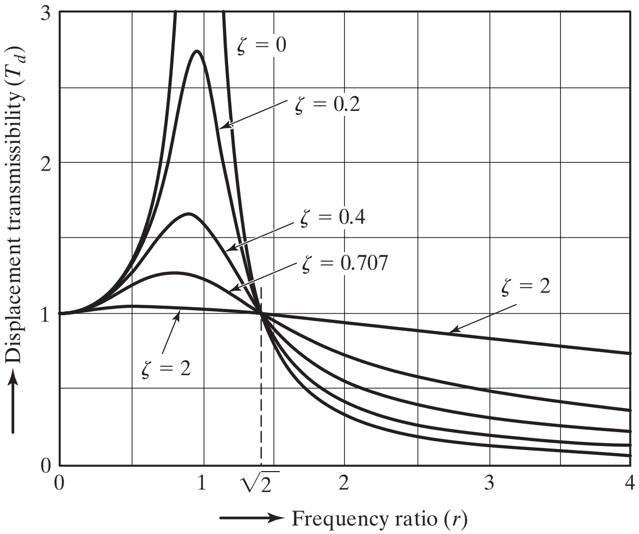


FIGURE 9.25 Variation of T_d with r (for base motion).

If F_t denotes the magnitude of the force transmitted to the mass by the spring and the damper, the force transmissibility (T_f) of the system is given by Eq. (3.74):

$$T_f = \frac{F_t}{kY} = r^2 \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}} \tag{9.106}$$

where kY is used to make the force transmissibility dimensionless. Note that once the displacement transmissibility, T_d, or the displacement amplitude of the mass (X) is computed using Eq. (9.105), the force transmitted to the mass, F_t, can be determined using the relation

$$\frac{F_t}{kY} = r^2 \frac{X}{Y} \qquad \text{or} \qquad F_t = kr^2 X \tag{9.107}$$

The variation of the force transmissibility with the frequency ratio (r) for different values of the damping ratio (ζ) is shown in Fig. 9.26. The following observations can be made from Fig. 9.26:

1. The force transmissibility (T_f) will be 2 at the frequency ratio $r = \sqrt{2}$ for all values of the damping ratio (ζ).
2. For $r > \sqrt{2}$, a lower damping ratio corresponds to a lower value of force transmissibility.
3. For $r > \sqrt{2}$, for any specific value of the damping ratio, the force transmissibility increases with r. This behavior is opposite to that of displacement transmissibility.
4. The force transmissibility is close to zero at small values of the frequency ratio r and attains a maximum at values of r close to 1.

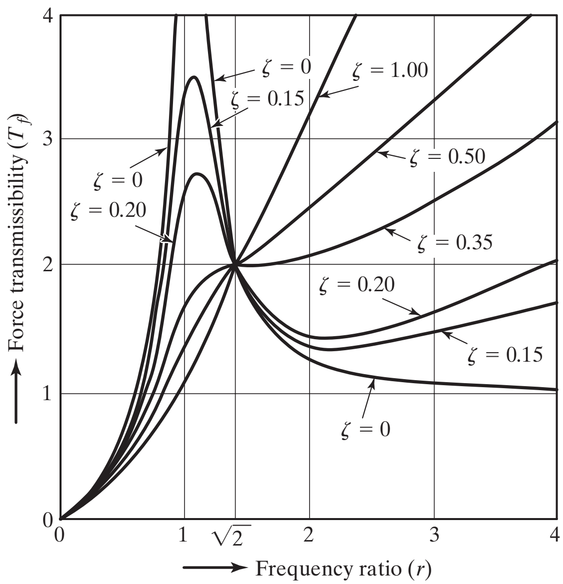


FIGURE 9.26 Variation of T_f with r (for base motion).

EXAMPLE 9.8

Isolation from Vibrating Base

A vibrating system is to be isolated from its vibrating base. Find the required damping ratio that must be achieved by the isolator to limit the displacement transmissibility to $T_d = 4$. Assume the system to have a single degree of freedom.

Solution: By setting $\omega = \omega_n$, Eq. (9.105) gives

$$T_d = \frac{\sqrt{1 + (2\zeta)^2}}{2\zeta}$$

or

$$\zeta = \frac{1}{2\sqrt{T_d^2 - 1}} = \frac{1}{2\sqrt{15}} = 0.1291$$

■

EXAMPLE 9.9

Design of Isolation for a Precision Machine with Base Motion

A precision machine used for the manufacture of integrated circuits, having a mass of 50 kg, is placed on a work bench (as base). The ground vibration transmitted by a nearby internal combustion engine causes the base (all four corners of the bench) to vibrate at a frequency of 1800 rpm. Helical springs, with a damping ratio of $\zeta = 0.01$ and a relationship of bilinear load (P) to deflection (x) given by

$$P = \begin{cases} 50{,}000x; & 0 \le x \le 8 \times 10^{-3} \\ 10^5 x - 4 \times 10^5; & 8 \times 10^{-3} \le x \le 13 \times 10^{-3} \end{cases} \tag{E.1}$$

(P in newtons and x in meters) are available for use as isolators at the four corners of the base. If no more than 10% of the vibration of the base is to be transmitted to the precision machine, determine a method of achieving the isolation.

Solution: Since the displacement transmissibility is required to be 0.1, Eq. (9.105), for $\zeta = 0.01$, gives

$$T_d = \frac{X}{Y} = 0.1 = \sqrt{\frac{1 + \{2(0.01)r\}^2}{(1 - r^2)^2 + \{2(0.01)r\}^2}} \tag{E.2}$$

The simplification of Eq. (E.2) yields a quadratic equation in r^2 as

$$r^4 - 2.0396r^2 - 99 = 0 \tag{E.3}$$

The solution of Eq. (E.3) gives

$$r^2 = 11.0218, \ -8.9822$$

which gives the positive value of r as 3.3199. Using the excitation frequency of

$$\omega = \frac{2\pi(1800)}{60} = 188.496 \text{ rad/s}$$

and the frequency ratio of $r = 3.3199$, the required natural frequency of the system can be determined as

$$r = 3.3199 = \frac{\omega}{\omega_n} = \frac{188.496}{\omega_n} \tag{E.4}$$

Equation (E.4) gives $\omega_n = 56.7776$ rad/s.

We assume that that one helical spring is installed at each corner of the base (under the four corners of the work bench). Because the expected deflection of the springs is unknown, the correct stiffness of the springs (out of the two possible values) is unknown. Hence we use the relation (see Eq. (2.28)):

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} \quad \text{or} \quad 56.7776 = \sqrt{\frac{9.81}{\delta_{st}}} \tag{E.5}$$

to find the static deflection of the system (δ_{st}) as

$$\delta_{st} = \frac{9.81}{56.7776^2} = 3.0431 \times 10^{-3} \text{ m}$$

Since all the four springs experience δ_{st}, the static load acting on each spring can be found from Eq. (E.1) as

$$P = 50000(3.0431 \times 10^{-3}) = 152.155 \text{ N}$$

The total load on the four springs is $4 \times 152.155 = 608.62$ N. Because the weight of the machine is $50\ g = 50(9.81) = 490.5$ N, in order to achieve the total load of 608.62 N, we need to add a weight of $609.62 - 490.50 = 118.12$ N to the system. This weight, in the form of a rectangular steel plate, is to be attached at the bottom of the machine, so that the total vibrating mass becomes 62.0408 kg (with a weight of 608.62 N).

■

EXAMPLE 9.10 Isolation System for a System with Base Motion

A printed circuit board (PCB) made of fiber-reinforced plastic composite material is used for the computer control of an automobile engine. It is attached to the chassis of the computer, which is fixed to the frame of the automobile as shown in Fig. 9.27(a). The frame of the automobile and the chassis of the computer are subject to vibration at the engine speed of 3000 rpm. If it is required to achieve a displacement transmissibility of no more than 10% at the PCB, design a suitable isolation system between the chassis of the computer and the frame of the automobile. Assume that the chassis of the computer is rigid with a mass of 0.25 kg.

Data of PCB: Length (l): 25 cm, width (w): 20 cm, thickness (t): 0.3 cm, mass per unit surface area: 0.005 kg/cm^2, Young's modulus (E): 15×10^9 N/m^2, damping ratio: 0.01.

Solution: The PCB is assumed to be fixed to the chassis of the computer as a cantilever beam. Its mass (m_{PCB}) is given by $25 \times 20 \times 0.005 = 2.5$ kg. The equivalent mass at the free end of the cantilever is m_b is (see Example 2.9):

$$m_b = \frac{33}{140} m_{PCB} = \frac{33}{140}(2.5) = 0.5893 \text{ kg}$$

Using the moment of inertia of the cross section of the PCB

$$I = \frac{1}{12} wt^3 = \frac{1}{12}(0.20)(0.003)^3 = 45 \times 10^{-8} \text{ m}^4$$

the stiffness of the PCB as a cantilever beam can be computed as

$$k_b = \frac{3EI}{l^3} = \frac{3(15 \times 10^9)(45 \times 10^{-8})}{(0.25)^3} = 1.296 \times 10^6 \text{ N/m}$$

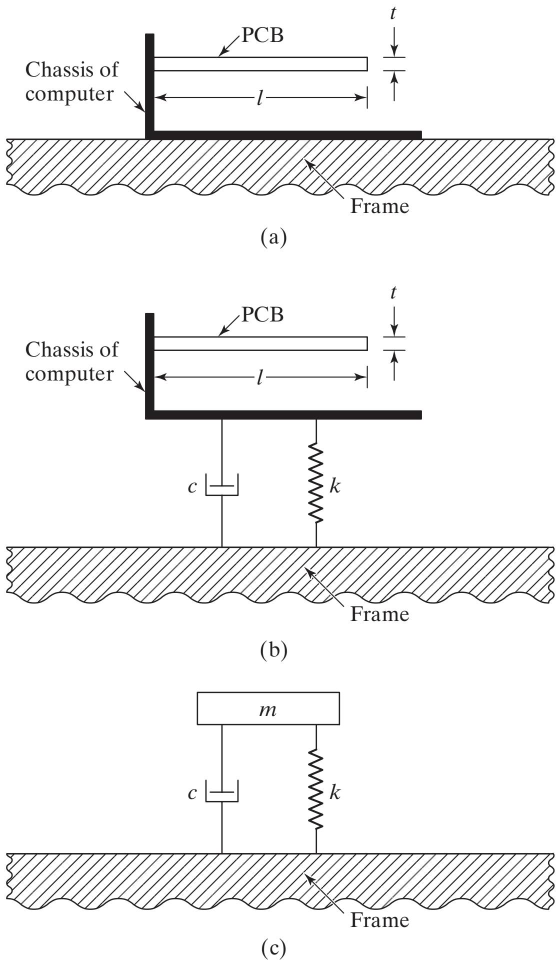


FIGURE 9.27

The natural frequency of the PCB is given by

$$\omega_n = \sqrt{\frac{k_b}{m_b}} = \sqrt{\frac{1.296 \times 10^6}{0.5893}} = 1482.99 \text{ rad/s}$$

The frequency of vibration of the base (chassis of the computer) is

$$\omega = \frac{2\pi(3000)}{60} = 312.66 \text{ rad/s}$$

The frequency ratio is given by

$$r = \frac{\omega}{\omega_n} = \frac{312.66}{1482.99} = 0.2108$$

Using the damping ratio $\zeta = 0.01$, the displacement transmissibility can be determined from Eq. (9.105):

$$T_d = \frac{X}{Y} = \left\{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1 + [2(0.01)(0.2108)]^2}{(1 - 0.2108^2)^2 + [2(0.01)(0.2108)]^2}\right\}$$

$$= 1.0465 \tag{E.1}$$

This value of $T_d = 104.65\%$ exceeds the maximum permissible value of 10%. Hence we design an isolator (with stiffness k and damping constant c) between the chassis of the computer and the frame of the automobile as shown in Fig. 9.27(b). If we model the PCB with stiffness k_b and mass m_b as before, the addition of the isolator makes the problem a two-degree-of-freedom system. For simplicity, we model the cantilever beam (PCB) as a rigid mass with no elasticity. This leads to the single-degree-of-freedom system shown in Fig. 9.27(c), where the equivalent mass m is given by

$$m = m_{\text{PCB}} + m_{\text{chassis}} = 2.5 + 0.25 = 2.75 \text{ kg}$$

Assuming a damping ratio of 0.01, for the required displacement transmissibility of 10%, the frequency ratio r can be determined from the relation

$$T_d = 0.1 = \left\{\frac{1 + [2(0.01)r]^2}{(1 - r^2)^2 + [2(0.01)r]^2}\right\}^{\frac{1}{2}} \tag{E.2}$$

By squaring both sides of Eq. (E.2) and rearranging the terms, we obtain

$$r^4 - 2.0396r^2 - 99 = 0 \tag{E.3}$$

The positive root of Eq. (E.3) is $r^2 = 11.0218$ or $r = 3.3199$. The stiffness of the isolator is given by

$$k = \frac{m\omega^2}{r^2} = \frac{(2.75)(312.66^2)}{11.0218} = 24{,}390.7309 \text{ N/m}$$

The damping constant of the isolator can be computed as

$$c = 2\zeta\sqrt{mk} = 2(0.01)\sqrt{(2.75)(24390.7309)} = 5.1797 \text{ N-s/m}$$

■

9.10.3 Vibration Isolation System with Flexible Foundation

In many practical situations, the structure or foundation to which the isolator is connected moves when the machine mounted on the isolator operates. For example, in the case of a turbine supported on the hull of a ship or an aircraft engine mounted on the wing of an airplane, the area surrounding the point of support also moves with the isolator. In such cases, the system can be represented as having two degrees of freedom. In Fig. 9.28, m_1 and m_2 denote the masses of the machine and the supporting structure that moves with the isolator, respectively. The isolator is represented by a spring k, and the damping is disregarded for the sake of simplicity. The equations of motion of the masses m_1 and m_2 are

$$\begin{aligned} m_1\ddot{x}_1 + k(x_1 - x_2) &= F_0 \cos \omega t \\ m_2\ddot{x}_2 + k(x_2 - x_1) &= 0 \end{aligned} \tag{9.108}$$

By assuming a harmonic solution of the form

$$x_j = X_j \cos \omega t, \qquad j = 1, 2$$

Eqs. (9.108) give

$$\left.\begin{aligned} X_1(k - m_1\omega^2) - X_2 k &= F_0 \\ -X_1 k + X_2(k - m_2\omega^2) &= 0 \end{aligned}\right\} \tag{9.109}$$

The natural frequencies of the system are given by the roots of the equation

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0 \tag{9.110}$$

The roots of Eq. (9.110) are given by

$$\omega_1^2 = 0, \qquad \omega_2^2 = \frac{(m_1 + m_2)k}{m_1 m_2} \tag{9.111}$$

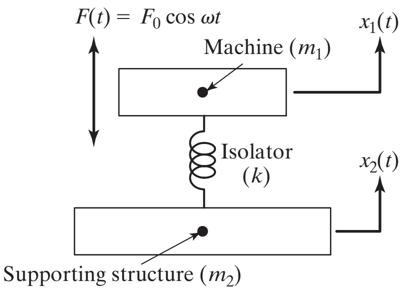


FIGURE 9.28 Machine with isolator on a flexible foundation.

The value $\omega_1 = 0$ corresponds to rigid-body motion, since the system is unconstrained. In the steady state, the amplitudes of m_1 and m_2 are governed by Eq. (9.109), whose solution yields

$$X_1 = \frac{(k - m_2\omega^2)F_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]}$$

$$X_2 = \frac{kF_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.112}$$

The force transmitted to the supporting structure (F_t) is given by the amplitude of $m_2\ddot{x}_2$:

$$F_t = -m_2\omega^2 X_2 = \frac{-m_2 k\omega^2 F_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.113}$$

The transmissibility of the isolator (T_f) is given by

$$T_f = \frac{F_t}{F_0}$$

$$= \frac{-m_2 k\omega^2}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]}$$

$$= \frac{1}{\left(\dfrac{m_1 + m_2}{m_2} - \dfrac{m_1\omega^2}{k}\right)}$$

$$= \frac{m_2}{(m_1 + m_2)}\left(\frac{1}{1 - \dfrac{\omega^2}{\omega_2^2}}\right) \tag{9.114}$$

where ω_2 is the natural frequency of the system given by Eq. (9.111). Equation (9.114) shows, as in the case of an isolator on a rigid base, that the force transmitted to the foundation becomes less as the natural frequency of the system ω_2 is reduced.

9.10.4 Vibration Isolation System with Partially Flexible Foundation

Figure 9.29 shows a more realistic situation in which the base of the isolator, instead of being completely rigid or completely flexible, is partially flexible [9.34]. We can define the mechanical impedance of the base structure, $Z(\omega)$, as the force at frequency ω required to produce a unit displacement of the base (as in Section 3.5):

$$Z(\omega) = \frac{\text{Applied force of frequency } \omega}{\text{Displacement}}$$

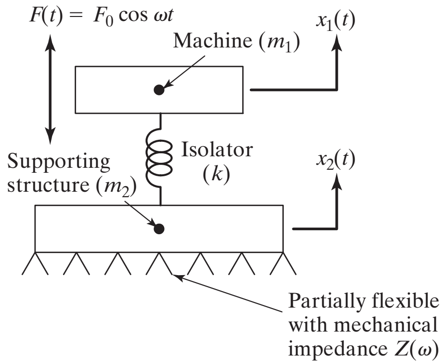


FIGURE 9.29 Machine with isolator on a partially flexible foundation.

The equations of motion are given by[7]

$$m_1 \ddot{x}_1 + k(x_1 - x_2) = F_0 \cos \omega t \tag{9.115}$$

$$k(x_2 - x_1) = -x_2 Z(\omega) \tag{9.116}$$

By substituting the harmonic solution

$$x_j(t) = X_j \cos \omega t, \qquad j = 1, 2 \tag{9.117}$$

into Eqs. (9.115) and (9.116), X_1 and X_2 can be obtained as in the previous case:

$$X_1 = \frac{[k + Z(\omega)]X_2}{k} = \frac{[k + Z(\omega)]F_0}{[Z(\omega)\,(k - m_1\omega^2) - km_1\omega^2]}$$

$$X_2 = \frac{kF_0}{[Z(\omega)(k - m_1\omega^2) - km_1\omega^2]} \tag{9.118}$$

The amplitude of the force transmitted is given by

$$F_t = X_2 Z(\omega) = \frac{kZ(\omega)F_0}{[Z(\omega)\,(k - m_1\omega^2) - km_1\omega^2]} \tag{9.119}$$

[7]If the base is completely flexible with an unconstrained mass of m_2, $Z(\omega) = -\omega^2 m_2$, and Eqs. (9.115) to (9.117) lead to Eq. (9.109).

and the transmissibility of the isolator by

$$T_f = \frac{F_t}{F_0} = \frac{kZ(\omega)}{[Z(\omega)(k - m_1\omega^2) - km_1\omega^2]} \tag{9.120}$$

In practice, the mechanical impedance $Z(\omega)$ depends on the nature of the base structure. It can be found experimentally by measuring the displacement produced by a vibrator that applies a harmonic force on the base structure. In some cases—for example, if an isolator is resting on a concrete raft on soil—the mechanical impedance at any frequency ω can be found in terms of the spring-mass-dashpot model of the soil.

9.10.5 Shock Isolation

As stated earlier, a shock load involves the application of a force for a short duration, usually for a period of less than one natural time period of the system. The forces involved in forge hammers, punch presses, blasts, and explosions are examples of shock loads. Shock isolation can be defined as a procedure by which the undesirable effects of shock are reduced. We noted that vibration isolation under a harmonic disturbance (input) occurs for the frequency ratio $r > \sqrt{2}$, with a smaller value of the damping ratio (ζ) leading to better isolation. On the other hand, shock isolation must occur over a wide range of frequencies, usually with large values of ζ. Thus a good vibration isolation design proves to be a poor shock isolation design and vice versa. In spite of the differences, the basic principles involved in shock isolation are similar to those of vibration isolation; however, the equations are different due to the transient nature of the shock.

A short-duration shock load $F(t)$, applied over a time period T, can be treated as an impulse $\boldsymbol{F}$:

$$\boldsymbol{F} = \int_0^T F(t)\, dt \tag{9.121}$$

Since this impulse acts on the mass m, the principle of impulse-momentum can be applied to find the velocity imparted to the mass (v) as

$$v = \frac{\boldsymbol{F}}{m} \tag{9.122}$$

This indicates that the application of a short-duration shock load can be considered as equivalent to giving an initial velocity to the system. Thus the response of the system under the shock load can be determined as the free-vibration solution with a specified initial velocity. By assuming the initial conditions as $x(0) = x_0 = 0$ and $\dot{x}(0) = \dot{x}_0 = v$, the free-vibration solution of a viscously damped single-degree-of-freedom system (displacement of the mass m) can be found from Eq. (2.72) as

$$x(t) = \frac{ve^{-\zeta\omega_n t}}{\omega_d} \sin \omega_d t \tag{9.123}$$

where $\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$ is the frequency of damped vibrations. The force transmitted to the foundation, $F_t(t)$, due to the spring and the damper is given by

$$F_t(t) = kx(t) + c\dot{x}(t) \tag{9.124}$$

Using Eq. (9.123), $F_t(t)$ can be expressed as

$$F_t(t) = \frac{v}{\omega_d}\sqrt{(k - c\zeta\omega_n)^2 + (c\omega_d)^2}\, e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{9.125}$$

where

$$\phi = \tan^{-1}\left(\frac{c\omega_d}{k - c\zeta\omega_n}\right) \tag{9.126}$$

Equations (9.125) and (9.126) can be used to find the maximum value of the force transmitted to the foundation.

For longer-duration shock loads, the maximum transmitted force can occur while the shock is being applied. In such cases, the shock spectrum, discussed in Section 4.6, can be used to find the maximum force transmitted to the foundation.

The following examples illustrate different approaches that can be used for the design of shock isolators.

EXAMPLE 9.11 Isolation Under Shock

An electronic instrument of mass 20 kg is subject to a shock in the form of a step velocity of 2 m/s. If the maximum allowable values of deflection (due to clearance limit) and acceleration are specified as 20 mm and 25 g, respectively, determine the spring constant of an undamped shock isolator.

Solution: The electronic instrument supported on the spring can be considered as an undamped system subject to base motion (in the form of step velocity). The mass vibrates at the natural frequency of the system with the magnitudes of velocity and acceleration given by

$$\dot{x}_{\max} = X\,\omega_n \tag{E.1}$$

$$\ddot{x}_{\max} = -X\,\omega_n^2 \tag{E.2}$$

where X is the amplitude of displacement of the mass. Since the maximum value of (step) velocity is specified as 2 m/s and the maximum allowable value of X is given to be 0.02 m, Eq. (E.1) yields

$$X = \frac{\dot{x}_{\max}}{\omega_n} < 0.02 \quad \text{or} \quad \omega_n > \frac{\dot{x}_{\max}}{X} = \frac{2}{0.02} = 100 \text{ rad/s} \tag{E.3}$$

Similarly, using the maximum specified value of $\ddot{x}_{max}$ as 25 g, Eq. (E.2) gives

$$X\omega_n^2 \le 25\,(9.81) = 245.25 \text{ m/s}^2 \qquad \text{or} \qquad \omega_n \le \sqrt{\frac{\ddot{x}_{max}}{X}} = \sqrt{\frac{245.25}{0.02}} = 110.7362 \text{ rad/s} \qquad \text{(E.4)}$$

Equations (E.3) and (E.4) give 100 rad/s $\le \omega_n \le$ 110.7362 rad/s. By selecting the value of ω_n in the middle of the permissible range as 105.3681 rad/s, the stiffness of the spring (isolator) can be found as

$$k = m\omega_n^2 = 20\,(105.3681)^2 = 2.2205 \times 10^5 \text{ N/m} \qquad \text{(E.5)}$$

■

EXAMPLE 9.12 Isolation Under Step Load

A sensitive electronic instrument of mass 100 kg is supported on springs and packaged for shipment. During shipping, the package is dropped from a height that effectively applied a shock load of intensity F_0 to the instrument, as shown in Fig. 9.30(a). Determine the stiffness of the springs used in the package if the maximum deflection of the instrument is required to be less than 2 mm. The response spectrum of the shock load is shown in Fig. 9.30(b) with $F_0 = 1000$ N and $t_0 = 0.1$ s.

Solution: The response spectrum, indicating the maximum response of an undamped single-degree-of-freedom system subject to the given shock, is given by

$$\frac{x_{max}k}{F_0} = 1 + \frac{1}{\omega_n t_0}\sqrt{2\,(1 - \cos 2\omega_n t_0)} \qquad \text{(E.1)}$$

where ω_n is the natural frequency of the system:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{100}} = 0.1\sqrt{k} \qquad \text{(E.2)}$$

$F_0 = 1000$ N, $t_0 = 0.1$ s, and k is the stiffness of the springs used in the package. Using the known data, Eq. (E.1) can be expressed as

$$\frac{x_{max}k}{1000} = 1 + \frac{1}{0.1\sqrt{k}\,(0.1)}\sqrt{2\,(1 - \cos 2(0.1\sqrt{k})\,(0.1))} \le \frac{2}{1000}\left(\frac{k}{1000}\right) \qquad \text{(E.3)}$$

By using the equality sign, Eq. (E.3) can be rearranged as

$$\frac{100}{\sqrt{k}}\sqrt{2(1 - \cos 0.02\sqrt{k})} - 2 \times 10^{-6}k + 1 = 0 \qquad \text{(E.4)}$$

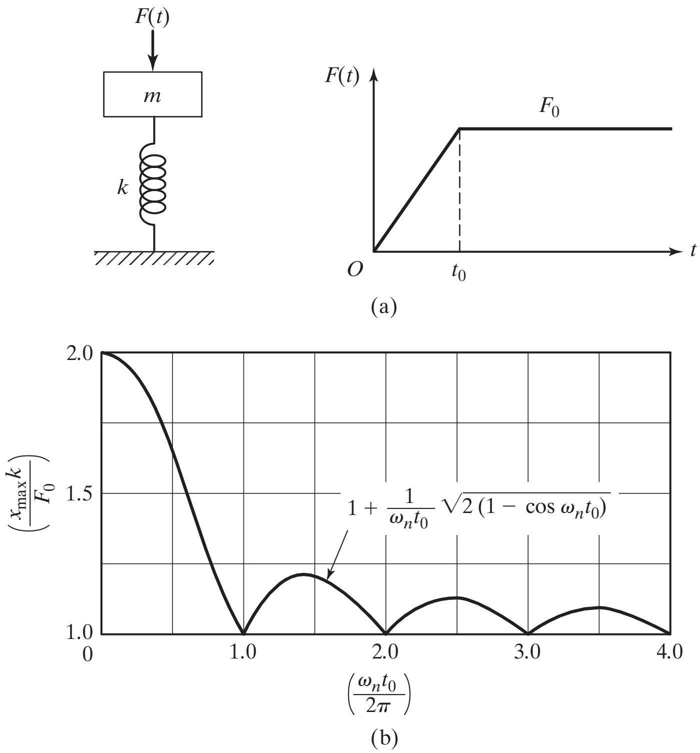


FIGURE 9.30 Shock load on electronic instrument.

The root of Eq. (E.4) gives the desired stiffness value as $k = 6.2615 \times 10^5$ N/m. The following MATLAB program can be used to find the root of Eq. (E.4):

```
>> x=1000:1:10000000;
>> f='(100/sqrt(x))*sqrt(2*(1-cos(0.02*sqrt(x))))-0.000002*x+1';
>> root=fzero(f,100000)

root =

 6.2615e+005

>>
```

■

9.10.6 Active Vibration Control

A vibration isolation system is called active if it uses external power to perform its function. It consists of a servomechanism with a sensor, signal processor, and an actuator, as shown schematically in Fig. 9.31 [9.31–9.33]. This system maintains a constant distance (l) between the vibrating mass and the reference plane. As the force $F(t)$ applied to the system (mass) varies, the distance l tends to vary. This change in l is sensed by the sensor and a signal, proportional to the magnitude of the excitation (or response) of the vibrating

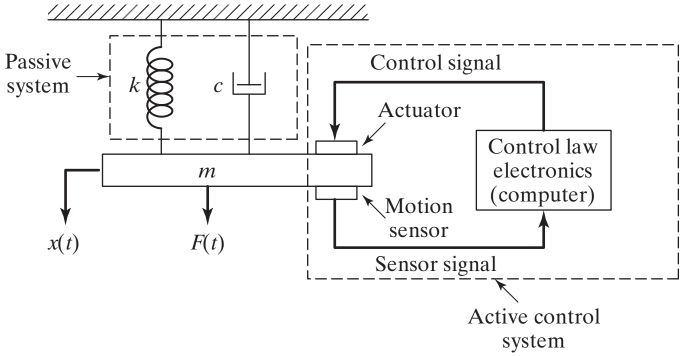


FIGURE 9.31 Active vibration isolation system.

body, is produced. The signal processor produces a command signal to the actuator based on the sensor signal it receives. The actuator develops a motion or force proportional to the command signal. The actuator motion or force will control the base displacement such that the distance l is maintained at the desired constant value.

Different types of sensors are available to create feedback signals based on the displacement, velocity, acceleration, jerk, or force. The signal processor may consist of a passive mechanism, such as a mechanical linkage, or an active electronic or fluidic network that can perform functions such as addition, integration, differentiation, attenuation, or amplification. The actuator may be a mechanical system such as a rack-and-pinion or ball screw mechanism, a fluidic system, or piezoelectric and electromagnetic force generating system. Depending on the types of sensor, signal processor, and actuator used, an active vibration control system can be called *electromechanical*, *electrofluidic*, *electromagnetic*, *piezoelectric*, or *fluidic*.

Analysis: Consider a single-degree-of-freedom system in which the mass m is subjected to an applied force $f(t)$ as shown in Fig. 9.31. If we use an active control system to control the vibration of the mass m, the actuator will be designed to exert a control force $f_c(t)$ so that the equation of motion of the system becomes

$$m\ddot{x} + c\dot{x} + kx = F(t) = f(t) + f_c(t) \tag{9.127}$$

Most commonly, the sensor (computer) measures the displacement x and the velocity $\dot{x}$ of the mass in real time (continuously). The computer computes the control force $f_c(t)$ necessary to control the motion and commands the actuator to exert the force $f_c(t)$ on the mass m.

Usually the computer is programmed to generate the control force proportional to the displacement $x(t)$ and the displacement derivative or velocity $\dot{x}(t)$ of the mass so that

$$f_c(t) = -g_p x - g_d \dot{x} \tag{9.128}$$

where g_p and g_d are constants whose values are to be determined and programmed into the computer by the designer. The constants g_p and g_d are known as control gains, with g_p denoting the proportional gain and g_d indicating the derivative or rate gain. The control algorithm in this case is known as the proportional and derivative (PD) control. By substituting Eq. (9.128) into Eq. (9.127), we obtain

$$m\ddot{x} + (c + g_d)\dot{x} + (k + g_p)x = f(t) \tag{9.129}$$

which shows that g_d acts like additional (or artificial) damping and g_p like additional (or artificial) stiffness. Equation (9.129), known as the closed-loop equation, can be solved to find the response characteristics of the system. For example, the new (effective) natural frequency is given by

$$\omega_n = \left(\frac{k + g_p}{m}\right)^{\frac{1}{2}} \tag{9.130}$$

and the new (effective) damping ratio by

$$\zeta = \frac{c + g_d}{2\sqrt{m(k + g_p)}} \tag{9.131}$$

The new (effective) time constant of the system, for $\zeta \leq 1$, is given by

$$\tau = \frac{2m}{c + g_d} \tag{9.132}$$

Thus the functioning of the active vibration control system can be described as follows: Given the values of m, c, and k, compute the control gains g_p and g_d to achieve the desired values of ω_n, ζ, or τ. In practice, the response of the system is continuously monitored, the computations are done, and the actuator is made to apply the control force f_c to the mass in real time so that the response of the system lies within the stated limits. Note that the gains g_p and g_d can be positive or negative depending on the measured and desired responses.

EXAMPLE 9.13 Vibration Control of a Precision Electronic System

It is proposed to control the vibration of a precision electronic system supported on an elastic pad (with no damping) by either a passive or an active method. The system has a mass of 15 kg and a natural frequency of 20 rad/s. It is estimated that the system requires a damping ratio of $\zeta = 0.85$ to control the vibration. Assume that the available dashpots can provide damping constants only in the range $0 \leq c \leq 400$ N-s/m.

Solution: First, we investigate the use of an available dashpot to control the vibration (passive control). From the known natural frequency of the system, we can find the stiffness of the elastic pad as

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{or} \quad k = m\omega_n^2 = 15(20)^2 = 6000 \text{ N/m} \tag{E.1}$$

The required damping ratio of the system gives the necessary damping constant (c) as

$$\zeta = \frac{c}{2\sqrt{km}} = 0.85 \quad \text{or} \quad c = 2\zeta\sqrt{km} = 2(0.85)\sqrt{6000(15)} = 510 \text{ N-s/m} \tag{E.2}$$

Since the available dashpots can provide damping constant values up to 400 N-s/m only, we cannot achieve the desired control using passive damping.

Thus we consider an active control system to create the required amount of damping into the system. Let the control force be of the form $f_c = -g_d\dot{x}$, so that the damping ratio, alternate form of Eq. (9.131), can be expressed (with $g_p = 0$):

$$2\zeta\omega_n = \frac{c + g_d}{m} \tag{E.3}$$

By adding the available dashpot, with a damping constant of 400 N-s/m, Eq. (E.3) can be rewritten as

$$400 + g_d = 2m\zeta\omega_n = 2(15)(0.85)(20) = 510 \text{ N-s/m}$$

or

$$g_d = 110 \text{ N-s/m}$$

This gives the value of the damping constant to be provided by the active control (also known as derivative gain) as $g_d = 110$ N-s/m.

■

EXAMPLE 9.14 Active Control of a System with Rotating Unbalance

A single-degree-of-freedom system consists of a mass (m) = 150 kg, damping constant (c) = 4000 N-s/m, and stiffness (k) = 6×10^6 N/m. The mass is subjected to a rotating unbalanced force given by $f(t) = 100 \sin 60\pi t$ N. The following observations can be made from the given data:

(i) The natural frequency of the system, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{6(10^6)}{150}} = 200$ rad/s, is close to the frequency of the disturbance, $\omega = 60\pi = 188.4955$ rad/s.

(ii) The damping ratio of the system is small with a value of

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{4000}{2\sqrt{[6(10^6)(160)]}} = 0.06667$$

It is desired to change the natural frequency of the system to 100 rad/s and the damping ratio to 0.5. Because the values of k and c of the system cannot be altered, it is proposed to use an active control system. Determine the control gains required to achieve the desired values of ω_n and ζ. Also find the magnitude of the response and the actuator force of the system in the steady state.

Solution: When an active control system is used with control gains g_p and g_d, the natural frequency of the system can be expressed as

$$\omega_n = 100 = \sqrt{\frac{6(10^6) + g_p}{150}}$$

or

$$g_p = 150(10^4) - 6(10^6) = -4.5(10^6) \text{ N/m}$$

This implies that the stiffness of the system is to be reduced to 1.5×10^6 N/m. The new damping ratio of the system is given by

$$\zeta = 0.5 = \frac{c + g_d}{2\sqrt{km}} = \frac{4000 + g_d}{2\sqrt{[1.5(10^6)](150)}}$$

or

$$g_d = 15000 - 4000 = 11000 \text{ N-s/m}$$

This implies that the damping of the system is to be increased to 15000 N-s/m.

The equation of motion of the actively controlled system can be written as

$$m\ddot{x} + c\dot{x} + kx = f(t) = f_0 \sin \omega t \tag{E.1}$$

which, in this case, takes the form

$$150\ddot{x} + 15000\dot{x} + 1.5(10^6)x = f(t) = 100 \sin 60\pi t \tag{E.2}$$

From Eq. (E.1), the general transfer function of the system can be expressed as (see Section 3.12)

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \tag{E.3}$$

The magnitude of the steady-state response of the system corresponding to Eq. (E.3) is given by (see Section 3.13)

$$X = \frac{f_0}{\left[(k - m\omega^2)^2 + (c\omega)^2\right]^{\frac{1}{2}}} \tag{E.4}$$

In the present case, $f_0 = 100$ N, $m = 150$ kg, $c = 15000$ N-s/m, $k = 1.5 \times 10^6$ N-s/m, and $\omega = 188.4955$ rad/s. Thus Eq. (E.4) gives

$$X = \frac{150}{\left[\{1.5(10^6) - 150(188.4955)^2\}^2 + \{15000(188.4955)\}^2\right]^{\frac{1}{2}}}$$

$$= \frac{150}{4.7602(10^6)}$$

$$= 31.5113(10^{-6}) \text{ N}$$

The actuator (control) force, F_t, at steady state can be obtained from the relation

$$\frac{F_t(s)}{F(s)} = \frac{F_t(s)}{X(s)}\frac{X(s)}{F(s)} = \frac{k + cs}{ms^2 + cs + k} \tag{E.5}$$

as

$$F_t(i\omega) = |4.5(10^6) - 11000i\omega|X(i\omega)$$

$$= |4.5(10^6) - 11000(188.4955)i|(31.5113(10^{-6}))$$

$$= \sqrt{\{4.5(10^6)\}^2 + \{11000(188.4955)\}^2}\,(31.5113(10^{-6}))$$

$$= 156.1289 \text{ N}$$

■

9.11 Vibration Absorbers

The *vibration absorber*, also called *dynamic vibration absorber*, is a mechanical device used to reduce or eliminate unwanted vibration. It consists of another mass and stiffness attached to the main (or original) mass that needs to be protected from vibration. Thus the main mass and the attached absorber mass constitute a two-degree-of-freedom system, hence the vibration absorber will have two natural frequencies. The vibration absorber is commonly used in machinery that operates at constant speed, because the vibration absorber is tuned to one particular frequency and is effective only over a narrow band of frequencies. Common applications of the vibration absorber include reciprocating tools, such as sanders, saws, and compactors, and large reciprocating internal combustion engines which run at constant speed (for minimum fuel consumption). In these systems, the vibration absorber helps balance the reciprocating forces. Without a vibration absorber, the unbalanced reciprocating forces might make the device impossible to hold or control. Vibration absorbers are also used on high-voltage transmission lines. In

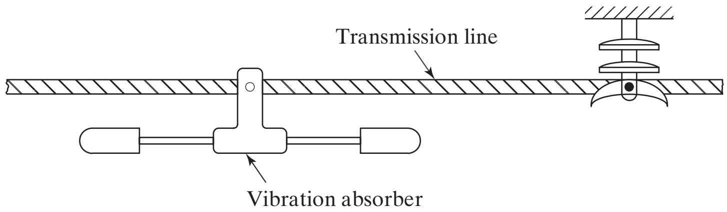


FIGURE 9.32

this case, the dynamic vibration absorbers, in the form of dumbbell-shaped devices (Fig. 9.32), are hung from transmission lines to mitigate the fatigue effects of wind-induced vibration.

A machine or system may experience excessive vibration if it is acted upon by a force whose excitation frequency nearly coincides with a natural frequency of the machine or system. In such cases, the vibration of the machine or system can be reduced by using a *vibration neutralizer* or *dynamic vibration absorber,* which is simply another spring-mass system. The dynamic vibration absorber is designed such that the natural frequencies of the resulting system are away from the excitation frequency. We shall consider the analysis of a dynamic vibration absorber by idealizing the machine as a single-degree-of-freedom system.

9.11.1 Undamped Dynamic Vibration Absorber

When we attach an auxiliary mass m_2 to a machine of mass m_1 through a spring of stiffness k_2, the resulting two-degree-of-freedom system will look as shown in Fig. 9.33. The equations of motion of the masses m_1 and m_2 are

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F_0 \sin \omega t$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \tag{9.133}$$

By assuming harmonic solution,

$$x_j(t) = X_j \sin \omega t, \qquad j = 1, 2 \tag{9.134}$$

we can obtain the steady-state amplitudes of the masses m_1 and m_2 as

$$X_1 = \frac{(k_2 - m_2\omega^2)F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2} \tag{9.135}$$

$$X_2 = \frac{k_2F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2} \tag{9.136}$$

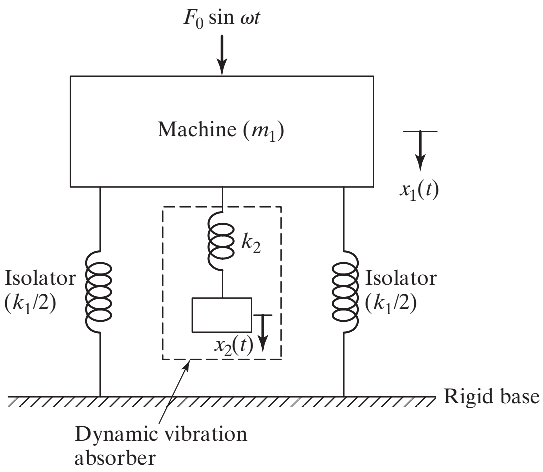


FIGURE 9.33 Undamped dynamic vibration absorber.

We are primarily interested in reducing the amplitude of the machine (X_1). In order to make the amplitude of m_1 zero, the numerator of Eq. (9.135) should be set equal to zero. This gives

$$\omega^2 = \frac{k_2}{m_2} \tag{9.137}$$

If the machine, before the addition of the dynamic vibration absorber, operates near its resonance, $\omega^2 \simeq \omega_1^2 = k_1/m_1$. Thus if the absorber is designed such that

$$\omega^2 = \frac{k_2}{m_2} = \frac{k_1}{m_1} \tag{9.138}$$

the amplitude of vibration of the machine, while operating at its original resonant frequency, will be zero. By defining

$$\delta_{st} = \frac{F_0}{k_1}, \qquad \omega_1 = \left(\frac{k_1}{m_1}\right)^{1/2}$$

as the natural frequency of the machine or main system, and

$$\omega_2 = \left(\frac{k_2}{m_2}\right)^{1/2} \tag{9.139}$$

as the natural frequency of the absorber or auxiliary system, Eqs. (9.135) and (9.136) can be rewritten as

$$\frac{X_1}{\delta_{st}} = \frac{1 - \left(\frac{\omega}{\omega_2}\right)^2}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right] - \frac{k_2}{k_1}} \tag{9.140}$$

$$\frac{X_2}{\delta_{st}} = \frac{1}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right] - \frac{k_2}{k_1}} \tag{9.141}$$

Figure 9.34 shows the variation of the amplitude of vibration of the machine (X_1/δ_{st}) with the machine speed (ω/ω_1). The two peaks correspond to the two natural frequencies of the composite system. As seen before, $X_1 = 0$ at $\omega = \omega_1$. At this frequency, Eq. (9.141) gives

$$X_2 = -\frac{k_1}{k_2}\delta_{st} = -\frac{F_0}{k_2} \tag{9.142}$$

This shows that the force exerted by the auxiliary spring is opposite to the impressed force $(k_2 X_2 = -F_0)$ and neutralizes it, thus reducing X_1 to zero. The size of the dynamic vibration absorber can be found from Eqs. (9.142) and (9.138):

$$k_2 X_2 = m_2 \omega^2 X_2 = -F_0 \tag{9.143}$$

Thus the values of k_2 and m_2 depend on the allowable value of X_2.

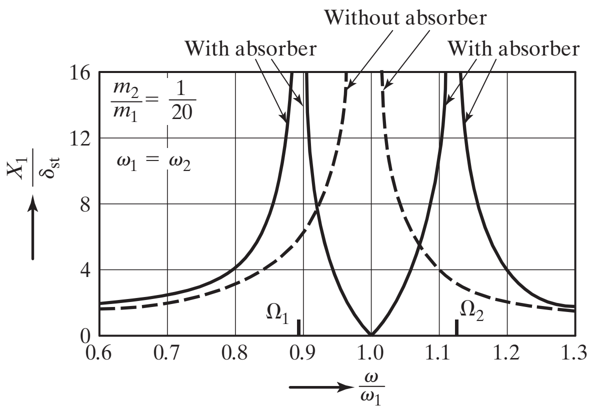


FIGURE 9.34 Effect of undamped vibration absorber on the response of machine.

It can be seen from Fig. 9.34 that the dynamic vibration absorber, while eliminating vibration at the known impressed frequency ω, introduces two resonant frequencies Ω_1 and Ω_2, at which the amplitude of the machine is infinite. In practice, the operating frequency ω must therefore be kept away from the frequencies Ω_1 and Ω_2. The values of Ω_1 and Ω_2 can be found by equating the denominator of Eq. (9.134) to zero. Noting that

$$\frac{k_2}{k_1} = \frac{k_2}{m_2}\frac{m_2}{m_1}\frac{m_1}{k_1} = \frac{m_2}{m_1}\left(\frac{\omega_2}{\omega_1}\right)^2 \tag{9.144}$$

and setting the denominator of Eq. (9.140) to zero leads to

$$\left(\frac{\omega}{\omega_2}\right)^4\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_2}\right)^2\left[1 + \left(1 + \frac{m_2}{m_1}\right)\left(\frac{\omega_2}{\omega_1}\right)^2\right] + 1 = 0 \tag{9.145}$$

The two roots of this equation are given by

$$\left.\begin{array}{l}\left(\dfrac{\Omega_1}{\omega_2}\right)^2\\[2ex] \left(\dfrac{\Omega_2}{\omega_2}\right)^2\end{array}\right\} = \frac{\left\{\left[1 + \left(1 + \dfrac{m_2}{m_1}\right)\left(\dfrac{\omega_2}{\omega_1}\right)^2\right] \mp \left\{\left[1 + \left(1 + \dfrac{m_2}{m_1}\right)\left(\dfrac{\omega_2}{\omega_1}\right)^2\right]^2 - 4\left(\dfrac{\omega_2}{\omega_1}\right)^2\right\}^{1/2}\right\}}{2\left(\dfrac{\omega_2}{\omega_1}\right)^2} \tag{9.146}$$

which can be seen to be functions of (m_2/m_1) and (ω_2/ω_1).

Notes

1. It can be seen, from Eq. (9.146), that Ω_1 is less than and Ω_2 is greater than the operating speed (which is equal to the natural frequency, ω_1) of the machine. Thus the machine must pass through Ω_1 during start-up and stopping. This results in large amplitudes.
2. Since the dynamic absorber is tuned to one excitation frequency (ω), the steady-state amplitude of the machine is zero only at that frequency. If the machine operates at other frequencies or if the force acting on the machine has several frequencies, then the amplitude of vibration of the machine may become large.
3. The variations of Ω_1/ω_2 and Ω_2/ω_2 as functions of the mass ratio m_2/m_1 are shown in Fig. 9.35 for three different values of the frequency ratio ω_2/ω_1. It can be seen that the difference between Ω_1 and Ω_2 increases with increasing values of m_2/m_1.

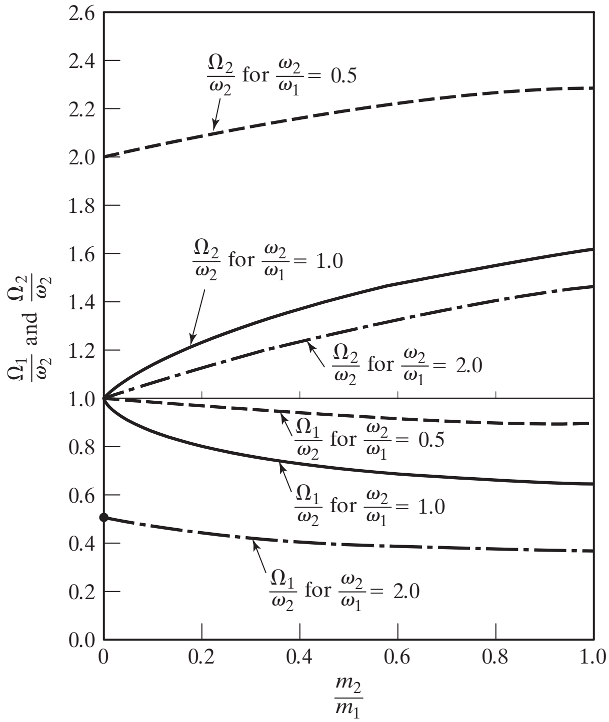


FIGURE 9.35 Variations of Ω_1 and Ω_2 given by Eq. (9.146).

EXAMPLE 9.15

Vibration Absorber for Diesel Engine

A diesel engine, weighing 3000 N, is supported on a pedestal mount. It has been observed that the engine induces vibration into the surrounding area through its pedestal mount at an operating speed of 6000 rpm. Determine the parameters of the vibration absorber that will reduce the vibration when mounted on the pedestal. The magnitude of the exciting force is 250 N, and the amplitude of motion of the auxiliary mass is to be limited to 2 mm.

Solution: The frequency of vibration of the machine is

$$f = \frac{6000}{60} = 100 \text{ Hz} \qquad \text{or} \qquad \omega = 628.32 \text{ rad/s}$$

Since the motion of the pedestal is to be made equal to zero, the amplitude of motion of the auxiliary mass should be equal and opposite to that of the exciting force. Thus from Eq. (9.143), we obtain

$$|F_0| = m_2 \omega^2 X_2 \tag{E.1}$$

Substitution of the given data yields

$$250 = m_2 (628.32)^2 (0.002)$$

Therefore $m_2 = 0.31665$ kg. The spring stiffness k_2 can be determined from Eq. (9.138):

$$\omega^2 = \frac{k_2}{m_2}$$

Therefore, $k_2 = (628.32)^2 (0.31665) = 125009$ N/m.

■

EXAMPLE 9.16 Absorber for Motor-Generator Set

A motor-generator set, shown in Fig. 9.36, is designed to operate in the speed range of 2000 to 4000 rpm. However, the set is found to vibrate violently at a speed of 3000 rpm due to a slight unbalance in the rotor. It is proposed to attach a cantilever mounted lumped-mass absorber system to eliminate the problem. When a cantilever carrying a trial mass of 2 kg tuned to 3000 rpm is attached to the set, the resulting natural frequencies of the system are found to be 2500 rpm and 3500 rpm. Design the absorber to be attached (by specifying its mass and stiffness) so that the natural frequencies of the total system fall outside the operating-speed range of the motor-generator set.

Solution: The natural frequencies ω_1 of the motor-generator set and ω_2 of the absorber are given by

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}, \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}} \tag{E.1}$$

The resonant frequencies Ω_1 and Ω_2 of the combined system are given by Eq. (9.146). Since the absorber $(m = 2 \text{ kg})$ is tuned, $\omega_1 = \omega_2 = 314.16$ rad/s (corresponding to 3000 rpm). Using the notation

$$\mu = \frac{m_2}{m_1}, \qquad r_1 = \frac{\Omega_1}{\omega_2}, \qquad \text{and} \qquad r_2 = \frac{\Omega_2}{\omega_2}$$

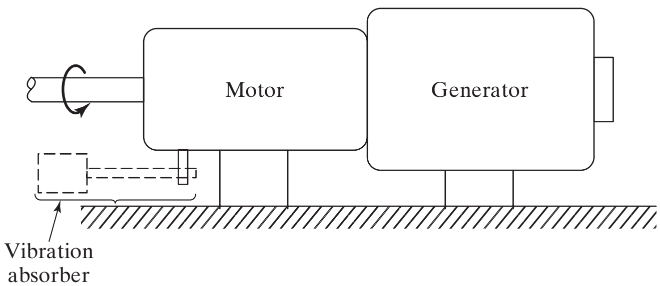


FIGURE 9.36 Motor-generator set.

Eq. (9.146) becomes

$$r_1^2, r_2^2 = \left(1 + \frac{\mu}{2}\right) \mp \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \tag{E.2}$$

Since Ω_1 and Ω_2 are known to be 261.80 rad/s (or 2500 rpm) and 366.52 rad/s (or 3500 rpm), respectively, we find that

$$r_1 = \frac{\Omega_1}{\omega_2} = \frac{261.80}{314.16} = 0.8333$$

$$r_2 = \frac{\Omega_2}{\omega_2} = \frac{366.52}{314.16} = 1.1667$$

Hence

$$r_1^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1}$$

or

$$\mu = \left(\frac{r_1^4 + 1}{r_1^2}\right) - 2 \tag{E.3}$$

Since $r_1 = 0.8333$, Eq. (E.3) gives $\mu = m_2/m_1 = 0.1345$ and $m_1 = m_2/0.1345 = 14.8699$ kg. The specified lower limit of Ω_1 is 2000 rpm or 209.44 rad/s, and so

$$r_1 = \frac{\Omega_1}{\omega_2} = \frac{209.44}{314.16} = 0.6667$$

With this value of r_1, Eq. (E.3) gives $\mu = m_2/m_1 = 0.6942$ and $m_2 = m_1(0.6942) = 10.3227$ kg. With these values, the second resonant frequency can be found from

$$r_2^2 = \left(1 + \frac{\mu}{2}\right) + \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} = 2.2497$$

which gives $\Omega_2 \simeq 4499.4$ rpm, larger than the specified upper limit of 4000 rpm. The spring stiffness of the absorber is given by

$$k_2 = \omega_2^2 m_2 = (314.16)^2(10.3227) = 1.0188 \times 10^6 \text{ N/m}$$

■

9.11.2 Damped Dynamic Vibration Absorber

The dynamic vibration absorber described in the previous section removes the original resonance peak in the response curve of the machine but introduces two new peaks. Thus the machine experiences large amplitudes as it passes through the first peak during start-up and stopping. The amplitude of the machine can be reduced by adding a damped vibration absorber, as shown in Fig. 9.37. The equations of motion of the two masses are given by

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) = F_0 \sin \omega t \tag{9.147}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) = 0 \tag{9.148}$$

By assuming the solution to be

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1, 2 \tag{9.149}$$

the steady-state solution of Eqs. (9.147) and (9.148) can be obtained:

$$X_1 = \frac{F_0(k_2 - m_2\omega^2 + ic_2\omega)}{[(k_1 - m_1\omega^2)(k_2 - m_2\omega^2) - m_2k_2\omega^2] + i\omega c_2(k_1 - m_1\omega^2 - m_2\omega^2)} \tag{9.150}$$

$$X_2 = \frac{X_1(k_2 + i\omega c_2)}{(k_2 - m_2\omega^2 + i\omega c_2)} \tag{9.151}$$

By defining

$$\mu = m_2/m_1 = \text{Mass ratio} = \text{Absorber mass/main mass}$$
$$\delta_{st} = F_0/k_1 = \text{Static deflection of the system}$$
$$\omega_a^2 = k_2/m_2 = \text{Square of natural frequency of the absorber}$$
$$\omega_n^2 = k_1/m_1 = \text{Square of natural frequency of main mass}$$
$$f = \omega_a/\omega_n = \text{Ratio of natural frequencies}$$

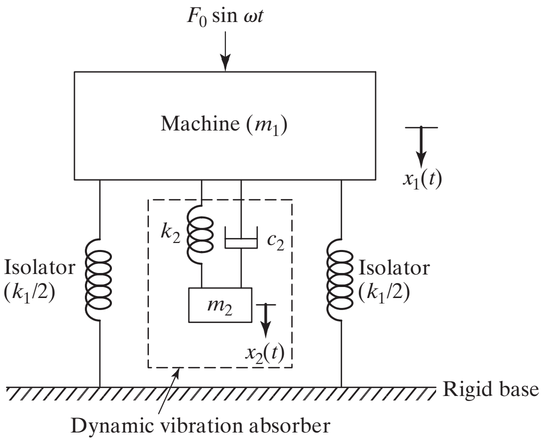


FIGURE 9.37 Damped dynamic vibration absorber.

$$g = \omega/\omega_n = \text{Forced frequency ratio}$$
$$c_c = 2m_2\omega_n = \text{Critical damping constant}$$
$$\zeta = c_2/c_c = \text{Damping ratio}$$

the magnitudes, X_1 and X_2, can be expressed as

$$\frac{X_1}{\delta_{st}} = \left[\frac{(2\zeta g)^2 + (g^2 - f^2)^2}{(2\zeta g)^2(g^2 - 1 + \mu g^2)^2 + \{\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)\}^2}\right]^{1/2} \tag{9.152}$$

and

$$\frac{X_2}{\delta_{st}} = \left[\frac{(2\zeta g)^2 + f^4}{(2\zeta g)^2(g^2 - 1 + \mu g^2)^2 + \{\mu f^2 g^2 - (g^2 - 1)(g^2 - f^2)\}^2}\right]^{1/2} \tag{9.153}$$

Equation (9.152) shows that the amplitude of vibration of the main mass is a function of μ, f, g, and ζ. The graph of

$$\left|\frac{X_1}{\delta_{st}}\right|$$

against the forced frequency ratio $g = \omega/\omega_n$ is shown in Fig. 9.38 for $f = 1$ and $\mu = 1/20$ for a few different values of ζ.

If damping is zero ($c_2 = \zeta = 0$), then resonance occurs at the two undamped resonant frequencies of the system, a result that is already indicated in Fig. 9.34. When the damping becomes infinite ($\zeta = \infty$), the two masses m_1 and m_2 are virtually clamped together, and the system behaves essentially as a single-degree-of-freedom system with a mass of $(m_1 + m_2) = (21/20)\,m$ and stiffness of k_1. In this case also, resonance occurs with $X_1 \rightarrow \infty$ at

$$g = \frac{\omega}{\omega_n} = \frac{1}{\sqrt{1 + \mu}} = 0.9759$$

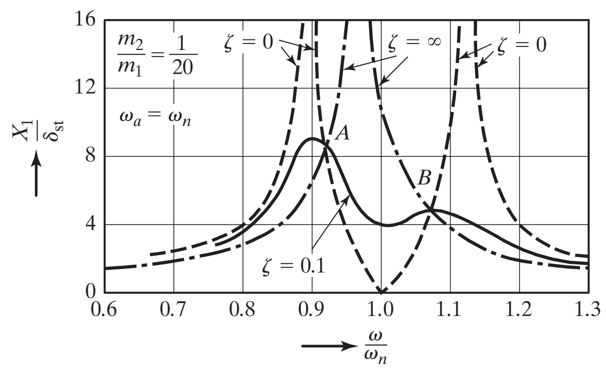


FIGURE 9.38 Effect of damped vibration absorber on the response of the machine.

Thus the peak of X_1 is infinite for $c_2 = 0$ as well as for $c_2 = \infty$. Somewhere in between these limits, the peak of X_1 will be a minimum.

Optimally Tuned Vibration Absorber. It can be seen from Fig. 9.38 that all the curves intersect at points A and B regardless of the value of damping. These points can be located by substituting the extreme cases of $\zeta = 0$ and $\zeta = \infty$ into Eq. (9.152) and equating the two. This yields

$$g^4 - 2g^2\left(\frac{1 + f^2 + \mu f^2}{2 + \mu}\right) + \frac{2f^2}{2 + \mu} = 0 \tag{9.154}$$

The two roots of Eq. (9.154) indicate the values of the frequency ratio, $g_A = \omega_A/\omega$ and $g_B = \omega_B/\omega$, corresponding to the points A and B. The ordinates of A and B can be found by substituting the values of g_A and g_B, respectively, into Eq. (9.146). It has been observed [9.35] that the most efficient vibration absorber is one for which the ordinates of the points A and B are equal. This condition requires that [9.35]

$$f = \frac{1}{1 + \mu} \tag{9.155}$$

An absorber satisfying Eq. (9.155) can be correctly called the *tuned vibration absorber*. Although Eq. (9.155) indicates how to tune an absorber, it does not indicate the optimal value of the damping ratio ζ and the corresponding value of X_1/δ_{st}. The optimal value of ζ can be found by making the response curve X_1/δ_{st} as flat as possible at peaks A and B. This can be achieved by making the curve horizontal at either A or B, as shown in Fig. 9.39. For this, first Eq. (9.155) is substituted into Eq. (9.152) to make the resulting equation applicable to the case of optimum tuning. Then the modified Eq. (9.152) is differentiated with

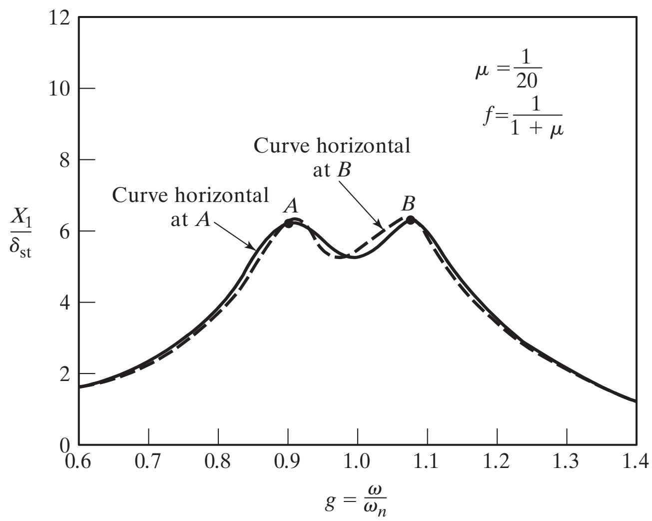


FIGURE 9.39 Tuned vibration absorber.

respect to g to find the slope of the curve of X_1/δ_{st}. By setting the slope equal to zero at points A and B, we obtain

$$\zeta^2 = \frac{\mu\left\{3 - \sqrt{\dfrac{\mu}{\mu + 2}}\right\}}{8(1 + \mu)^3} \quad \text{for point } A \tag{9.156}$$

and

$$\zeta^2 = \frac{\mu\left\{3 + \sqrt{\dfrac{\mu}{\mu + 2}}\right\}}{8(1 + \mu)^3} \quad \text{for point } B \tag{9.157}$$

A convenient average value of ζ^2 given by Eqs. (9.156) and (9.157) is used in design so that

$$\zeta^2_{\text{optimal}} = \frac{3\mu}{8(1 + \mu)^3} \tag{9.158}$$

The corresponding optimal value of $\left(\dfrac{X_1}{\delta_{st}}\right)$ becomes

$$\left(\frac{X_1}{\delta_{st}}\right)_{\text{optimal}} = \left(\frac{X_1}{\delta_{st}}\right)_{\max} = \sqrt{1 + \frac{2}{\mu}} \tag{9.159}$$

Notes

1. It can be seen from Eq. (9.153) that the amplitude of the absorber mass (X_2) is always much greater than that of the main mass (X_1). Thus the design should be able to accommodate the large amplitudes of the absorber mass.
2. Since the amplitudes of m_2 are expected to be large, the absorber spring (k_2) needs to be designed from a fatigue point of view.
3. Most vibration absorbers used in practical applications are undamped. If damping is added, it defeats the purpose of the vibration absorber, which is to eliminate unwanted vibration. In a damped vibration absorber, the amplitude of vibration of the main mass will be nonzero. Damping is to be added only in situations in which the frequency band in which the absorber is effective is too narrow for operation.
4. Additional work relating to the optimum design of vibration absorbers can be found in references [9.36–9.39].

9.12 Examples Using MATLAB

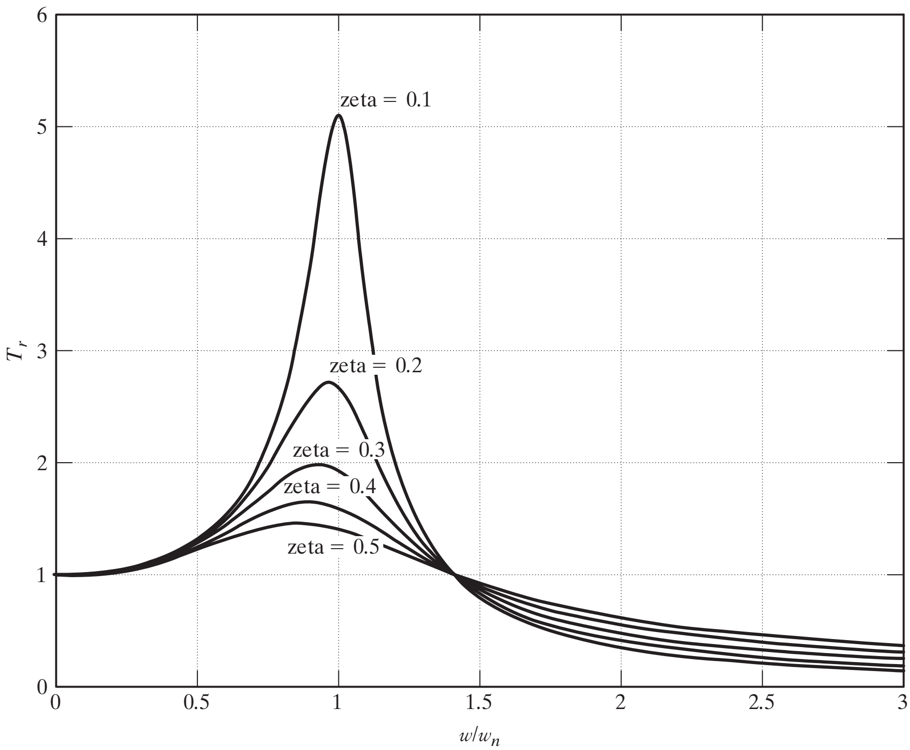
EXAMPLE 9.17 Plotting of Transmissibility

Using MATLAB, plot the variation of transmissibility of a single-degree-of-freedom system with the frequency ratio, given by Eq. (9.94), corresponding to $\zeta = 0.0, 0.1, 0.2, 0.3, 0.4$, and 0.5.

Solution: The following MATLAB program plots the variation of transmissibility as a function of the frequency ratio using Eq. (9.94):

```
%Exam 9-17
for j = 1 : 5
    kesi = j * 0.1;
    for i = 1 : 1001
        w_wn(i) = 3 * (i - 1)/1000;
        T(i) = sqrt((1 + (2 * kesi * w_wn(i)) ^ 2)/((1 - w_wn(i) ^ 2)
        ^ 2 + ↙
2 * kesi * w_wn(i) ^2));
    end;
    plot(w_wn, T);
    hold on;
end;

xlabel ('w/w_n');
ylabel('Tr');
gtext('zeta = 0.1');
gtext('zeta = 0.2');
gtext('zeta = 0.3');
gtext('zeta = 0.4');
gtext('zeta = 0.5');
title('Ex9.2');
grid on;
```

■



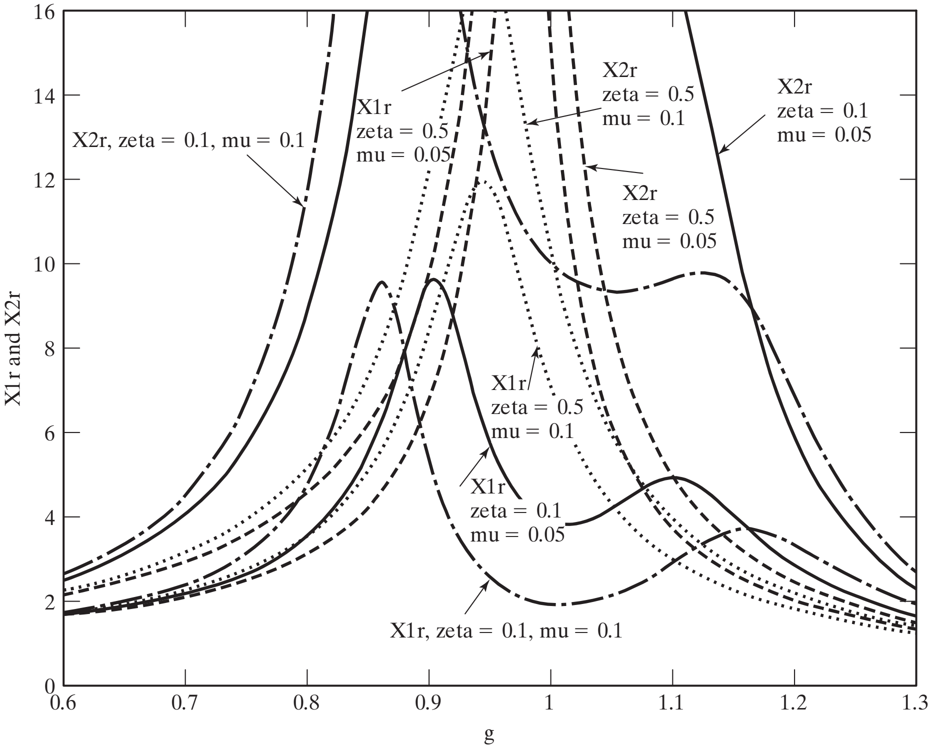
EXAMPLE 9.18 Vibration Amplitudes of Masses of Vibration Absorber

Using MATLAB, plot the variations of vibration amplitudes of the main and auxiliary masses of a vibration absorber, Eqs. (9.140) and (9.141), as functions of the frequency ratio.

Solution: Equations (9.140) and (9.141) are plotted for the following data: $f = \omega_a/\omega_n = 1$, $\zeta = 0.1$ and 0.5, $\mu = m_2/m_1 = 0.05$ and 0.1.

```
f = 1;
%------    zeta = 0.1, mu=0.05 ------------------------------------↙
-------
zeta = 0.1;
mu = 0.05;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%---    tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2–1+mu.*g.^2).^2;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2–1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r)
hold on
plot(g,x2r);
hold on
%------    zeta = 0.1, mu=0.01 ------------------------------------↙
------
zeta = 0.1;
mu = 0.1; 0.001:1.3;
g = 0.6:

tzg2 = (2.*zeta.*g).^2 ;%—-- tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2–1+mu.*g.^2).^2;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2–1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r,'-.');
hold on
plot(g,x2r,'-.');
hold on
%------   zeta = 0.5, mu=0.05 ------------------------------------↙
------
zeta = 0.5;
mu = 0.05;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%—-- tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2) ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2–1+mu.*g.^2).^2;
muf2g2 = mu.*f.^2*g.^2;
g2_1 = g.^2–1 ;
g2_f2 = g.^2-f.^2;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r,'–');
hold on
```

```
plot(g,x2r,'-');
hold on
%------    zeta = 0.5, mu=0.1 --------------------------------------↙
------
zeta = 0.5;
mu = 0.1;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%---- tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2-1+mu.*g.^2).^2 ;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2-1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2 1mug2 2+(muf2g2-g2 1.*g2 f2).^2));
plot(g,x1r,':');
hold on
plot(g,x2r,':');
xlabel('g')
ylabel('X1r and X2r')
axis ([0.6 1.3 0 16])
```



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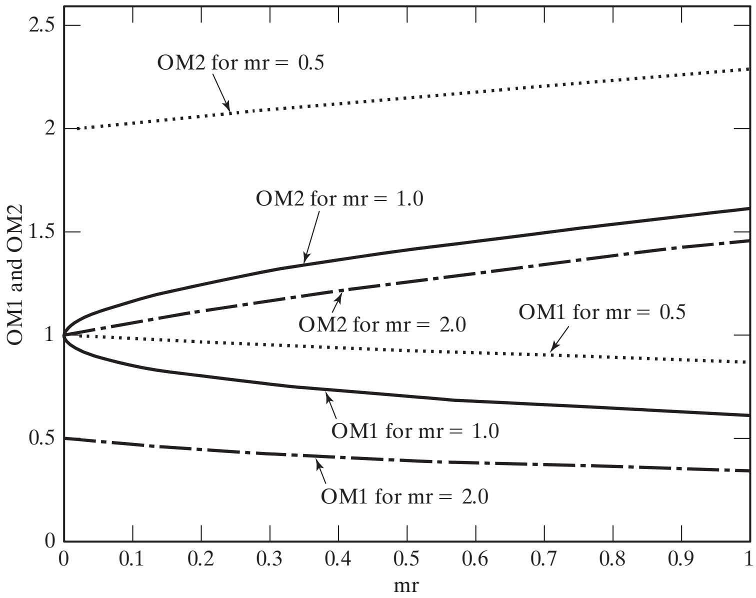
EXAMPLE 9.19 Resonant Frequencies of Vibration Absorber

Using MATLAB, plot the variations of the resonant frequency ratios given by Eq. (9.146) with the mass ratio, m_2/m_1.

Solution: The ratios Ω_1/ω_2 and Ω_2/ω_2, given by Eq. (9.146), are plotted for $\omega_2/\omega_1 = 0.5, 1.0$, and 2.0 over the range of $m_2/m_1 = 0$ to 1.

```
%------ omega2/omega1=0.5 ------------------------------------------↙
---
omega21=0.5
m21 = 0:0.001:1.0
X11 = sqrt(((1 + (1+m21)*omega21.^2) +((1+(1+m21).*omega21.^2).^2-↙
4.*omega21.^2).^0.5)...
    /(2.*omega21.^2))
plot(m21,X11,':')
axis([0 1.0 0.0 2.6])
hold on

X12 = sqrt(((1+(1+m21)*omega21.^2) - ((1 + (1+m21).*omega21.^2).^2-↙
4.*omega21.^2).^0.5)...
    /(2.*omega21.^2))
plot(m21,X12,':')
hold on

%------ omega2/omega1=1.0 ------------------------------------------↙
---

omega21=1.0
m21 = 0:0.001:1.0
X21 = sqrt(((1+(1+m21)*omega21.^2) +((1+(1+m21).*omega21.^2).^2-↙
4.*omega21.^2).^0.5)...
    /(2.*omega21.^2))
plot(m21,X21,'-')
axis([0 1.0 0.0 2.6])
hold on

X22 = sqrt(((1+(1+m21)*omega21.^2) -((1+(1+m21).*omega21.^2).^2-↙
4.*omega21.^2).^0.5)...
    /(2.*omega21.^2))
plot(m21,X22,'-')
hold on

%------ omega2/omega1=2.0 ------------------------------------------↙
---

omega21=2.0
m21 = 0 : 0.001 : 1.0
X31 = sqrt(((1+(1+m21)*omega21.^2) +((1+(1+m21).*omega21.^2).^2).↙
^2-4.*omega21.^2).^0.5)...
    / (2.*omega21.^2))
plot(m21,X31,'-.')
axis([0 1.0 0.0 2.6])
hold on

X32 = sqrt(((1+(1+m21)*omega21.^2) -((1+(1+m21).*omega21.^2).^2-4.↙
*omega21.^2) .^0.5)...
    /(2.*omega21.^2))
plot(m21,X32,'-.')
hold on
xlabel ('mr')
ylabel ('OM1 and OM2')
```



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EXAMPLE 9.20 Two-Plane Balancing

Develop a general MATLAB program called **Program13.m** for the two-plane balancing of rotating machines. Use the program to solve Example 9.2.

Solution: **Program13.m** is developed to accept the vectors $\vec{V}_A$, $\vec{V}_B$, $\vec{V}_A'$, $\vec{V}_B'$, $\vec{V}_A''$, $\vec{V}_B''$, $\vec{W}_L$, and $\vec{W}_R$ as input in the form of two-dimensional arrays *VA, VB, VAP, VBP, VAPP, VBPP, WL, WR, BL,* and *BR*, respectively. The program gives the vectors B_L and B_R as output in the form of two-dimensional arrays *BL* and *BR* indicating the magnitude and position of the balancing weights in the left and right planes, respectively. The listing of the program and the output are given below.

```
%======================================================================
%
% Program13.m
% Two-plane balancing
%
%======================================================================
% Run "Program13" in MATLAB command window. Progrm13.m, balan.m,
vsub.m,
% vdiv.m and vmult.m should be in the same folder,and set the
MATLAB path
% to this folder.
% following 8 lines contain problem-dependent data
va=[8.5 60];
vap=[6 125];
wl=[10 270];
vb=[6.5 205];
```

```
vbp=[4.5 230];
vapp=[6 35];
vbpp=[10.5 160];
wr=[12 180];
% end of problem-dependent data
[bl,br]=balan(va,vb,vap,vbp,vapp,vbpp,wl,wr);
fprintf('            Results of two-plane balancing \n\n');
fprintf('Left-plane balancing weight  Right-plane balancing weight');
fprintf('\n\n');
fprintf('Magnitude=%8.6f          Magnitude=%8.6f \n\n',bl(1),br(1));
fprintf('Angel=%8.6f              Angle=%8.6f \n\n',bl(2),br(2));
%======================================================================
%
%Function Balan.m
%
%======================================================================
function [bl,br]=balan(va,vb,vap,vbp,vapp,vbpp,wl,wr);
pi=180/3.1415926;
va(2)=va(2)/pi;
p(1)=va(1);
p(2)=va(2);
va(1)=p(1)*cos(p(2));
va(2)=p(1)*sin(p(2));
vb(2)=vb(2)/pi;
p(1)=vb(1);
p(2)=vb(2);
vb(1)=p(1)*cos(p(2));
vb(2)=p(1)*sin(p(2));
vap(2)=vap(2)/pi;
p(1)=vap(1);
p(2)=vap(2);
vap(1)=p(1)*cos(p(2));
vap(2)=p(1)*sin(p(2));
vbp(2)=vbp(2)/pi;
p(1)=vbp(1);
p(2)=vbp(2);
vbp(1)=p(1)*cos(p(2));
vbp(2)=p(1)*sin(p(2));
vapp(2)=vapp(2)/pi;
p(1)=vapp(1);
p(2)=vapp(2);
vapp(1)=p(1)*cos(p(2));
vapp(2)=p(1)*sin(p(2));
vbpp(2)=vbpp(2)/pi;
p(1)=vbpp(1);
p(2)=vbpp(2);
vbpp(1)=p(1)*cos(p(2));
vbpp(2)=p(1)*sin(p(2));
wl(2)=wl(2)/pi;
p(1)=wl(1);
p(2)=wl(2);
wl(1)=p(1)*cos(p(2));
wl(2)=p(1)*sin(p(2));
wr(2)=wr(2)/pi;
p(1)=wr(1);
p(2)=wr(2);
wr(1)=p(1)*cos(p(2));
wr(2)=p(1)*sin(p(2));
[r]=vsub(vap,va);
[aal]=vdiv(r,wl);
[s]=vsub(vbp,vb);
[abl]=vdiv(s,wl);
[p]=vsub(vapp,va);
[aar]=vdiv(p,wr);
[q]=vsub(vbpp,vb);
[abr]=vdiv(q,wr);
```

```
[ar1]=sqrt(aar(1)^2+aar(2)^2);
[ar2]=atan(aar(2)/aar(1))*pi;
[al1]=sqrt(aal(1)^2+aal(2)^2);
[al2]=atan(aal(2)/aal(1))*pi;
[r]=vmult(abl,va);
[s]=vmult(aal,vb);
[vap]=vsub(r,s);
[r]=vmult(aar,abl);
[s]=vmult(aal,abr);
[vbp]=vsub(r,s);
[ur]=vdiv(vap,vbp);
[r]=vmult(abr,va);
[s]=vmult(aar,vb);
[vap]=vsub(r,s);
[r]=vmult(abr,aal);
[s]=vmult(aar,abl);
[vbp]=vsub(r,s);
[ul]=vdiv(vap,vbp);
bl(1)=sqrt(ul(1)^2+ul(2)^2);
a1=ul(2)/ul(1);
bl(2)=atan(ul(2)/ul(1));
br(1)=sqrt(ur(1)^2+ur(2)^2);
a2=ur(2)/ur(1);
br(2)=atan(ur (2)/ur (1));
bl(2)=bl(2)*pi;
br(2)=br(2)*pi;
bl(2)=bl(2)+180;
br(2)=br(2)+180;

%=====================================================================
%
%Function vdiv.m
%
%=====================================================================
function [c]=vdiv(a,b);
c(1)=(a(1)*b(1)+a(2)*b(2))/(b(1)^2+b(2)^2);
c(2)=(a(2)*b(1)-a(1)*b(2))/(b(1)^2+b(2)^2);
%=====================================================================
%
%Function vmult.m
%
%=====================================================================
function [c]=vmult(a,b);
c(1)=a(1)*b(1)-a(2)*b(2);
c(2)=a(2)*b(1)+a(1)*b(2);

%=====================================================================
%
%Function vsub.m
%
%=====================================================================
function [c]=vsub(a,b);
c(1)=a(1)-b(1);
c(2)=a(2)-b(2);
```

```
                    Results of two-plane balancing

Left-plane balancing weight          Right-plane balancing weight

Magnitude=10.056139                  Magnitude=5.877362

Angle=145.554799                     Angle=248.255931
```

■

CHAPTER SUMMARY

We discussed the use of vibration nomographs and vibration criteria to determine acceptable levels of vibration. We presented several methods, such as balancing of rotating and reciprocating machines, to eliminate/reduce vibration at the source. We outlined methods of changing mass and/or stiffness and dissipating energy by adding damping. We discussed methods of designing vibration isolators, vibration absorbers, and active vibration-control systems. We presented the solution of vibration-control problems using MATLAB.

Now that you have finished this chapter, you should be able to answer the review questions and solve the problems given below.

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REVIEW QUESTIONS

9.1 Give brief answers to the following:

1. Name some sources of industrial vibration.
2. What are the various methods available for vibration control?
3. What is single-plane balancing?
4. Describe the two-plane balancing procedure.
5. What is whirling?
6. What is the difference between stationary damping and rotary damping?
7. How is the critical speed of a shaft determined?
8. What causes instability in a rotor system?
9. What considerations are to be taken into account for the balancing of a reciprocating engine?
10. What is the function of a vibration isolator?
11. What is a vibration absorber?
12. What is the difference between a vibration isolator and a vibration absorber?
13. Does spring mounting always reduce the vibration of the foundation of a machine?
14. Is it better to use a soft spring in the flexible mounting of a machine? Why?
15. Is the shaking force proportional to the square of the speed of a machine? Does the vibratory force transmitted to the foundation increase with the speed of the machine?
16. Why does dynamic balancing imply static balancing?
17. Explain why dynamic balancing can never be achieved by a static test alone.
18. Why does a rotating shaft always vibrate? What is the source of the shaking force?
19. Is it always advantageous to include a damper in the secondary system of a dynamic vibration absorber?
20. What is active vibration isolation?
21. Explain the difference between passive and active isolation.

9.2 Indicate whether each of the following statements is true or false:

1. Vibration can cause structural and mechanical failures.
2. The response of a system can be reduced by the use of isolators and absorbers.
3. Vibration control means the elimination or reduction of vibration.
4. The vibration caused by a rotating unbalanced disc can be eliminated by adding a suitable mass to the disc.
5. Any unbalanced mass can be replaced by two equivalent unbalanced masses in the end planes of the rotor.
6. The oil whip in the bearings can cause instability in a rotor system.
7. The natural frequency of a system can be changed by varying its damping.
8. The stiffness of a rotating shaft can be altered by changing the location of its bearings.
9. All practical systems have damping.

10. High loss factor of a material implies less damping.
11. Passive isolation systems require external power to function.
12. The transmissibility is also called the transmission ratio.
13. The force transmitted to the foundation of an isolator with rigid foundation can never be infinity.
14. Internal and external friction can cause instability in a rotating shaft at speeds above the first critical speed.

9.3 Fill in each of the following blanks with the appropriate word:

1. Even a small excitation force can cause an undesirably large response near _____.
2. The use of close tolerances and better surface finish for machine parts tends to make a machine _____ susceptible to vibration.
3. The presence of unbalanced mass in a rotating disc causes _____.
4. When the speed of rotation of a shaft equals one of the natural frequencies of the shaft, it is called _____ speed.
5. The moving elements of a reciprocating engine are the crank, the connecting rod, and the _____.
6. The vertical component of the inertia force of a reciprocating engine has primary and _____ parts.
7. Laminated structures have _____ damping.
8. Materials with a large value of the loss factor are subject to _____ stress.
9. Vibration isolation involves insersion of a resilient member between the vibrating mass and the _____ of vibration.
10. Cork is a _____ isolator.
11. An active isolator consists of a sensor, a signal processor, and an _____.
12. Vibration neutralizer is also known as dynamic vibration _____.
13. Although an undamped vibration absorber removes the original resonance peak of the response, it introduces _____ new peaks.
14. The single-plane balancing is also known as _____ balancing.
15. Phase marks are used in _____ plane balancing using a vibration analyzer.
16. Machine errors can cause _____ in rotating machines.
17. The combustion instabilities are a source of _____ in engines.
18. The deflection of a rotating shaft becomes very large at the _____ speed.
19. Oil whip in bearings can cause _____ in a flexible rotor system.

9.4 Select the most appropriate answer out of the multiple choices given:

1. An example of a source of vibration that cannot be altered is:
   a. atmospheric turbulance
   b. hammer blow
   c. tire stiffness of an automobile.
2. The two-plane balancing is also known as:
   a. static balancing
   b. dynamic balancing
   c. proper balancing
3. The unbalanced force caused by an eccentric mass m rotating at an angular speed ω and located at a distance r from the axis of rotation is
   a. $mr^2\omega^2$ **b.** $mg\omega^2$ **c.** $mr\omega^2$

4. The following material has high internal damping:
a. cast iron **b.** copper **c.** brass

5. Transmissibility is the ratio of
a. force transmitted and exciting force
b. force applied and the resulting displacement
c. input displacement and output displacement

6. Mechanical impedance is the ratio of
a. force transmitted and exciting force
b. force applied and force transmitted
c. applied force and displacement

7. Vibration can be eliminated on the basis of theoretical analysis
a. sometimes **b.** always **c.** never

8. A long rotor can be balanced by adding weights in
a. a single plane **b.** any two planes **c.** two specific planes

9. The damping caused by the internal friction of a shaft material is called
a. stationary damping
b. external damping
c. rotary damping

10. The damping caused by the bearing support structure of a rotating shaft is called
a. stationary damping
b. internal damping
c. rotary damping

11. An undamped vibration absorber removes the original resonance peak but introduces
a. one new peak **b.** two new peaks **c.** several new peaks

9.5 Match the items in the two columns below.

1. Control natural frequency	**a.** Introduce damping
2. Avoid excessive response at resonance	**b.** Use vibration isolator
3. Reduce transmission of excitation force from one part to another	**c.** Add vibration absorber
4. Reduce response of the system	**d.** Avoid resonance

PROBLEMS

Section 9.2 Vibration Criteria

9.1 An automobile moving on a rough road, in the form of a sinusoidal surface, is modeled as a spring-mass system, as shown in Fig. 9.40. The sinusoidal surface has a wave length of 5 m and an amplitude of $Y = 1$ mm. If the mass of the automobile, including the passengers, is 1500 kg and the stiffness of the suspension system (k) is 400 kN/m, determine the range of speed (v) of the automobile in which the passengers perceive the vibration. Suggest possible methods of improving the design for a more comfortable ride of the passengers.

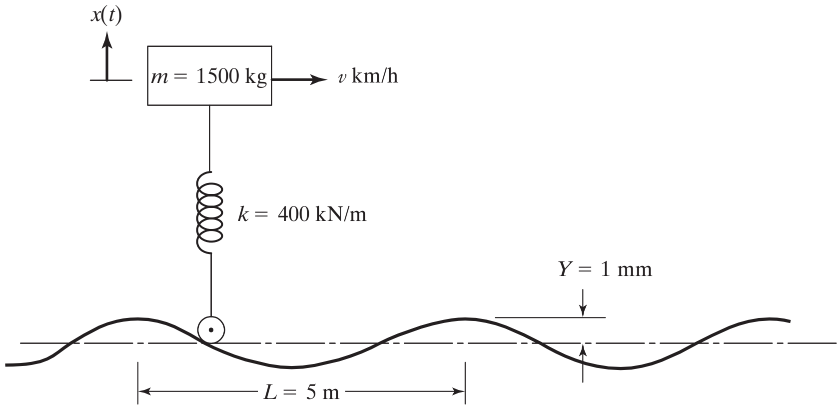

FIGURE 9.40

9.2 The root mean square value of a signal $x(t)$, x_{rms}, is defined as

$$x_{\text{rms}} = \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t)\, dt \right\}^{1/2}$$

Using this definition, find the root mean square values of the displacement (x_{rms}), velocity $(\dot{x}_{\text{rms}})$, and acceleration $(\ddot{x}_{\text{rms}})$ corresponding to $x(t) = X \cos \omega t$.

Section 9.4 Balancing of Rotating Machines

9.3 Two identical discs are connected by four bolts of different sizes and mounted on a shaft, as shown in Fig. 9.41. The masses and locations of three bolts are as follows:

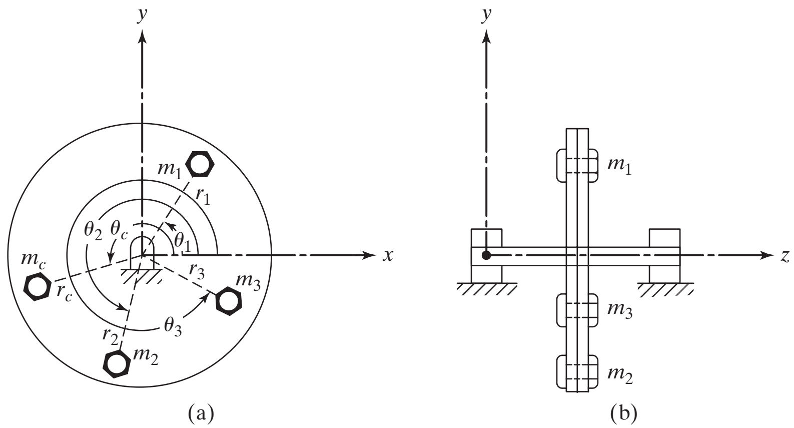

FIGURE 9.41

$m_1 = 35$ grams, $r_1 = 110$ mm, and $\theta_1 = 40°$; $m_2 = 15$ grams, $r_2 = 90$ mm, and $\theta_2 = 220°$; and $m_3 = 25$ grams, $r_3 = 130$ mm, $\theta_3 = 290°$. Find the mass and location of the fourth bolt (m_c, r_c, and θ_c), which results in the static balance of the discs.

9.4 Four holes are drilled in a uniform circular disc at a radius of 4 in. and angles of 0°, 60°, 120°, and 180°. The weight removed at holes 1 and 2 is 4 oz each and the weight removed at holes 3 and 4 is 5 oz each. If the disc is to be balanced statically by drilling a fifth hole at a radius of 5 in., find the weight to be removed and the angular location of the fifth hole.

9.5 Three masses, weighing 0.5 lb, 0.7 lb, and 1.2 lb, are attached around the rim, of diameter 30 in., of a flywheel at the angular locations $\theta = 10°$, 100°, and 190°, respectively. Find the weight and the angular location of the fourth mass to be attached on the rim that leads to the dynamic balance of the flywheel.

9.6 The amplitude and phase angle due to original unbalance in a grinding wheel operating at 1200 rpm are found to be 10 mils and 40° counterclockwise from the phase mark. When a trial weight $W = 6$ oz is added at 65° clockwise from the phase mark and at a radial distance 2.5 in. from the center of rotation, the amplitude and phase angle are observed to be 19 mils and 150° counterclockwise. Find the magnitude and angular position of the balancing weight if it is to be located 2.5 in. radially from the center of rotation.

9.7 An unbalanced flywheel shows an amplitude of 6.5 mils and a phase angle of 15° clockwise from the phase mark. When a trial weight of magnitude 2 oz is added at an angular position 45° counterclockwise from the phase mark, the amplitude and the phase angle become 8.8 mils and 35° counterclockwise, respectively. Find the magnitude and angular position of the balancing weight required. Assume that the weights are added at the same radius.

9.8 In order to determine the unbalance in a grinding wheel, rotating clockwise at 2400 rpm, a vibration analyzer is used and an amplitude of 4 mils and a phase angle of 45° are observed with the original unbalance. When a trial weight $W = 4$ oz is added at 20° clockwise from the phase mark, the amplitude becomes 8 mils and the phase angle 145°. If the phase angles are measured counterclockwise from the right-hand horizontal, calculate the magnitude and location of the necessary balancing weight.

9.9 A turbine rotor is run at the natural frequency of the system. A stroboscope indicates that the maximum displacement of the rotor occurs at an angle 229° in the direction of rotation. At what angular position must mass be removed from the rotor in order to improve its balancing?

9.10 A rotor, having three eccentric masses in different planes, is shown in Fig. 9.42. The axial, radial, and angular locations of mass m_i are given by l_i, r_i, and θ_i, respectively, for $i = 1, 2, 3$. If the rotor is to be dynamically balanced by locating two masses m_{b1} and m_{b2} at radii r_{b1} and r_{b2} at the angular locations θ_{b1} and θ_{b2}, as shown in Fig. 9.42, derive expressions for $m_{b1}r_{b1}$, $m_{b2}r_{b2}$, θ_{b1}, and θ_{b2}.

9.11 The rotor shown in Fig. 9.43(a) is balanced temporarily in a balancing machine by adding the weights $W_1 = W_2 = 0.2$ lb in the plane A and $W_3 = W_4 = 0.2$ lb in the plane D at a radius of 3 in., as shown in Fig. 9.43(b). If the rotor is permanently balanced by drilling holes at a radius of 4 in. in planes B and C, determine the position and amount of material to be removed from the rotor. Assume that the adjustable weights W_1 to W_4 will be removed from the planes A and D.

9.12 Weights of 2 lb, 4 lb, and 3 lb are located at radii 2 in., 3 in., and 1 in. in the planes C, D, and E, respectively, on a shaft supported at the bearings B and F, as shown in Fig. 9.44. Find the weights and angular locations of the two balancing weights to be placed in the end planes A and G so that the dynamic load on the bearings will be zero.

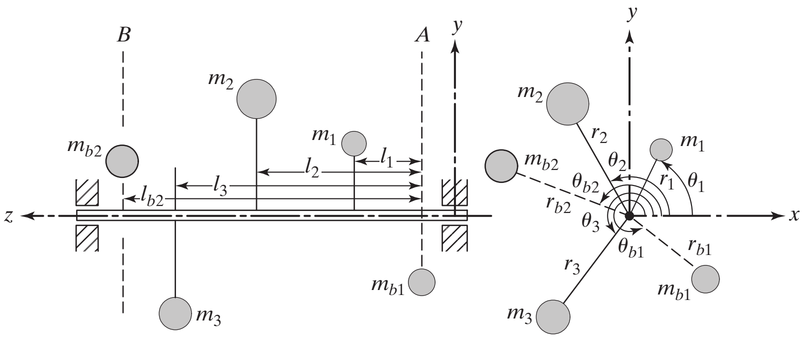

FIGURE 9.42

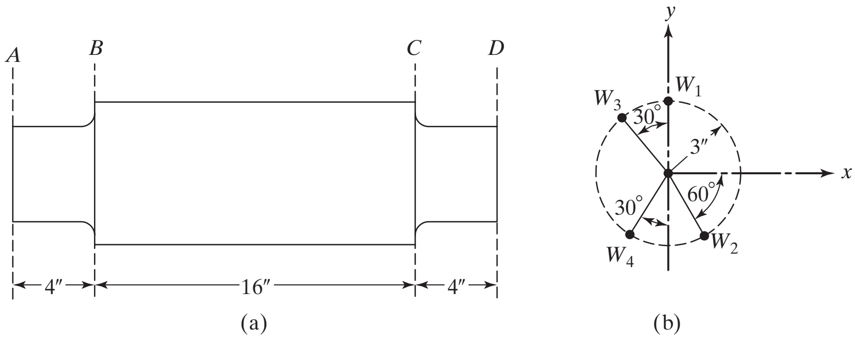

FIGURE 9.43

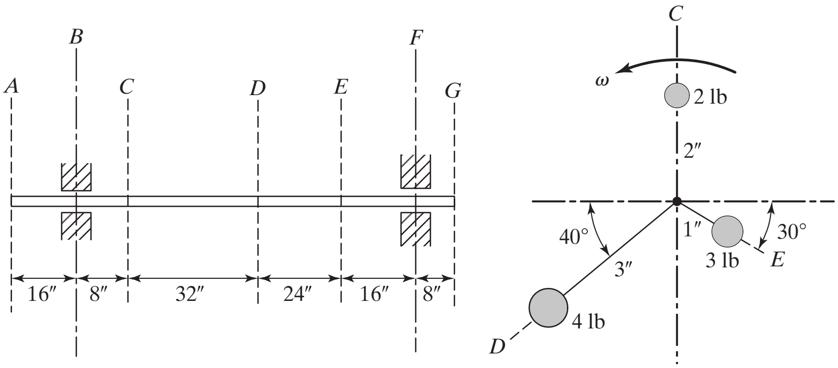

FIGURE 9.44

9.13 The data obtained in a two-plane balancing procedure are given in the table below. Determine the magnitude and angular position of the balancing weights, assuming that all angles are measured from an arbitrary phase mark and all weights are added at the same radius.

	Amplitude (mils)		Phase Angle	
Condition	**Bearing *A***	**Bearing *B***	**Bearing *A***	**Bearing *B***
Original unbalance	5	4	100°	180°
W_L = 2 oz added at 30° in the left plane	6.5	4.5	120°	140°
W_R = 2 oz added at 0° in the right plane	6	7	90°	60°

9.14 Figure 9.45 shows a rotating system in which the shaft is supported in bearings at *A* and *B*. The three masses m_1, m_2, and m_3 are connected to the shaft as indicated in the figure. (a) Find the bearing reactions at *A* and *B* if the speed of the shaft is 1000 rpm. (b) Determine the locations and magnitudes of the balancing masses to be placed at a radius of 0.25 m in the planes *L* and *R*, which can be assumed to pass through the bearings *A* and *B*.

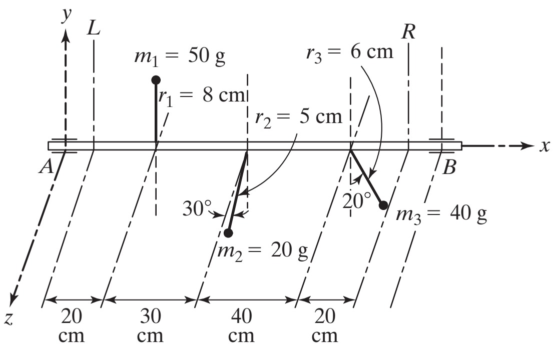

FIGURE 9.45

Section 9.5 Whirling of Rotating Shafts

9.15 A flywheel, with a weight of 100 lb and an eccentricity of 0.5 in., is mounted at the center of a steel shaft of diameter 1 in. If the length of the shaft between the bearings is 30 in. and the rotational speed of the flywheel is 1200 rpm, find (a) the critical speed, (b) the vibration amplitude of the rotor, and (c) the force transmitted to the bearing supports.

9.16 Derive the expression for the stress induced in a shaft with an unbalanced concentrated mass located midway between two bearings.

9.17 A steel shaft of diameter 2.5 cm and length 1 m is supported at the two ends in bearings. It carries a turbine disc, of mass 20 kg and eccentricity 0.005 m, at the middle and operates at 6000 rpm. The damping in the system is equivalent to viscous damping with $\zeta = 0.01$. Determine the whirl amplitude of the disc at (a) operating speed, (b) critical speed, and (c) 1.5 times the critical speed.

9.18 Find the bearing reactions and the maximum bending stress induced in the shaft at (a) operating speed, (b) critical speed, and (c) 1.5 times the critical speed for the shaft-rotor system described in Problem 9.17.

9.19 Solve Problem 9.17 by assuming that the material of the shaft is aluminum rather than steel.

9.20 Solve Problem 9.18 by assuming that the material of the shaft is aluminum rather than steel.

9.21 A shaft, having a stiffness of 3.75 MN/m, rotates at 3600 rpm. A rotor, having a mass of 60 kg and an eccentricity of 2000 microns, is mounted on the shaft. Determine (a) the steady-state whirl amplitude of the rotor and (b) the maximum whirl amplitude of the rotor during start-up and stopping conditions. Assume the damping ratio of the system as 0.05.

Section 9.6 Balancing of Reciprocating Engines

9.22 The cylinders of a four-cylinder in-line engine are placed at intervals of 12 in. in the axial direction. The cranks have the same length, 4 in., and their angular positions are given by 0°, 180°, 180°, and 0°. If the length of the connecting rod is 10 in. and the reciprocating weight is 2 lb for each cylinder, find the unbalanced forces and moments at a speed of 3000 rpm, using the center line through cylinder 1 as the reference plane.

9.23 The reciprocating mass, crank radius, and connecting-rod length of each of the cylinders in a two-cylinder in-line engine are given by m, r, and l, respectively. The crank angles of the two cylinders are separated by 180°. Find the unbalanced forces and moments in the engine.

9.24 A four-cylinder in-line engine has a reciprocating weight of 3 lb, a stroke of 6 in., and a connecting-rod length of 10 in. in each cylinder. The cranks are separated by 4 in. axially and 90° radially, as shown in Fig. 9.46. Find the unbalanced primary and secondary forces and moments with respect to the reference plane shown in Fig. 9.46 at an engine speed of 1500 rpm.

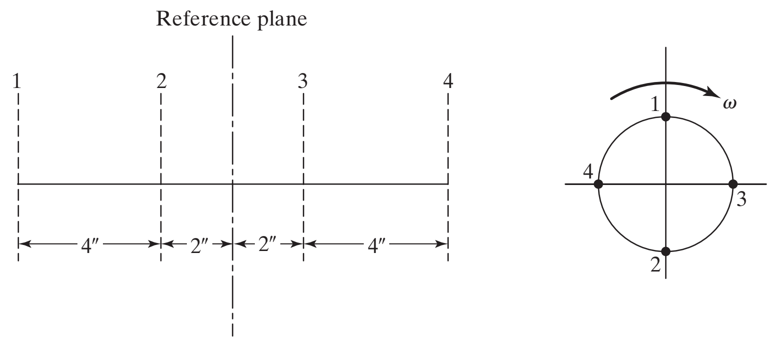

FIGURE 9.46

9.25 The arrangement of cranks in a six-cylinder in-line engine is shown in Fig. 9.47. The cylinders are separated by a distance a in the axial direction, and the angular positions of the cranks are given by $\alpha_1 = \alpha_6 = 0°$, $\alpha_2 = \alpha_5 = 120°$, and $\alpha_3 = \alpha_4 = 240°$. If the crank length, connecting-rod length, and the reciprocating mass of each cylinder are r, l, and m, respectively, find the primary and secondary unbalanced forces and moments with respect to the reference plane indicated in Fig. 9.47.

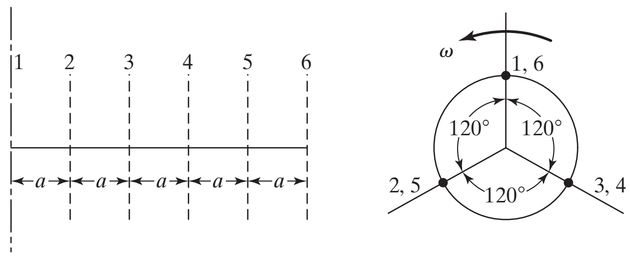

FIGURE 9.47

9.26 A single-cylinder engine has a total mass of 150 kg. Its reciprocating mass is 5 kg, and the rotating mass is 2.5 kg. The stroke ($2r$) is 15 cm, and the speed is 600 rpm. (a) If the engine is mounted floating on very weak springs, what is the amplitude of vertical vibration of the engine? (b) If the engine is mounted solidly on a rigid foundation, what is the alternating force amplitude transmitted? Assume the connecting rod to be of infinite length.

Section 9.10 Vibration Isolation

9.27 An electronic instrument is to be isolated from a panel that vibrates at frequencies ranging from 25 Hz to 35 Hz. It is estimated that at least 80 percent vibration isolation must be achieved to prevent damage to the instrument. If the instrument weights 85 N, find the necessary static deflection of the isolator.

9.28* An exhaust fan, having a small unbalance, weights 800 N and operates at a speed of 600 rpm. It is desired to limit the response to a transmissibility of 2.5 as the fan passes through resonance during start-up. In addition, an isolation of 90 percent is to be achieved at the operating speed of the fan. Design a suitable isolator for the fan.

9.29* An air compressor of mass 500 kg has an eccentricity of 50 kg-cm and operates at a speed of 300 rpm. The compressor is to be mounted on one of the following mountings: (a) an isolator consisting of a spring with negligible damping, and (b) a shock absorber having a damping ratio of 0.1 and negligible stiffness. Select a suitable mounting and specify the design details by considering the static deflection of the compressor, the transmission ratio, and the amplitude of vibration of the compressor.

9.30 The armature of a variable-speed electric motor, of mass 200 kg, has an unbalance due to manufacturing errors. The motor is mounted on an isolator having a stiffness of 10 kN/m and a dashpot having a damping ratio of 0.15. (a) Find the speed range over which the amplitude of the fluctuating force transmitted to the foundation will be larger than the exciting force. (b) Find the speed range over which the transmitted force amplitude will be less than 10 percent of the exciting force amplitude.

9.31 A dishwashing machine weighing 150 lb operates at 300 rpm. Find the minimum static deflection of an isolator that provides 60 percent isolation. Assume that the damping in the isolator is negligible.

9.32 A washing machine of mass 50 kg operates at 1200 rpm. Find the maximum stiffness of an isolator that provides 75 percent isolation. Assume that the damping ratio of the isolator is 7 percent.

*The asterisk denotes a problem with no unique answer.

9.33 It is found that an exhaust fan, of mass 80 kg and operating speed 1000 rpm, produces a repeating force of 10,000 N on its rigid base. If the maximum force transmitted to the base is to be limited to 2000 N using an undamped isolator, determine (a) the maximum permissible stiffness of the isolator that serves the purpose; (b) the steady-state amplitude of the exhaust fan with the isolator that has the maximum permissible stiffness; and (c) the maximum amplitude of the exhaust fan with isolation during start-up.

9.34 It has been found that a printing press, of mass 300 kg and operating speed 3000 rpm, produces a repeating force of 30,000 N when attached to a rigid foundation. Find a suitable viscously damped isolator to satisfy the following requirements: (a) the static deflection should be as small as possible; (b) the steady-state amplitude should be less than 2.5 mm; (c) the amplitude during start-up conditions should not exceed 20 mm; and (d) the force transmitted to the foundation should be less than 10,000 N.

9.35 A compressor of mass 120 kg has a rotating unbalance of 0.2 kg-m. If an isolator of stiffness 0.5 MN/m and damping ratio 0.06 is used, find the range of operating speeds of the compressor over which the force transmitted to the foundation will be less than 2500 N.

9.36 An internal combustion engine has a rotating unbalance of 1.0 kg-m and operates between 800 and 2000 rpm. When attached directly to the floor, it transmitted a force of 7,018 N at 800 rpm and 43,865 N at 2000 rpm. Find the stiffness of the isolator that is necessary to reduce the force transmitted to the floor to 6,000 N over the operating-speed range of the engine. Assume that the damping ratio of the isolator is 0.08, and the mass of the engine is 200 kg.

9.37 A small machine tool of mass 100 kg operates at 600 rpm. Find the static deflection of an undamped isolator that provides 90 percent isolation.

9.38 A diesel engine of mass 300 kg and operating speed 1800 rpm is found to have a rotating unbalance of 1 kg-m. It is to be installed on the floor of an industrial plant for purposes of emergency power generation. The maximum permissible force that can be transmitted to the floor is 8000 N and the only type of isolator available has a stiffness of 1 MN/m and a damping ratio of 5 percent. Investigate possible solutions to the problem.

9.39 The force transmitted by an internal combustion engine of mass 500 kg, when placed directly on a rigid floor, is given by

$$F_t(t) = (18000 \cos 300t + 3600 \cos 600\, t)\,\text{N}$$

Design an undamped isolator so that the maximum magnitude of the force transmitted to the floor does not exceed 12,000 N.

9.40 Design the suspension of an automobile such that the maximum vertical acceleration felt by the driver is less than $2g$ at all speeds between 40 and 80 mph while traveling on a road whose surface varies sinusoidally as $y(u) = 0.5 \sin 2u$ ft, where u is the horizontal distance in feet. The weight of the automobile, with the driver, is 1500 lb and the damping ratio of the suspension is to be 0.05. Use a single-degree-of-freedom model for the automobile.

9.41 Consider a single-degree-of-freedom system with Coulomb damping (which offers a constant friction force, F_c). Derive an expression for the force transmissibility when the mass is subjected to a harmonic force, $F(t) = F_0 \sin \omega t$.

9.42 Consider a single-degree-of-freedom system with Coulomb damping (which offers a constant friction force, F_c). Derive expressions for the absolute and relative displacement transmissibilities when the base is subjected to a harmonic displacement, $y(t) = Y \sin \omega t$.

9.43 When a washing machine, of mass 200 kg and an unbalance 0.02 kg-m, is mounted on an isolator, the isolator deflects by 5 mm under the static load. Find (a) the amplitude of the washing machine and (b) the force transmitted to the foundation at the operating speed of 1200 rpm.

9.44 An electric motor, of mass 60 kg, rated speed 3000 rpm, and an unbalance 0.002 kg-m, is to be mounted on an isolator to achieve a force transmissibility of less than 0.25. Determine (a) the stiffness of the isolator, (b) the dynamic amplitude of the motor, and (c) the force transmitted to the foundation.

9.45 An engine is mounted on a rigid foundation through four springs. During operation, the engine produces an excitation force at a frequency of 3000 rpm. If the weight of the engine causes the springs to deflect by 10 mm, determine the reduction in the force transmitted to the foundation.

9.46 A sensitive electronic system, of mass 30 kg, is supported by a spring-damper system on the floor of a building that is subject to a harmonic motion in the frequency range 10–75 Hz. If the damping ratio of the suspension is 0.25, determine the stiffness of the suspension if the amplitude of vibration transmitted to the system is to be less than 15 percent of the floor vibration over the given frequency range.

9.47 A machine weighing 2600 lb is mounted on springs. A piston of weight $w = 60$ lb moves up and down in the machine at a speed of 600 rpm with a stroke of 15 in. Considering the motion to be harmonic, determine the maximum force transmitted to the foundation if (a) $k = 10000$ lb/in., and (b) $k = 25000$ lb/in.

9.48 A printed circuit board of mass 1 kg is supported to the base through an undamped isolator. During shipping, the base is subjected to a harmonic disturbance (motion) of amplitude 2 mm and frequency 2 Hz. Design the isolator so that the displacement transmitted to the printed circuit board is to be no more than 5 percent of the base motion.

9.49 An electronic instrument of mass 10 kg is mounted on an isolation pad. If the base of the isolation pad is subjected to a shock in the form of a step velocity of 10 mm/s, find the stiffness of the isolation pad if the maximum permissible values of deflection and acceleration of the instrument are specified as 10 mm and 20g, respectively.

9.50 A water tank of mass 10^5 kg is supported on a reinforced cement concrete column, as shown in Fig. 9.48(a). When a projectile hits the tank, it causes a shock, in the form of a step force, as shown in Fig. 9.48(b). Determine the stiffness of the column if the maximum deflection of the tank is to be limited to 0.5 m. The response spectrum of the shock load is shown in Fig. 9.48(c).

9.51 A viscously damped single-degree-of-freedom system has a body (mass) weighing 60 lb with a spring constant of 400 lb/in. It's base is subjected to harmonic vibration. (a) When the base vibrates with an amplitude of 2.0 in. at resonance, the steady-state amplitude of the body is found to be 5.0 in. Find the damping ratio of the system. (b) When the base vibrates at a frequency of 10 Hz, the steady-state amplitude of the body is found to be 1.5 in. Find the magnitude of the force transmitted to the base.

9.52 A single-degree-of-freedom system is used to represent an automobile, of mass m, damping constant c, and stiffness k, which travels on a rough road that is in the form of a sinusoidal surface with an amplitude Y and wavelength l. If the automobile travels at a velocity v, derive an expression for the transmissibility of the vertical motion of the automobile mass (m).

9.53 A sensitive instrument of mass 100 kg is installed at a location that is subjected to harmonic motion with frequency 20 Hz and acceleration 0.5 m/s^2. If the instrument is supported on an isolator having a stiffness $k = 25 \times 10^4$ N/m and a damping ratio $\zeta = 0.05$, determine the maximum acceleration experienced by the instrument.

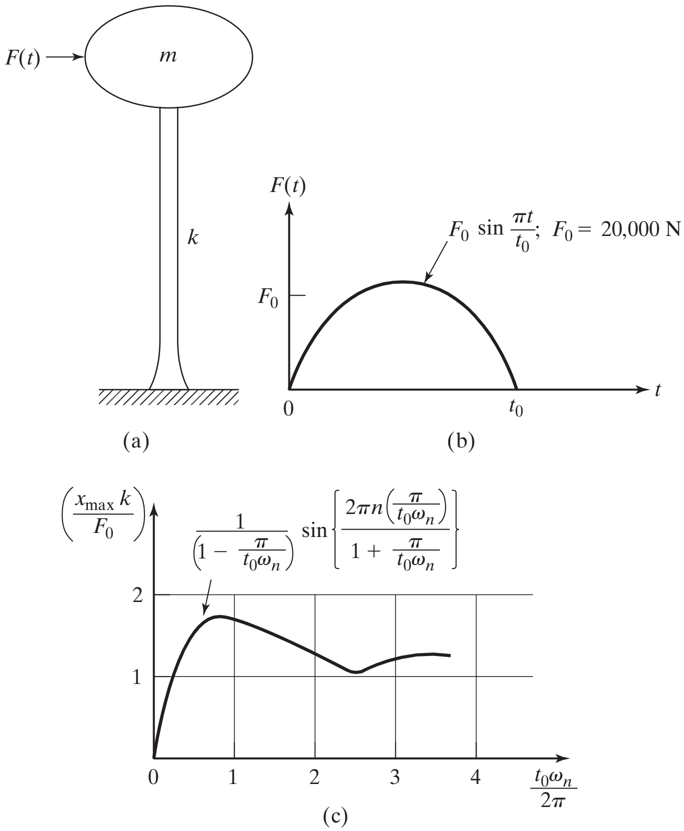

FIGURE 9.48

9.54 An electronic instrument of mass 20 kg is to be isolated from engine vibrations with frequencies ranging from 1000 rpm to 3000 rpm. Find the stiffness of the undamped isolator to be used to achieve a 90% isolation.

9.55 A delicate instrument weighing 200 N is suspended by four identical springs, each with stiffness 50,000 N/m, in a rigid box as shown in Fig. 9.49. The box is transported by a truck. If the truck is subjected to a vertical harmonic motion given by $y(t) = 0.02 \sin 10tm$, find the maximum displacement, velocity, and acceleration experienced by the instrument.

9.56 A damped torsional system is composed of a shaft and a rotor (disk). The torsional stiffness and the torsional damping constant of the shaft are given by $k_t = 6000$ N-m/rad and $c_t = 100$ N-m-s/rad. The mass moment of inertia of the rotor is $J_0 = 5$ kg-m^2. The rotor is subjected to a harmonically varying torque of magnitude $M_t = 500$ N-m, which results in a steady-state angular displacement of 5°. Find the frequency of the harmonically varying torque applied to the rotor and the maximum torque transmitted to the base or support of the system.

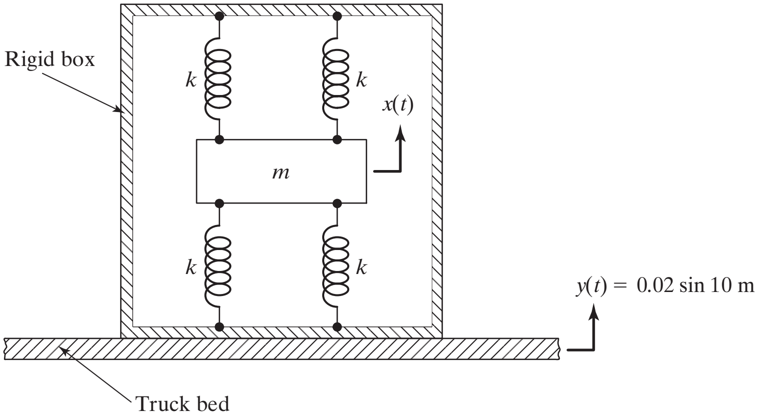

FIGURE 9.49

9.57 The force transmissibility of a damped single-degree-of-freedom system with base motion is given by Eq. (9.106):

$$T_f = \frac{F_t}{kY} = r^2 \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}}$$

where F_t is the magnitude of the force transmitted to the mass. Determine the frequency ratios (r) at which the force transmissibility attains maximum and minimum values. Discuss your results.

9.58 Derive an expression for the relative displacement transmissibility, $\frac{Z}{Y}$, where $Z = X - Y$, for a damped single-degree-of-freedom system subjected to the base motion, $y(t) = Y \sin \omega t$.

9.59 During operation, the compressor unit of a refrigerator, with mass 75 kg and rotational speed 900 rpm, experiences a dynamic force of 200 N. The compressor unit is supported on four identical springs, each with a stiffness of k and negligible damping. Find the value of k if only 15% of the dynamic force is to be transmitted to the support or base. Also, find the clearance space to be provided to the compressor unit.

9.60 An electronic instrument, of mass 20 kg, is to be isolated to achieve a natural frequency of 15 rad/s and a damping ratio of 0.95. The available dashpots can produce a damping constant (c) in the range 10 N-s/m to 80 N-s/m. Determine whether the desired damping ratio can be achieved using a passive system. If a passive system cannot be used, design a suitable active control system to achieve the desired damping ratio.

9.61 A damped single-degree-of-freedom system has a mass (m) of 5 kg, stiffness (k) of 20 N/m, and a damping constant (c) of 5 N-s/m. Design an active controller to achieve a settling time less than 15 s for the closed loop system.

Hint: The settling time is defined by Eqs. (4.68) and (4.69).

9.62 A damped single-degree-of-freedom system has an undamped natural frequency of 20 rad/s and a damping ratio of 0.20. Design an active control system which achieves an undamped natural frequency of 100 rad/s and a damping ratio of 0.8. Assume that the mass, stiffness, and damping constant of the original system remain in place.

Section 9.11 Vibration Absorbers

9.63 An air compressor of mass 200 kg, with an unbalance of 0.01 kg-m, is found to have a large amplitude of vibration while running at 1200 rpm. Determine the mass and spring constant of the absorber to be added if the natural frequencies of the system are to be at least 20 percent from the impressed frequency.

9.64 An electric motor, having an unbalance of 2 kg-cm, is mounted at the end of a steel cantilever beam, as shown in Fig. 9.50. The beam is observed to vibrate with large amplitudes at the operating speed of 1500 rpm of the motor. It is proposed to add a vibration absorber to reduce the vibration of the beam. Determine the ratio of the absorber mass to the mass of the motor needed in order to have the lower frequency of the resulting system equal to 75 percent of the operating speed of the motor. If the mass of the motor is 300 kg, determine the stiffness and mass of the absorber. Also find the amplitude of vibration of the absorber mass.

9.65* The pipe carrying feedwater to a boiler in a thermal power plant has been found to vibrate violently at a pump speed of 800 rpm. In order to reduce the vibrations, an absorber consisting of a spring of stiffness k_2 and a trial mass m_2' of 1 kg is attached to the pipe. This arrangement is found to give the natural frequencies of the system as 750 rpm and 1000 rpm. It is desired to keep the natural frequencies of the system outside the operating speed range of the pump, which is 700 rpm to 1040 rpm. Determine the values of k_2 and m_2 that satisfy this requirement.

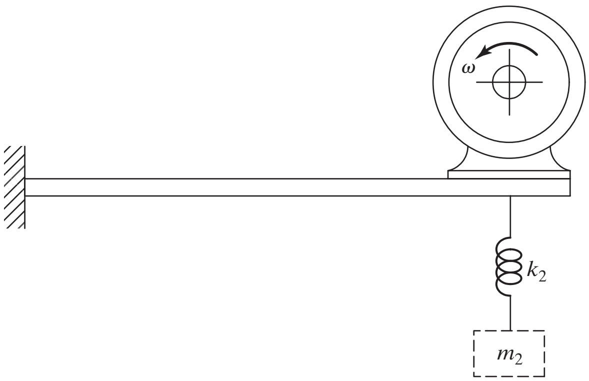

FIGURE 9.50

9.66 A reciprocating engine is installed on the first floor of a building, which can be modeled as a rigid rectangular plate resting on four elastic columns. The equivalent weight of the engine and the floor is 2000 lb. At the rated speed of the engine, which is 600 rpm, the operators experience large vibration of the floor. It has been decided to reduce these vibrations by suspending a spring-mass system from the bottom surface of the floor. Assume that the spring stiffness is $k_2 = 5000$ lb/in. (a) Find the weight of the mass to be attached to absorb the vibrations. (b) What will be the natural frequencies of the system after the absorber is added?

9.67* Find the values of k_2 and m_2 in Problem 9.54 in order to have the natural frequencies of the system at least 30 percent away from the forcing frequency.

9.68* A hollow steel shaft of outer diameter 2 in., inner diameter 1.5 in., and length 30 in. carries a solid disc of diameter 15 in. and weight 100 lb. Another hollow steel shaft of length 20 in., carrying a solid disc of diameter 6 in. and weight 20 lb, is attached to the first disc, as shown in Fig. 9.51. Find the inner and outer diameters of the shaft such that the attached shaft-disc system acts as an absorber.

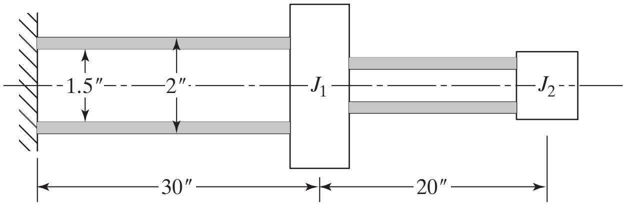

FIGURE 9.51

9.69* A rotor, having a mass moment of inertia $J_1 = 15$ kg-m^2, is mounted at the end of a steel shaft having a torsional stiffness of 0.6 MN-m/rad. The rotor is found to vibrate violently when subjected to a harmonic torque of 300 cos 200t N-m. A tuned absorber, consisting of a torsional spring and a mass moment of inertia (k_{t2} and J_2), is to be attached to the first rotor to absorb the vibrations. Find the values of k_{t2} and J_2 such that the natural frequencies of the system are away from the forcing frequency by at least 20 percent.

9.70 Plot the graphs of (Ω_1/ω_2) against (m_2/m_1) and (Ω_2/ω_2) against (m_2/m_1) as (m_2/m_1) varies from 0 to 1.0 when $\omega_2/\omega_1 = 0.1$ and 10.0.

9.71 Determine the operating range of the frequency ratio ω/ω_2 for an undamped vibration absorber to limit the value of $|X_1/\delta_{st}|$ to 0.5. Assume that $\omega_1 = \omega_2$ and $m_2 = 0.1 m_1$.

9.72 When an undamped vibration absorber, having a mass 30 kg and a stiffness k, is added to a spring-mass system, of mass 40 kg and stiffness 0.1 MN/m, the main mass (40 kg mass) is found to have zero amplitude during its steady-state operation under a harmonic force of amplitude 300 N. Determine the steady-state amplitude of the absorber mass.

9.73 An electric motor, of mass 20 kg and operating speed 1350 rpm, is placed on a fixed-fixed steel beam of width 15 cm and depth 12 cm, as shown in Fig. 9.52. The motor has a rotating unbalance of 0.1 kg-m. The amplitude of vibration of the beam under steady-state operation

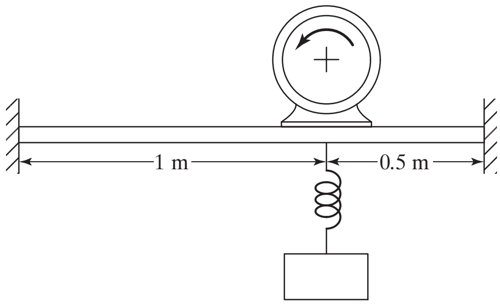

FIGURE 9.52

of the motor is suppressed by attaching an undamped vibration absorber underneath the motor, as shown in Fig. 9.52. Determine the mass and stiffness of the absorber such that the amplitude of the absorber mass is less than 2 cm.

9.74 A bridge is found to vibrate violently when a vehicle, producing a harmonic load of magnitude 600 N, crosses it. By modeling the bridge as an undamped spring-mass system with a mass 15,000 kg and a stiffness 2 MN/m, design a suitable tuned damped vibration absorber. Determine the improvement achieved in the amplitude of the bridge with the absorber.

9.75 A small motor, weighing 100 lb, is found to have a natural frequency of 100 rad/s. It is proposed that an undamped vibration absorber weighing 10 lb be used to suppress the vibrations when the motor operates at 80 rad/s. Determine the necessary stiffness of the absorber.

9.76 Consider the system shown in Fig. 9.53 in which a harmonic force acts on the mass m. Derive the condition under which the steady-state displacement of mass m will be zero.

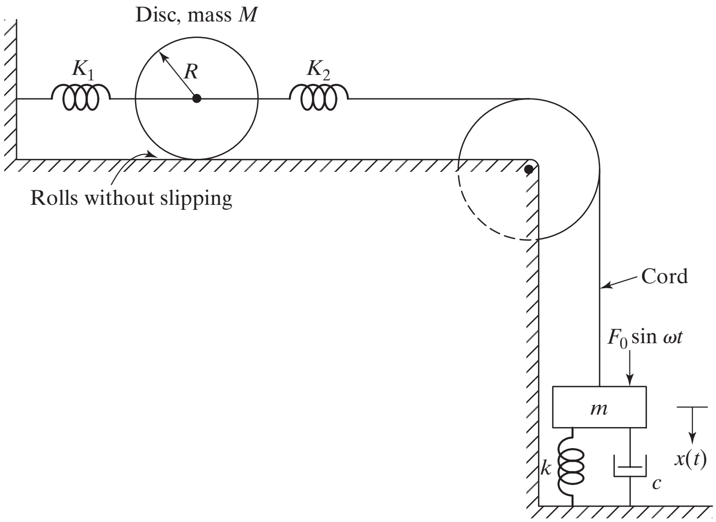

FIGURE 9.53

Section 9.12 MATLAB Problems

9.77 Using MATLAB, plot Eq. (9.94) for $\zeta = 0, 0.25, 0.5, 0.75$, and 1 over the range $0 \le r \le 3$.

9.78 Using MATLAB, plot Eqs. (9.140) and (9.141) for $f = 1$, $\zeta = 0.2, 0.3$, and 0.4, and $\mu = 0.2$ and 0.5 over the range $0.6 \le \omega/\omega_1$.

9.79 Using MATLAB, plot the ratios Ω_1/ω_2 and Ω_2/ω_2 given by Eq. (9.146) for $\omega_2/\omega_1 = 1.5$, 3.0, and 4.5 and $m_2/m_1 = 0$ to 1.

9.80 Using Program 13.m, solve Problem 9.13.

9.81 Write a computer program to find the displacement of the main mass and the auxiliary mass of a damped dynamic vibration absorber. Use this program to generate the results of Fig. 9.38.

DESIGN PROJECT

9.82 Ground vibrations from a crane operation, a forging press, and an air compressor are transmitted to a nearby milling machine and are found to be detrimental to achieving specified accuracies during precision milling operations. The ground vibrations at the locations of the crane, forging press, and air compressor are given by $x_c(t) = A_c e^{-\omega_c \zeta_c t} \sin \omega_c t$, $x_f(t) = A_f \sin \omega_f t$, and $x_a(t) = A_a \sin \omega_a t$, respectively, where $A_c = 20\ \mu$m, $A_f = 30\ \mu$m, $A_a = 25\ \mu$m, $\omega_c = 10$ Hz, $\omega_f = 15$ Hz, $\omega_a = 20$ Hz, and $\zeta_c = 0.1$. The ground vibrations travel at the shear wave velocity of the soil, which is equal to 980 ft/sec, and the amplitudes attenuate according to the relation $A_r = A_0 e^{-0.005r}$, where A_0 is the amplitude at the source and A_r is the amplitude at a distance of r ft from the source. The crane, forging press, and air compressor are located at a distance of 60 ft, 80 ft, and 40 ft, respectively, from the milling machine. The equivalent mass, stiffness, and damping ratio of the machine tool head in vertical vibration (at the location of the cutter) are experimentally determined to be 500 kg, 480 kN/m, and 0.15, respectively. The equivalent mass of the machine tool base is 1000 kg. It is proposed that an isolator for the machine tool be used, as shown in Fig. 9.54, to improve the cutting accuracies [9.2]. Design a suitable vibration isolator, consisting of a mass, spring, and damper, as shown in Fig. 9.54(b), for the milling machine such that the maximum vertical displacement of the milling cutter, relative to the horizontal surface being machined, due to ground vibration from all the three sources does not exceed 5 μm peak-to-peak.

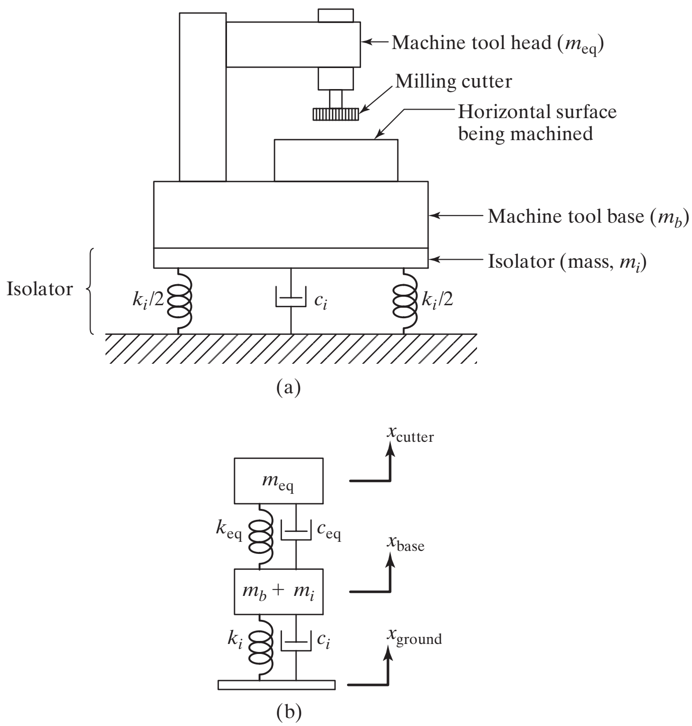

FIGURE 9.54