Agenda

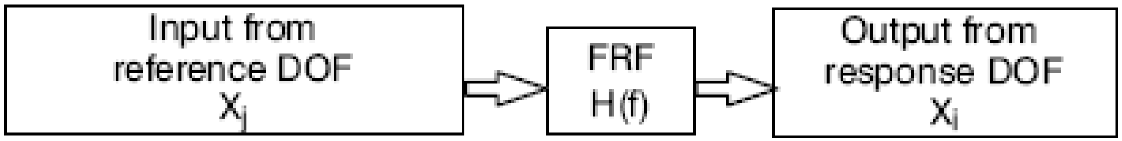
- 1. Introduction to Modal Analysis.
- 2. Modal Testing procedure.
- 3. SDOF and MDOF systems.
- 4. DSP overview: analog to digital conversion, windowing, leakage, aliasing, Fourier transform, spectrum, autopower, PSD, coherence, estimation of FRF.
- 5. Measurement overview: excitation impact, excitation shaker, accelerometer, trigger, pre-trigger, windowing for modal analysis.
- 6. Parameter estimation methods: SDOF and MDOF methods.
- 7. Validation methods: FRF synthesis, MAC, MOC, MPC.
- 8. Test. Lab Modal Analysis Overview
- 9. Practice: participants will practice in several modal analyses of automotive components by using up-to-date instrumentations (front-ends for data acquisition and processing, accelerometers, impact hammers, shakers, modal analysis software).

1. Introduction to Modal Analysis

Due to resonances

FRF

The Frequency Response Function (FRF) is a frequency domain function expressing the ratio between a response (output) signal and a reference (input) signal. The position and direction of the measurements are termed *Degrees Of Freedom* DOFs. An FRF thus always depends on 2 DOFs, the response DOF (numerator) and the reference DOF (denominator).



$$H(f) = \frac{X_i}{X_j}$$

For modal purposes the response signal is most commonly the acceleration at the response DOF due to a force input at another. In this case peaks in the FRF indicate that low input levels generate high response levels (resonances), while minima indicate low response levels, even for high inputs (anti-resonances).

What is modal analysis?

Modal is the process of characterizing the dynamic behavior of a structure in terms of its modal parameters

Frequency
Damping
Mode Shapes

What is Experimental modal analysis

Experimental modal testing involves the acquisition of point to point frequency response functions (FRF) at a set of points used to define the dynamic model.

There are many digital signal processing considerations that must be recognized when collecting FRFs.

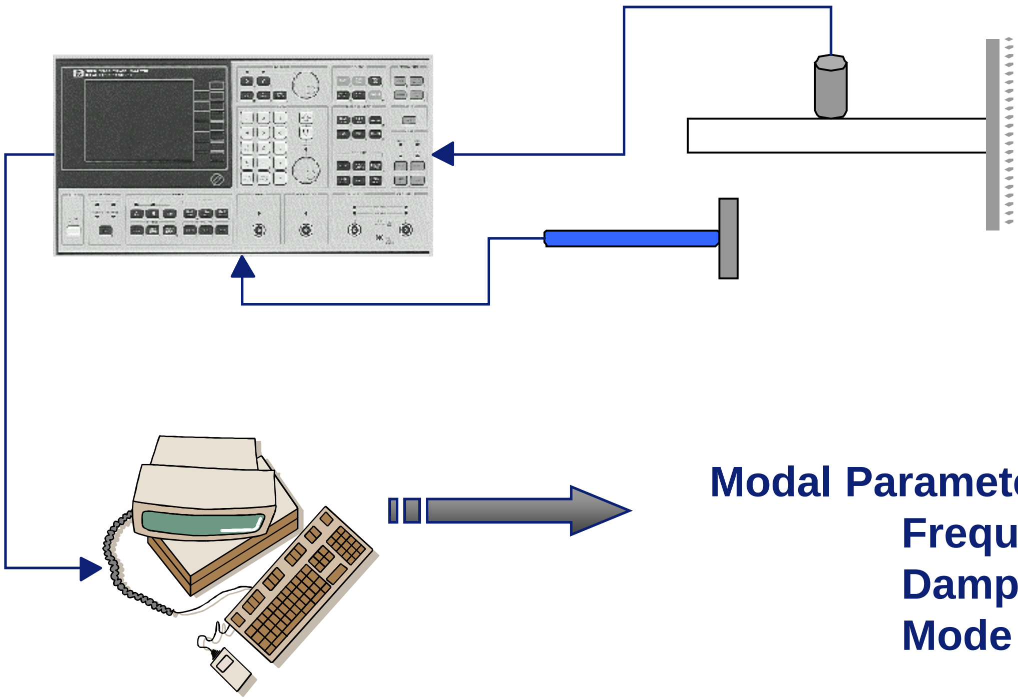
In addition, there many different excitation techniques that can typically be utilized in the estimation of the FRF.

Modal parameters are extracted from the measured FRF in a process referred to as curvefitting.

A variety of techniques are available to assit in the validation of the modal model.

Experimental Modal Analysis
Critical Elements

- Excitation techniques (hammer, shaker, operational excitation,…)
- DSP
- Frequency Response Functions (FRFs)
- Curve-fitting / (modal) parameter estimation
- Validation (MAC, MPC, MOV, synthetized FRFs, …)



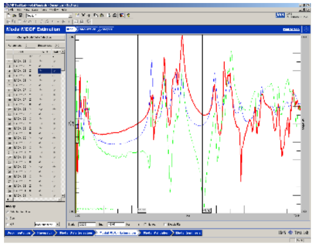
Modal Parameters:
Frequency
Damping
Mode shapes

Experimental Modal Analysis

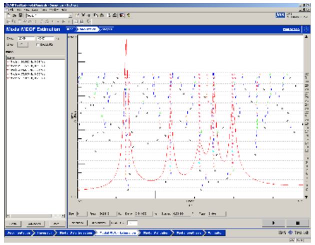
- Required knowledge for a successful modal test

Test Setup	Purpose of the test • Knowledge of expected modes of the system • Expected results • Transducers and excitation devices
Make measurements	• Knowledge of digital signal processing, parameters such as leakage, windows, time and frequency relationships, FFT, excitation techniques
Identify Parameters	• Knowledge of modal theory • Knowledge of modal parameter estimation techniques
Verify/document results	• Knowledge of modal theory • Synthesis. MAC

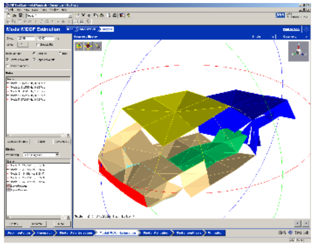
Experimental Modal Analysis



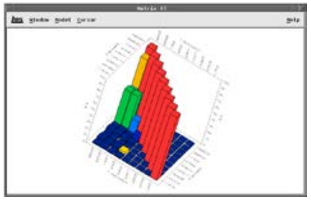
1. Measure FRFs



2. Estimate poles



3. Estimate mode shapes

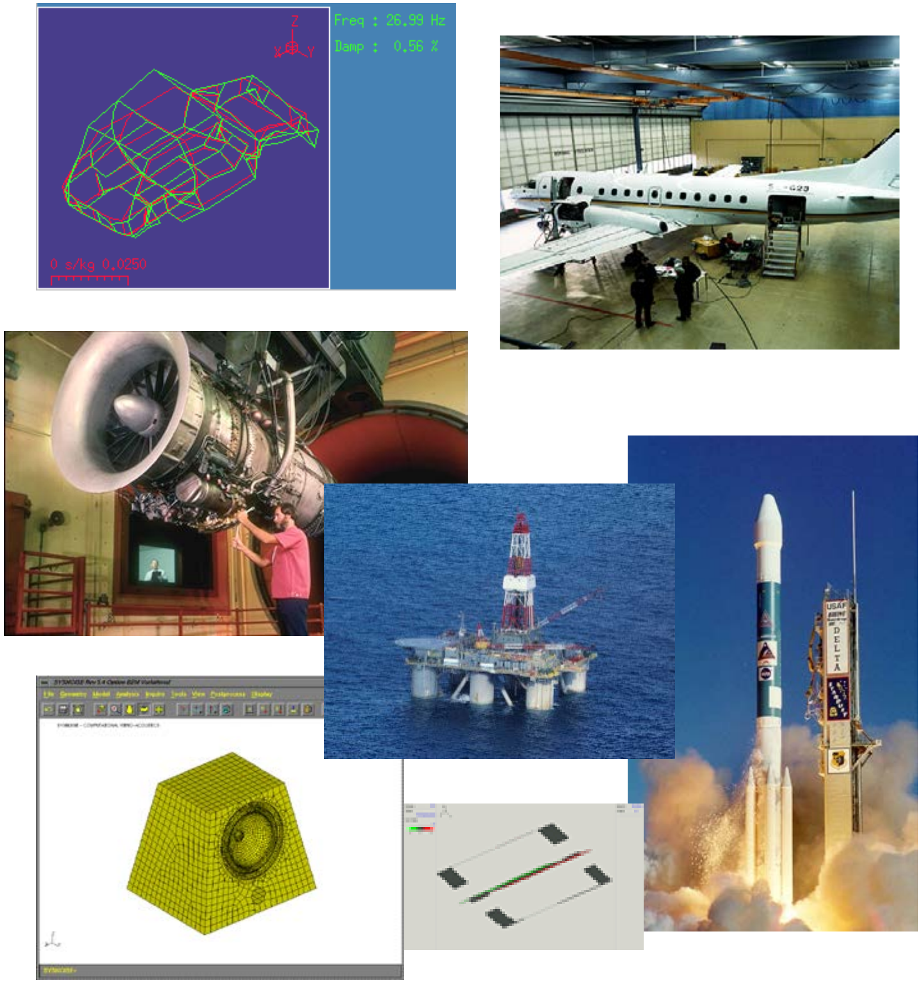


4. Validate

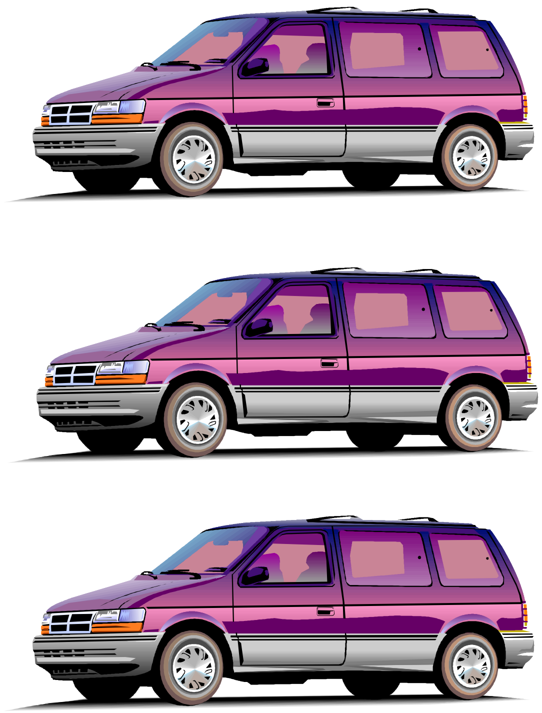
5. Use modal parameters
   - Troubleshooting
     - Check frequencies
     - Qualitative descriptions of mode shapes
   - Simulation and prediction
   - Design optimisation
   - Diagnostics and health monitoring
   - Finite Element model verification/improvement
   - Hybrid system model building

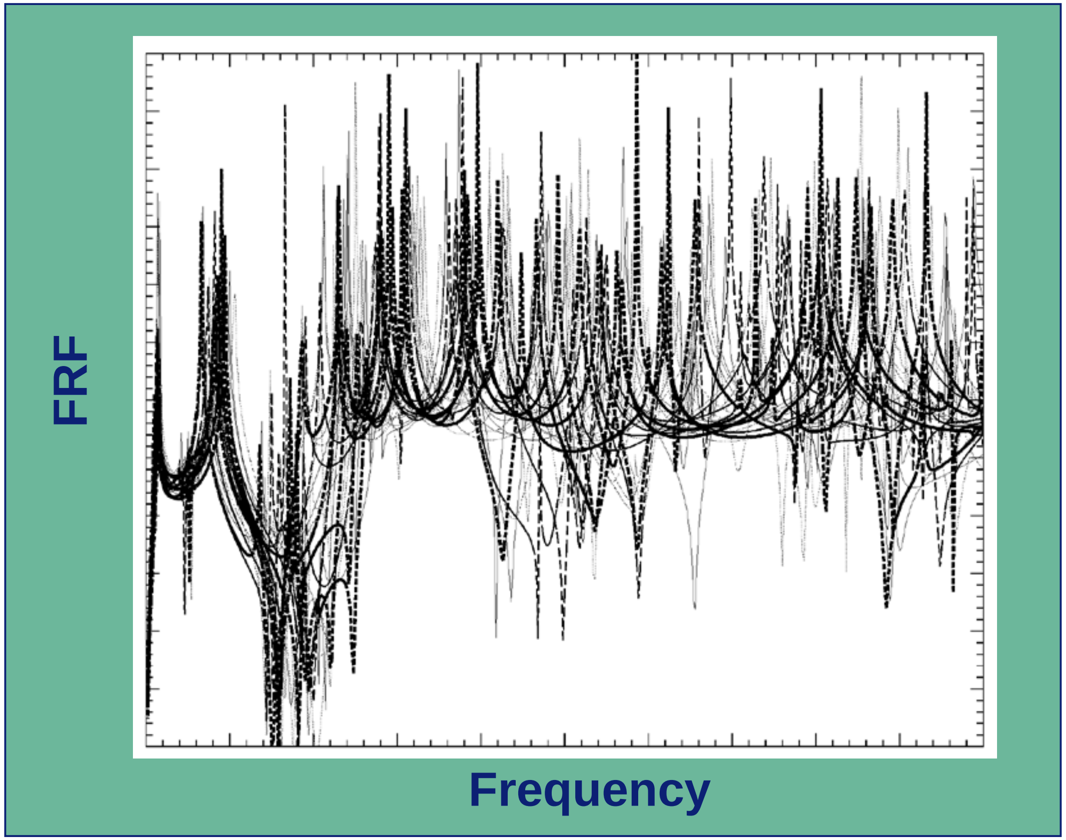
Experimental Modal Analysis Applications

- Car body, fully equipped car, car interior cavity, …
- Aircraft fuselage, full aircraft, interior cavity, …
- Components: engine block, suspension systems, brakes, antennas
- Processing plants: piping systems, equipment mounting
- Mechanical equipment: turbine blades, compressors, pumps
- Audio & household: CD-drive, washing machine, loudspeakers
- Infrastructure: bridge, off-shore platforms



- be aware of variability (and uncertainty) issue





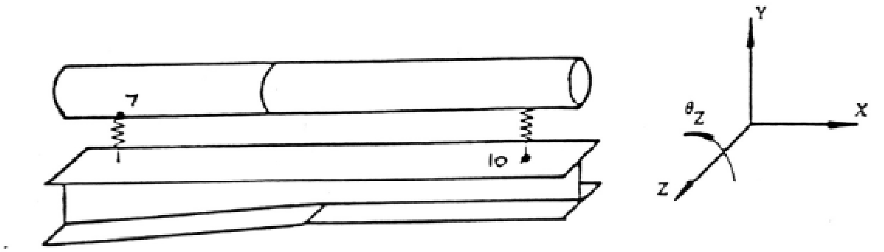
response variability of nominally identical vehicles

Uncertainty in Experimental/FE Structural analysis

DYNAS survey [2]

During the late eighties a survey, codenamed DYNAS (DYNamic Analysis of Strucures), was organised in order to access the reliability of structural dynamic analysis capabilities using Finite Element Methods.

Technical drawings of a I-beam/cylinder/spring structure were sent to a number of organisations who were invited to undertake its dynamic analysis using their own well-established in-house FEM techniques.



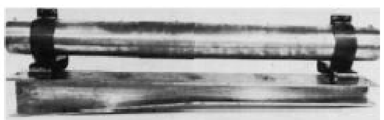
DYNAS I-beam/cylinder/spring DYNAS structure.

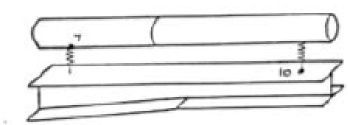
Among the several conclusions drawn from the survey it was noticed that *"what is seen is that inaccurate predictions can be made by poor analysis methods and this is probably the essence of the conclusions throughout. That being the case, there is clearly some need for procedures which could indicate the quality or reliability of a given prediction so that a good analysis can be identified (as, indeed, can a poor one)".*

Uncertainty in Experimental Structural analysis

Time Spent on Analysis

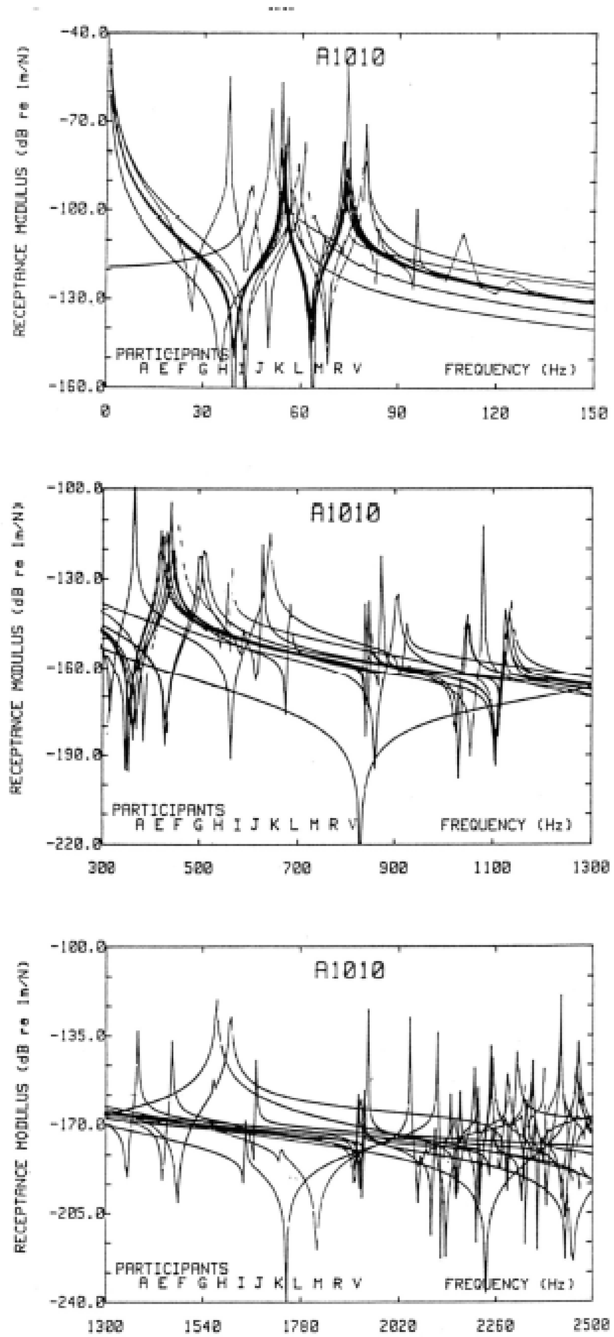
Participant	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
Man hours	50	80	75	300	200	100	25	200	80	110	NS	38





Natural frequencies (Hz)

Participant	S1	S2	S3	B1	B2	B3	C1	C2	C3	NOM
I	54	74	—	503	1464	2552	630	1670	—	12
II	45	61	—	448	1130	2110	518	1380	—	42
III	54	73	67	512	1380	2385	561	1553	2503	12
IV	57	35	80	434	1080	1645	544	1436	—	22
V	59	—	188	644	1608	2313	566	1565	2464	14
VI	54	74	84	440	1125	1920	530	1355	2384	15
VII	54	73	—	370	908	1580	553	1126	—	11
VIII	55	75	—	570	1574	2573	617	1588	—	8
IX	55	75	96	424	1052	1571	534	1435	—	24
X	55	75	110	428	1048	1559	545	1478	—	23
XI	55	75	123	445	1137	1935	537	1365	2379	18
XII	38	51	74	461	1140	1708	499	1358	2389	27
Min	38	35	67	370	908	1571	499	1126	2379	8
Max	59	75	188	644	1608	2573	630	1670	2503	42
Mean	54	68	103	473	1220	1998	553	1442	2420	19
Measured	59	80	118	426	1072	2002	532	1470		



Driving point receptances at position 10.

Modal analysis….which type?

- Analytical Modal analysis
- Finite Element Modal analysis
- Experimental Modal analysis

Experimental Modal Analysis is based on the use of experimentally determined data, which is obtained from a test structure. Modal Parameter Estimation methods are used to obtain modal parameters of the structure from the measured data.

Analytical Modal Analysis is based on the use of differential equations of motion of a structure. The resulting Equations are then decomposed into eigenvalues (frequencies) and eigenvectors (mode shapes).

Analytical Modal Analysis

- Only for simple cases
- E.g.: homogeneous cantilever beam
  - Dynamic equilibrium

$$\frac{d^4 w}{dx^4} - \omega^2 \frac{m}{EI} w = 0$$

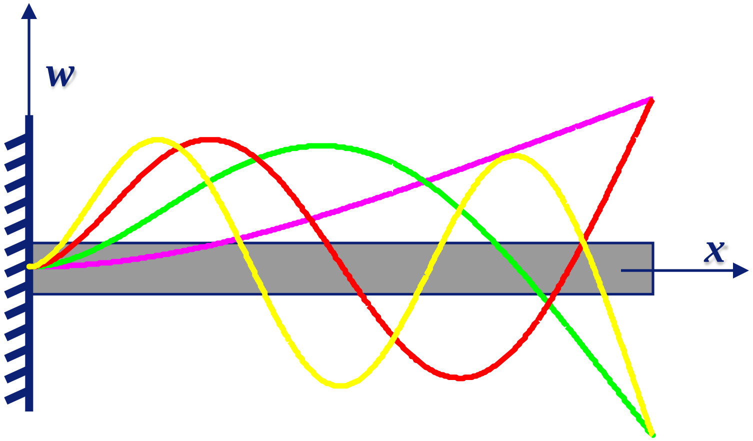
  - Eigenvalue solution
    - Transcendent equation

$$-\cos\mu = \frac{1}{\cosh\mu} \qquad \mu^4 = \frac{\omega^2 m l^4}{EI}$$

    - Numerical solutions

$$\mu_{1,2,3,\ldots} = 1.875,\ 4.694,\ 7.855,\ \ldots$$

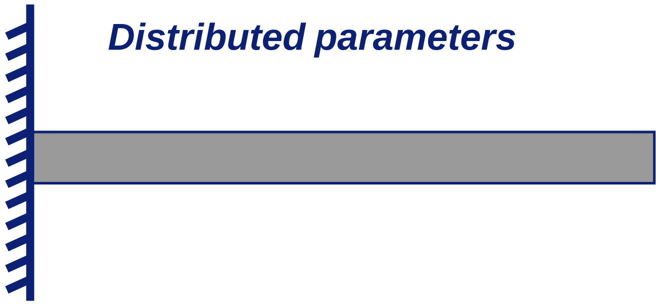
- Analytical mode shapes (Sum of sine, cosine, hyberbolic sine and hyberbolic cosine)



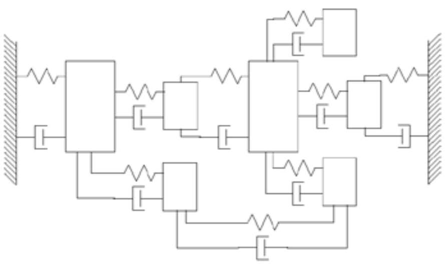
Analytical Modal Analysis

- OK for

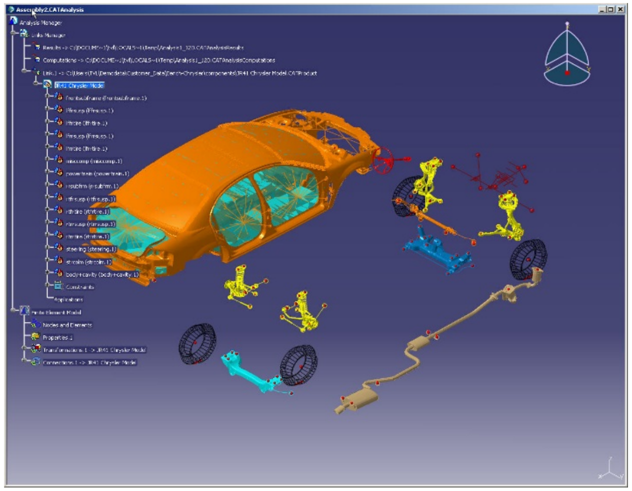
Distributed parameters

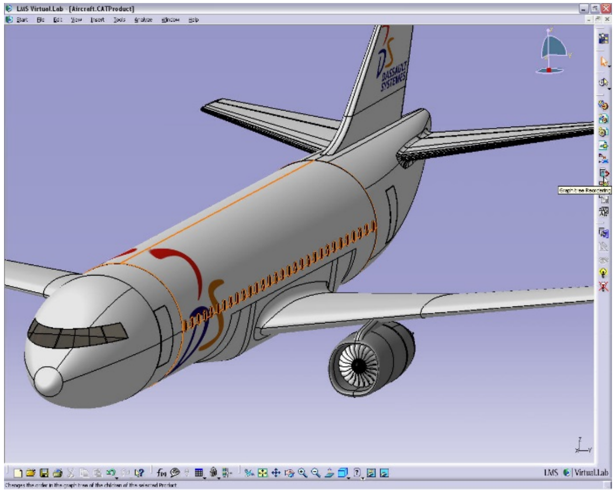


Lumped parameters



- But what about real-life structures?





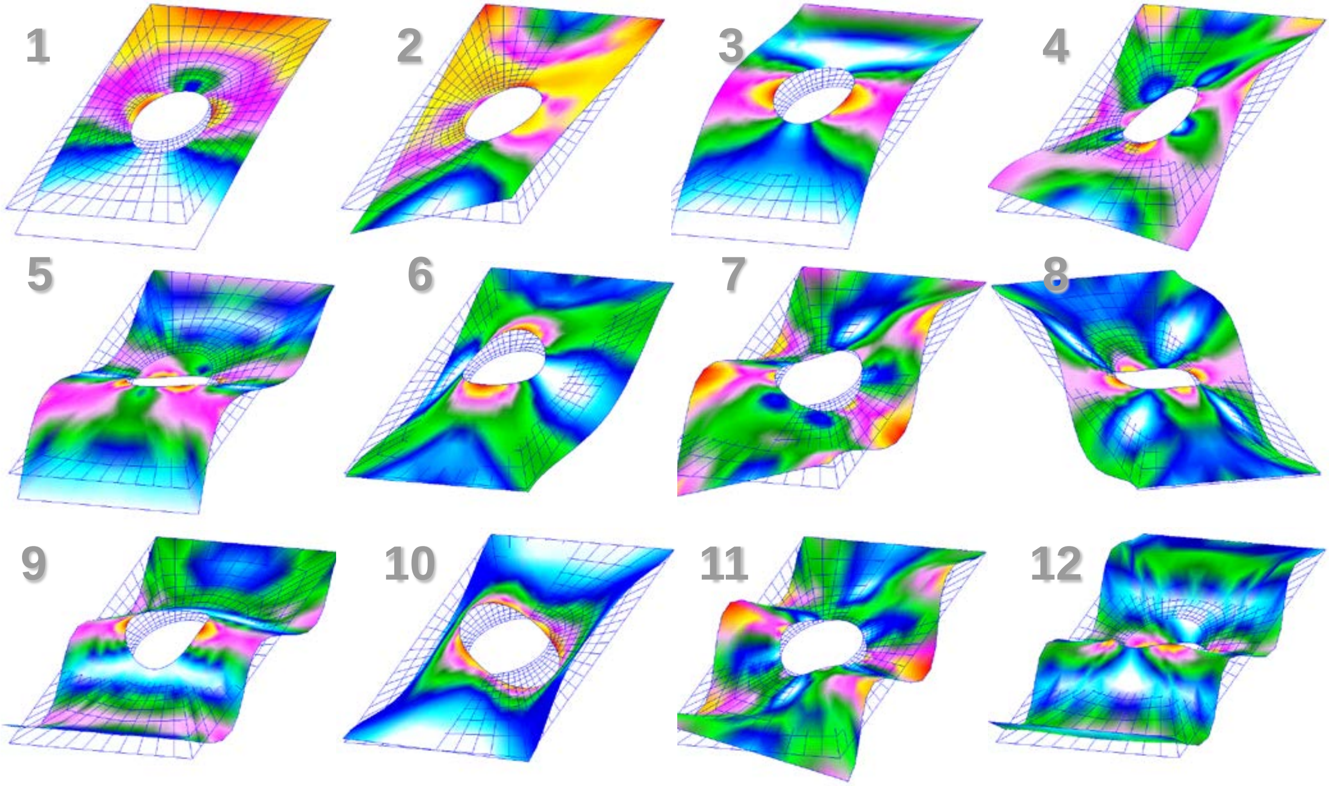
  - We have to look for other approaches
    - Virtual prototype: ***Finite Element Modal Analysis***
    - Physical prototype: ***Experimental Modal Analysis***

Finite Element Modal Analysis

- Important modelling issues
  - Discretization (meshing) is a critical phase as it determines accuracy but also calculation time
  - Automatic meshing procedures are developed, but it remains to a large extent a manual, time consuming effort
  - Element selection
  - Material parameter selection
  - Different numerical solvers exist, optimized for specific calculations (static, dynamic, non-linear…)
  - Possible to update from experiments

Finite Element Modal Analysis

- Example: plate with hole: mode shapes (displacements and stresses) for first 12 natural frequencies



Experimental Modal Analysis *vs.* Finite Element Modal Analysis

- Experimental

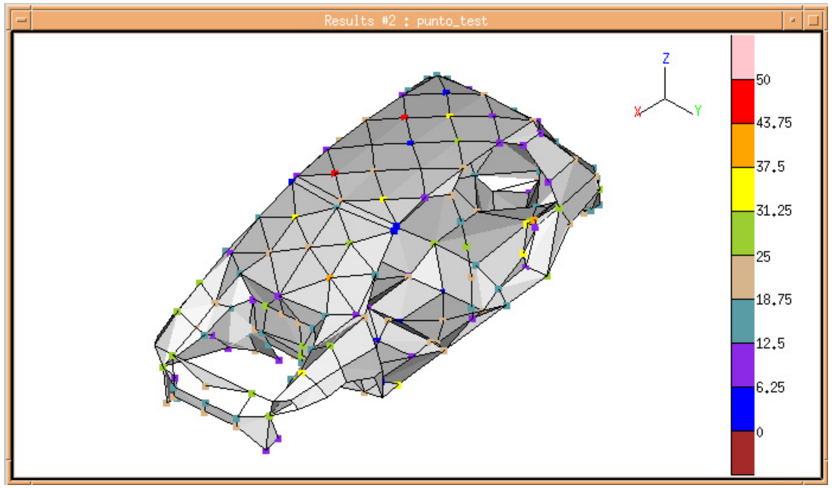
$$H(\omega) \Rightarrow \omega_{nk}, \xi_k, \{\phi\}_k, Q_k$$

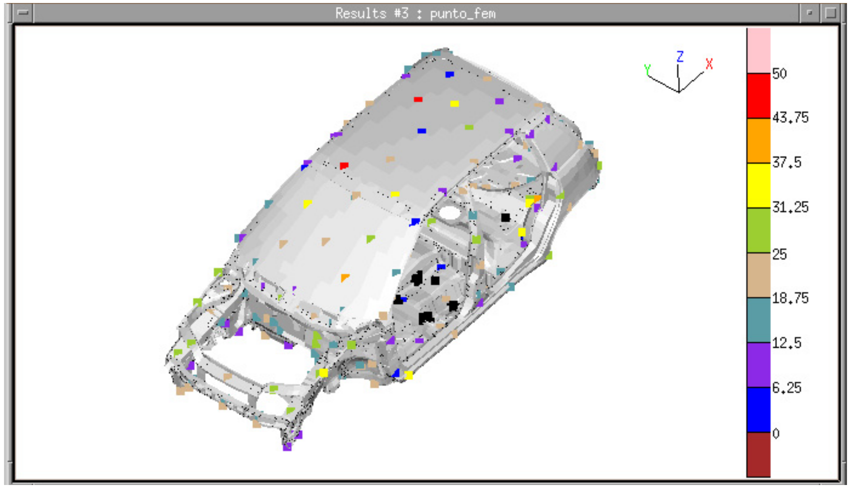
  - Requires prototype
  - Very fast (1-5 days)
  - Very accurate for frequency
  - More reliable for damping
  - Limited number of points

- Numerical

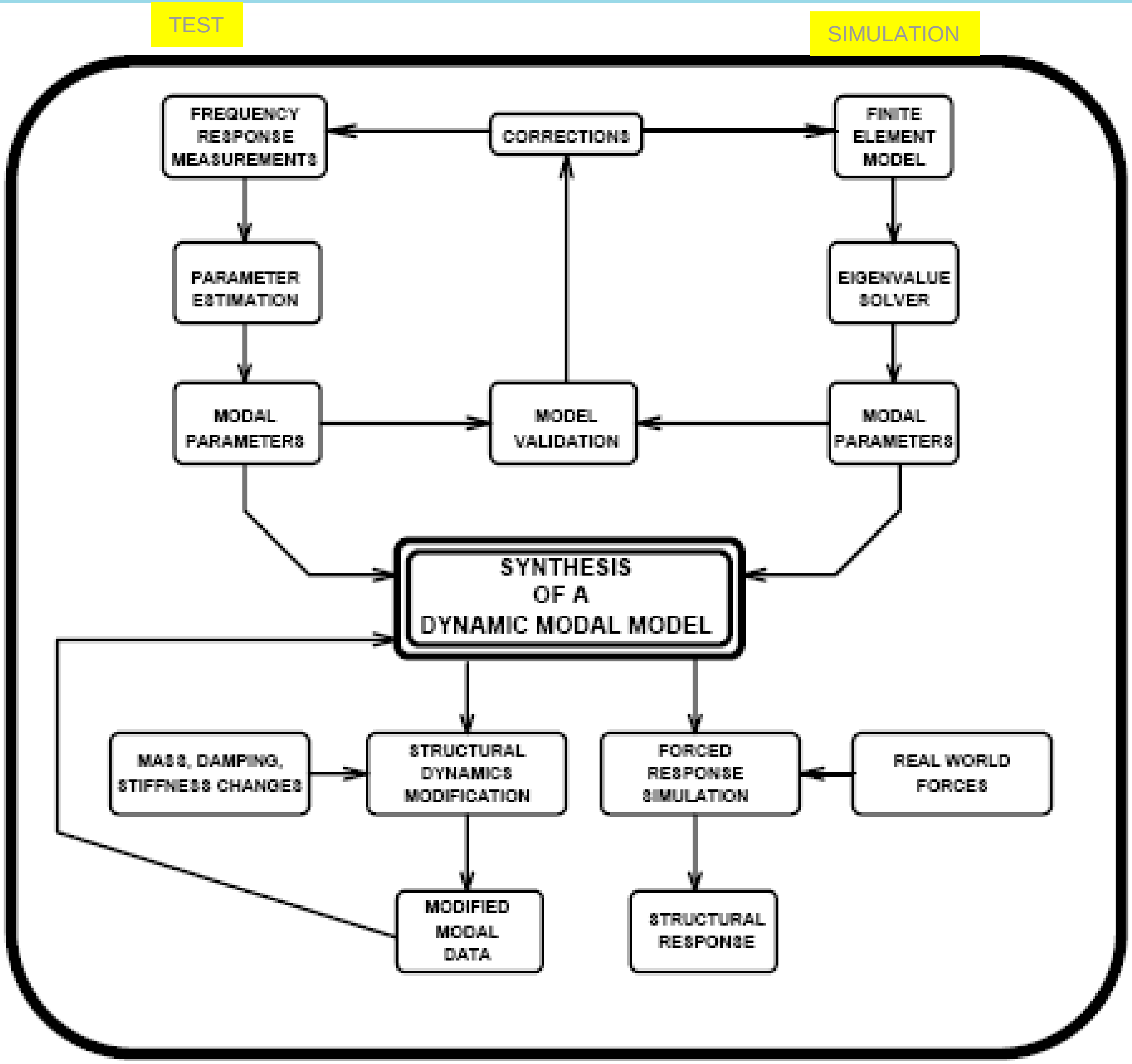
$$M, C, K \Rightarrow \omega_{nk}, \xi_k, \{\phi\}_k, Q_k$$

  - Requires FE model
  - Many days/weeks
  - Fast alternative evaluation
  - A lot of model uncertainties (joints / damping / …)
  - High number of points





The complete process – SIMULATION&TEST



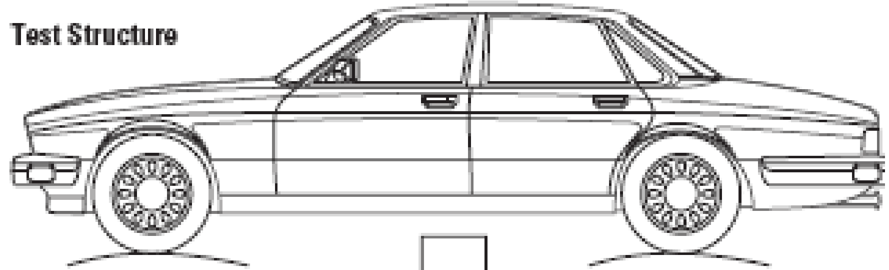
02 Modal testing procedures

Modal analysis core

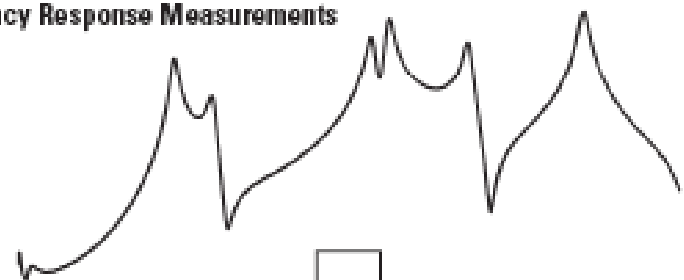
What is the purpose of the test?

- **Develop a dynamic model for structural modification**
- **Correlate with a finite element model**
- **Monitoring structure health**
- **Develop dynamic model for prediction**
- **Adjust an existing finite element model**

Test Structure



Frequency Response Measurements



Curve Fit Representation

I response point

j excitation point

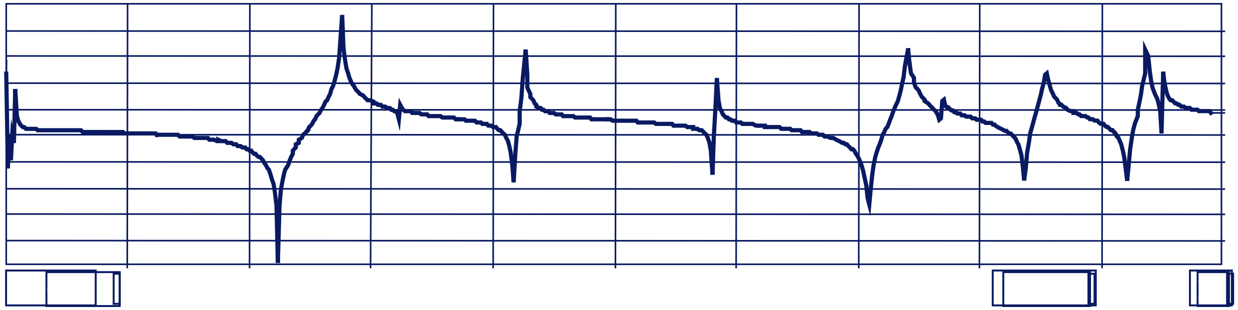
$$H_{ij}(j\omega) = \sum_{k=1}^{n} \frac{A_{ijk}}{j\omega - \lambda_k} + \frac{A_{ijk}^*}{j\omega - \lambda_k^*}$$

Modal Parameters

ω — Frequency
ζ — Damping
$\{\phi\}$ — Mode Shape

Experimental Modal Analysis
Test Setup

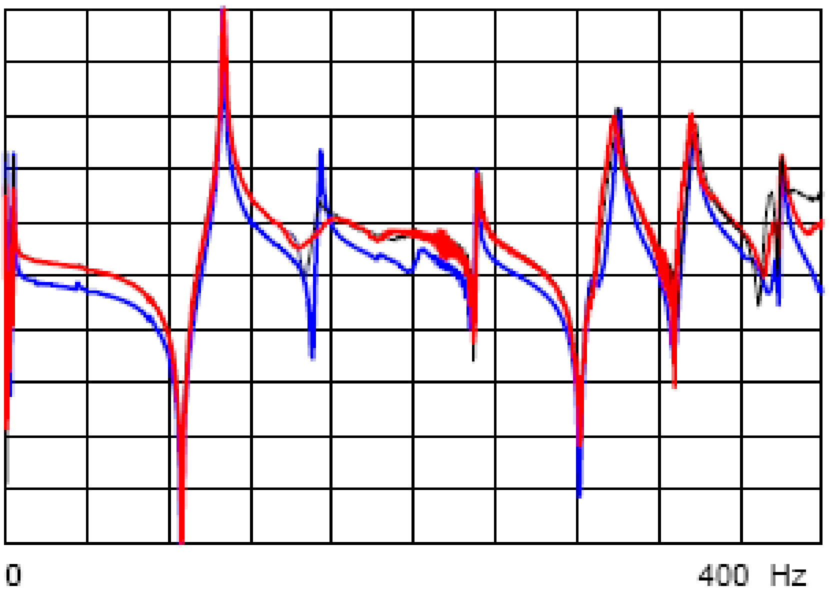
- Frequency range of interest

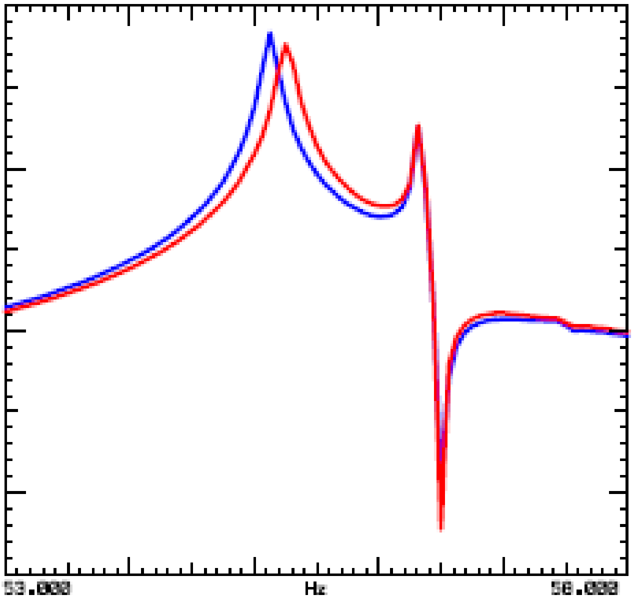


  - Attention to the frequency content of the excitation force during operation
  - Attention to the modal truncation of higher frequencies that could have an effect on the accuracy of the dynamic model

Experimental Modal Analysis
Test Setup

Identify the system characteristics





Is the system linear? Does reciprocity hold true?

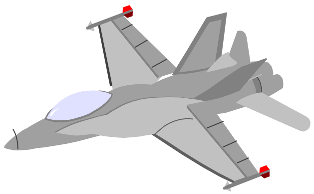
Is the measurement repeatable?

Will I get the same measurement tommorrow?

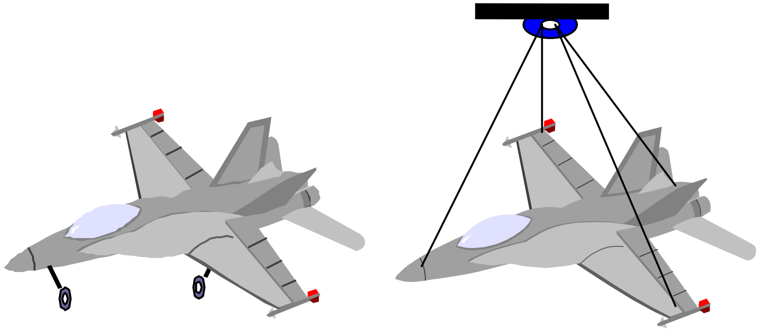
Experimental Modal Analysis
Test Setup

- Boundary conditions
  - How will the structure be “fixtured” for the test?
  - Is the structure to be tested free-free or fixed?
  - Does it matter?
  - Are the actual boundary conditions to be simulated?
  - Is data to be correlated with a model?
  - How is the excitation to be applied to the structure?

Flight conditions



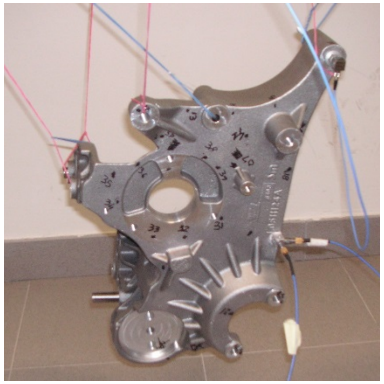
Modal test conditions

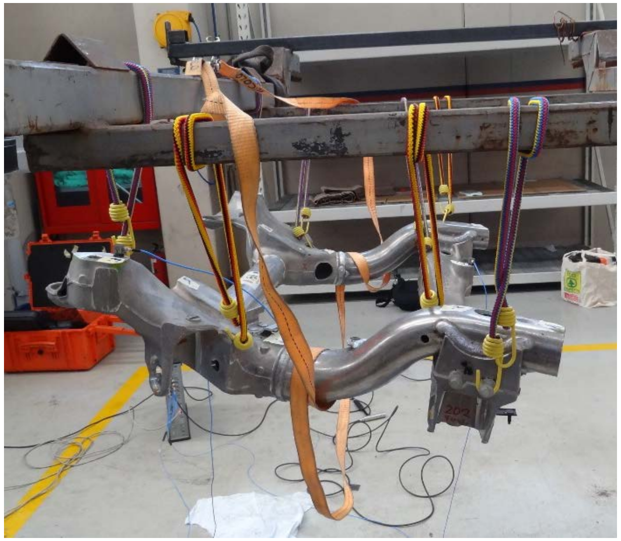


Experimental Modal Analysis
Test Setup

- Boundary conditions

Free-free conditions (bungee cords, elastic springs, air springs)

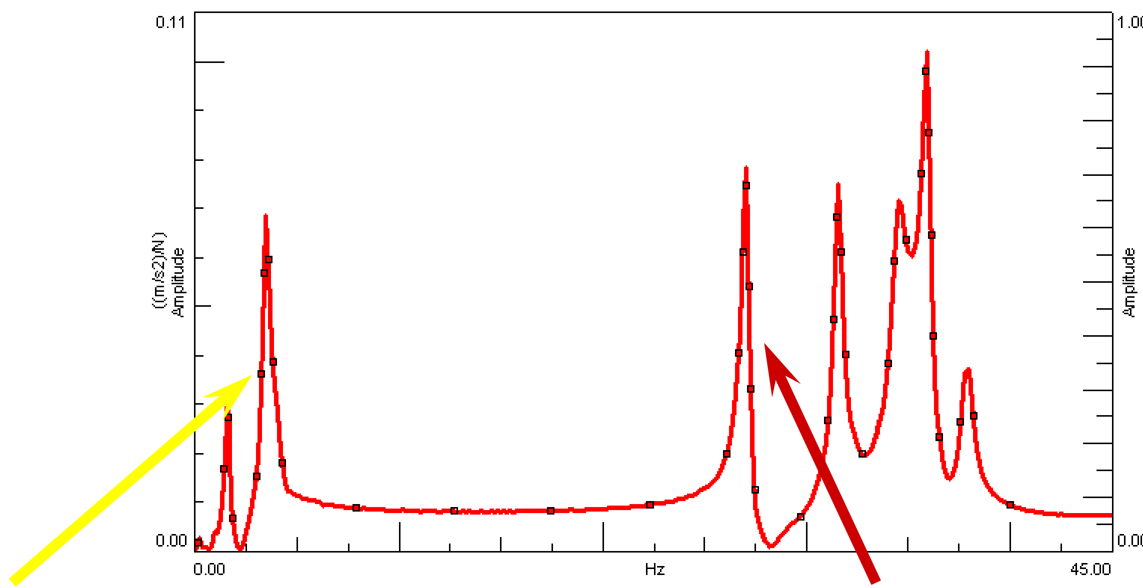




Boundary Conditions

- Fixed boundary conditions
  - Difficult to realise
    - Flexibility of fixtures
    - Added damping
    - Environmental noise

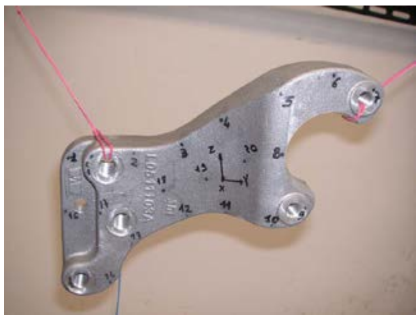
- Free-free suspension
  - In practice: almost free-free
    - Soft spring, elastic cord
    - Soft cushion

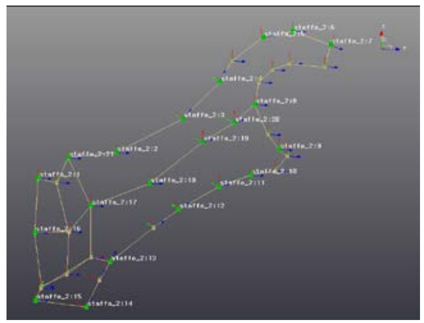


Rigid body mode frequency < 10 % of first flexible mode

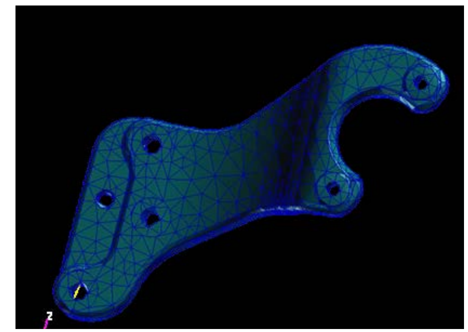
Experimental Modal Analysis
Test Setup

- Boundary conditions: the correct procedure for model validation



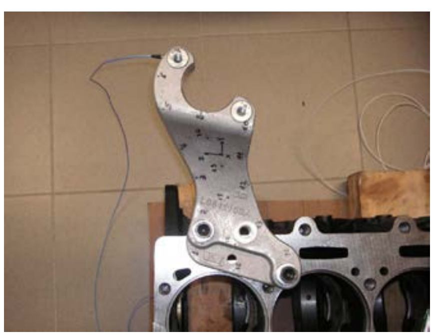


Free-free EMA

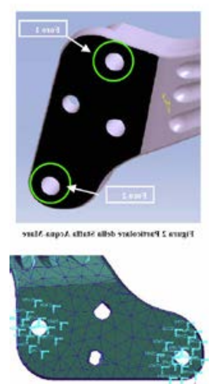


Modal analysis on Free-free FEM

Verify denisty and Young’s modulus



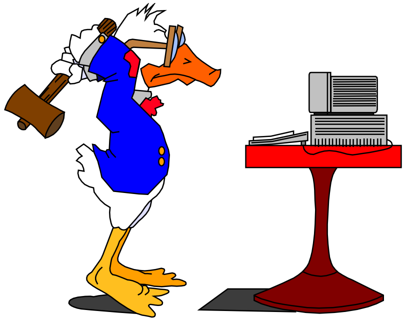
Clamped EMA

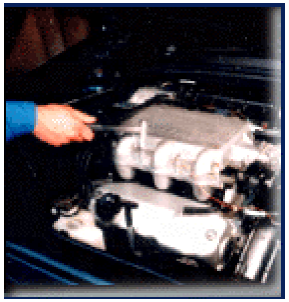


Modal analysis on
Clamped FEM

Verify connections to support structure (rigid connections or flexible ones)

Experimental Modal Analysis
Test Setup



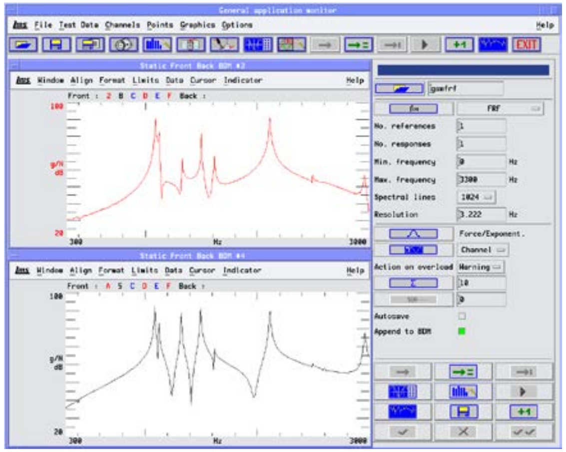


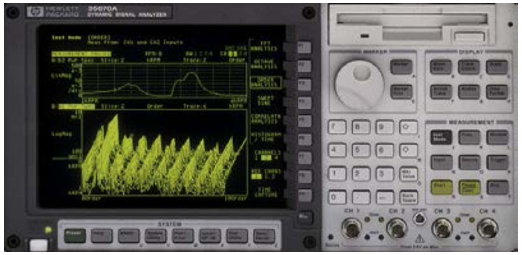
- Excitation techniques
  - Will impact, step relaxation, or shaker excitation be used?
  - Will the measurements be made using operating excitations?
    - road simulation, flight simulation, wind excitation
  - Will unusual excitations be employed?
    - loudspeaker, gun shot, etc
  - What advantage is there to using SISO *vs.* MIMO techniques?

Experimental Modal Analysis
Test Setup

- Measurement system
  - Frequency range and resolution
  - Number of averages
  - Time domain weighting (windows)
  - Trigger conditions
  - System Calibration

High # channels



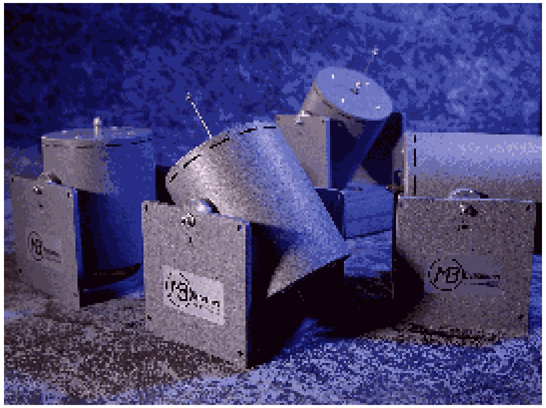


Low # channels

Experimental Modal Analysis
Test Setup

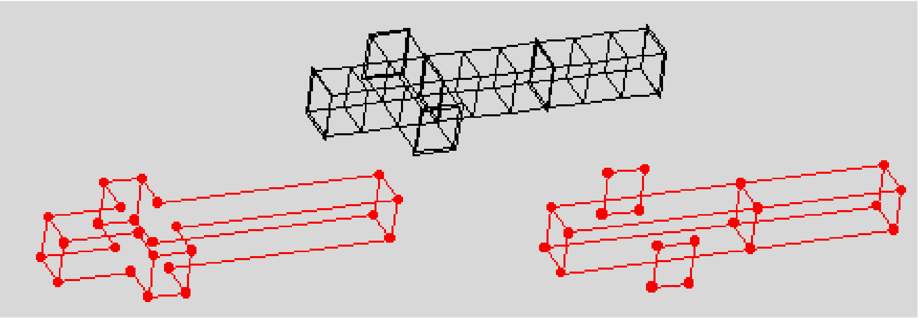
- Transducers and signal conditioning
  - Identify the transducers and signal conditioning to be used
  - Force range of the of the hammer or shaker to be used
  - Range and mass of the accelerometers to be used



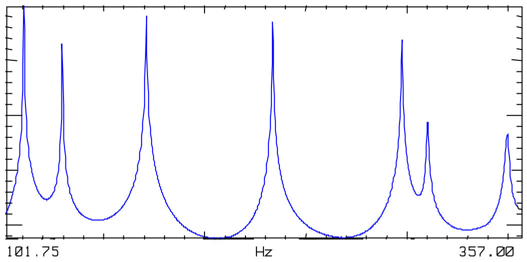
Experimental Modal Analysis
Identify Measurements DOFs

- Data to be acquired must include:
  - Points where the input force is applied
  - Points where response is needed
  - Enough spatial resolution to avoid an aliased view of the mode shapes
- All other points are for your viewing pleasure only!!

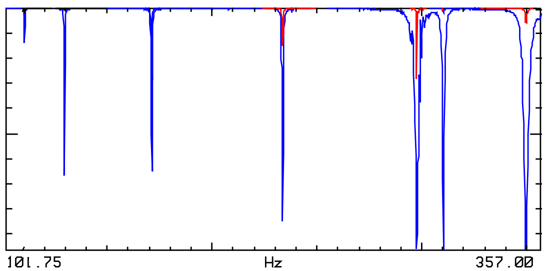


Identify the Modes

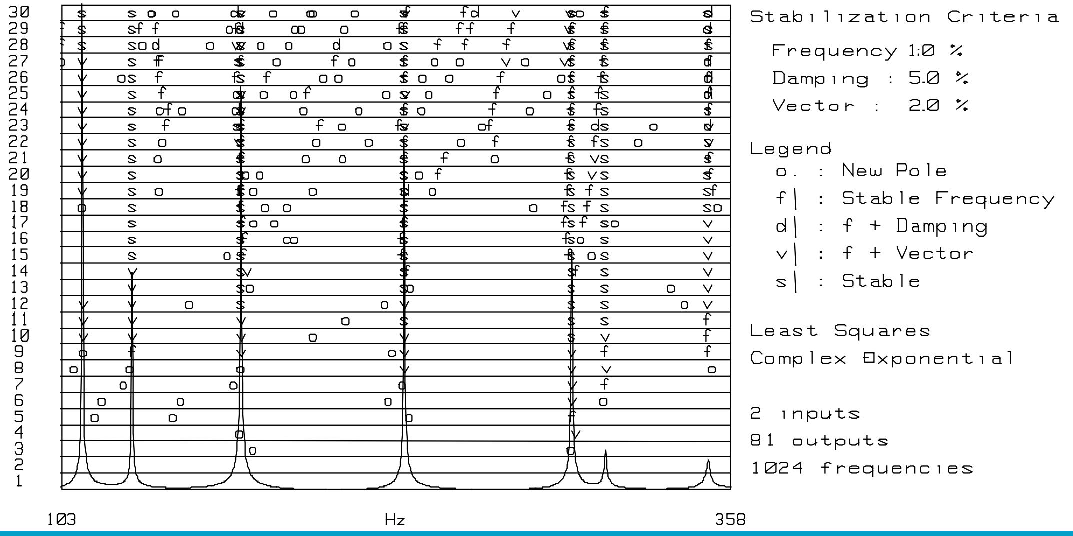
Summation function



Mode Indicator Function



Stability Diagram

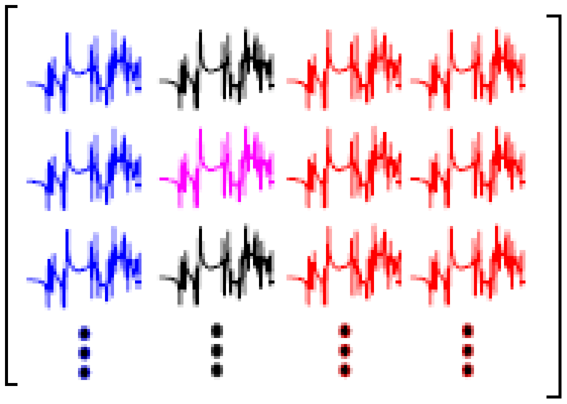


Identify Modal Parameters

Should SDOF or MDOF estimation be used?

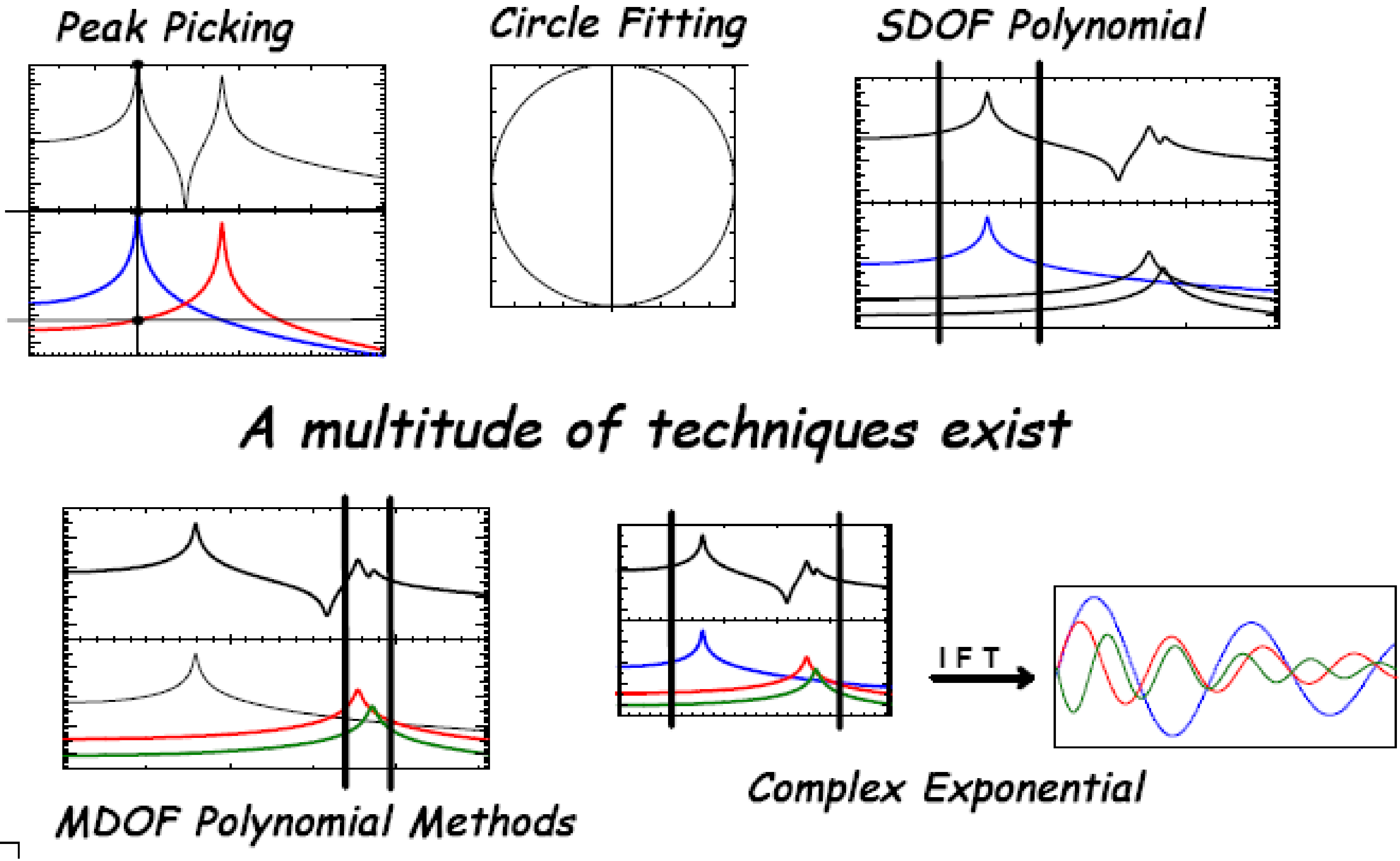
Should TIME or FREQUENCY domain estimation be used?

Should LOCAL or GLOBAL or PREF estimation be used?

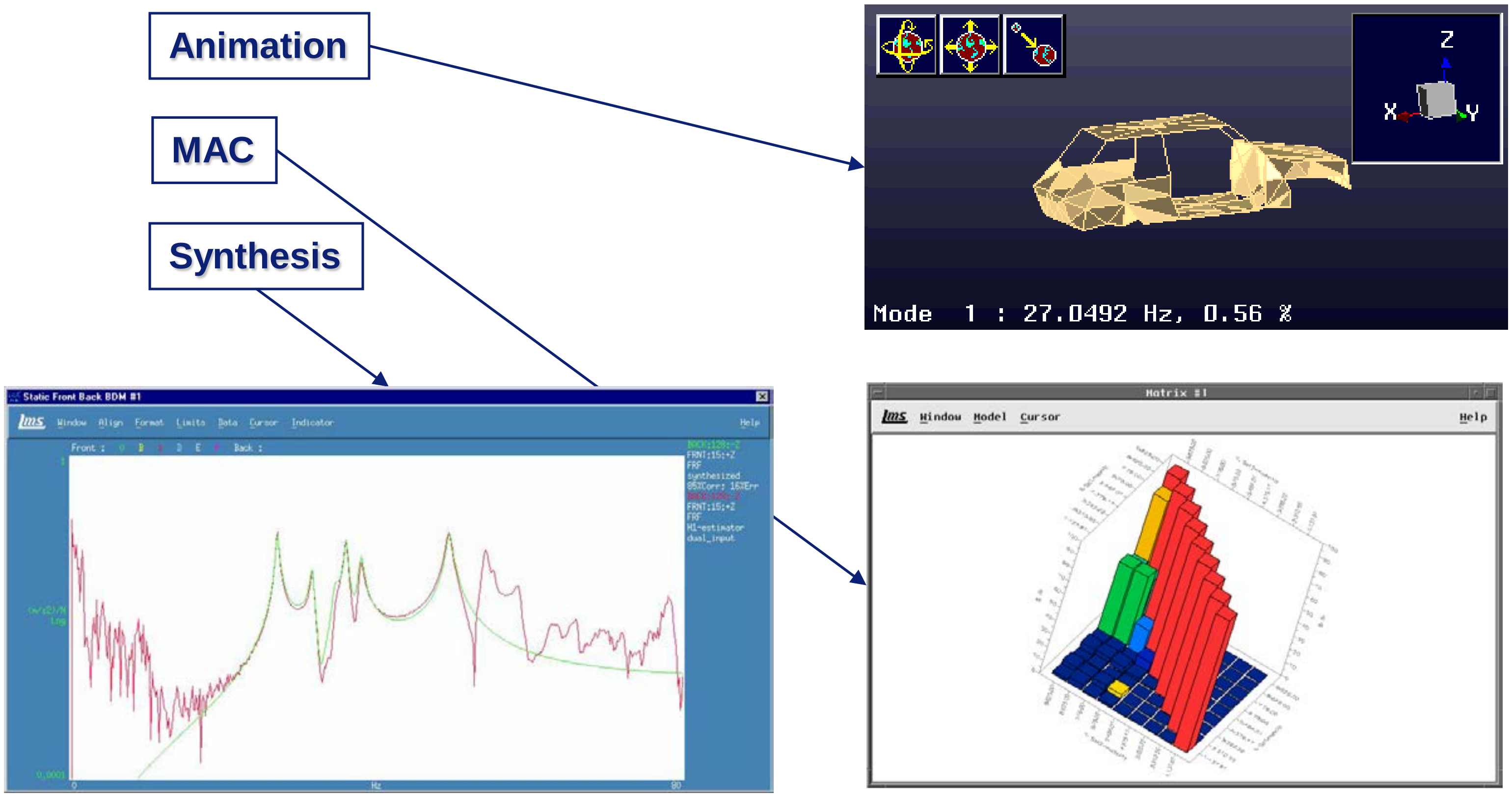


FITTER	TYPE	SDOF or MDOF	RESIDUAL	DAMPING
COINCIDENT	LOCAL	SDOF	NO	NO
QUADRADTURE	LOCAL	SDOF	NO	NO
PEAK	LOCAL	SDOF	NO	NO
POLYNOMIAL	LOCAL	SDOF/MDOF	YES	YES
Global F&D	GLOBAL	SDOF/MDOF	YES	YES
Global RESIDUE	GLOBAL	SDOF/MDOF	YES	YES
POLYREFERENCE	GLOBAL	SDOF/MDOF	YES/NO	YES

Estimate modal parameters

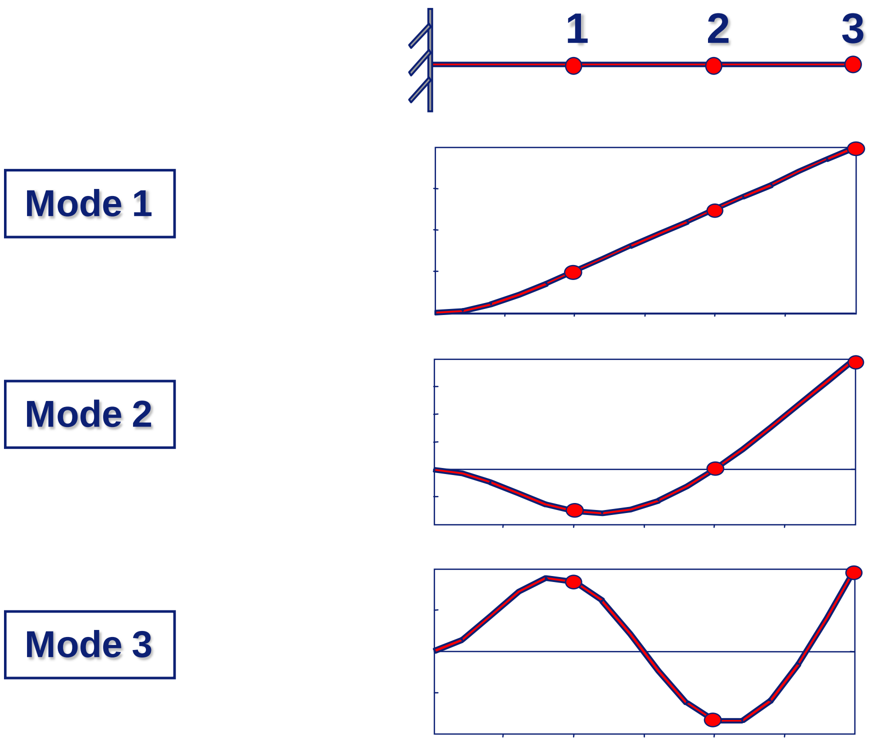


Experimental Modal Analysis
Validate the Model

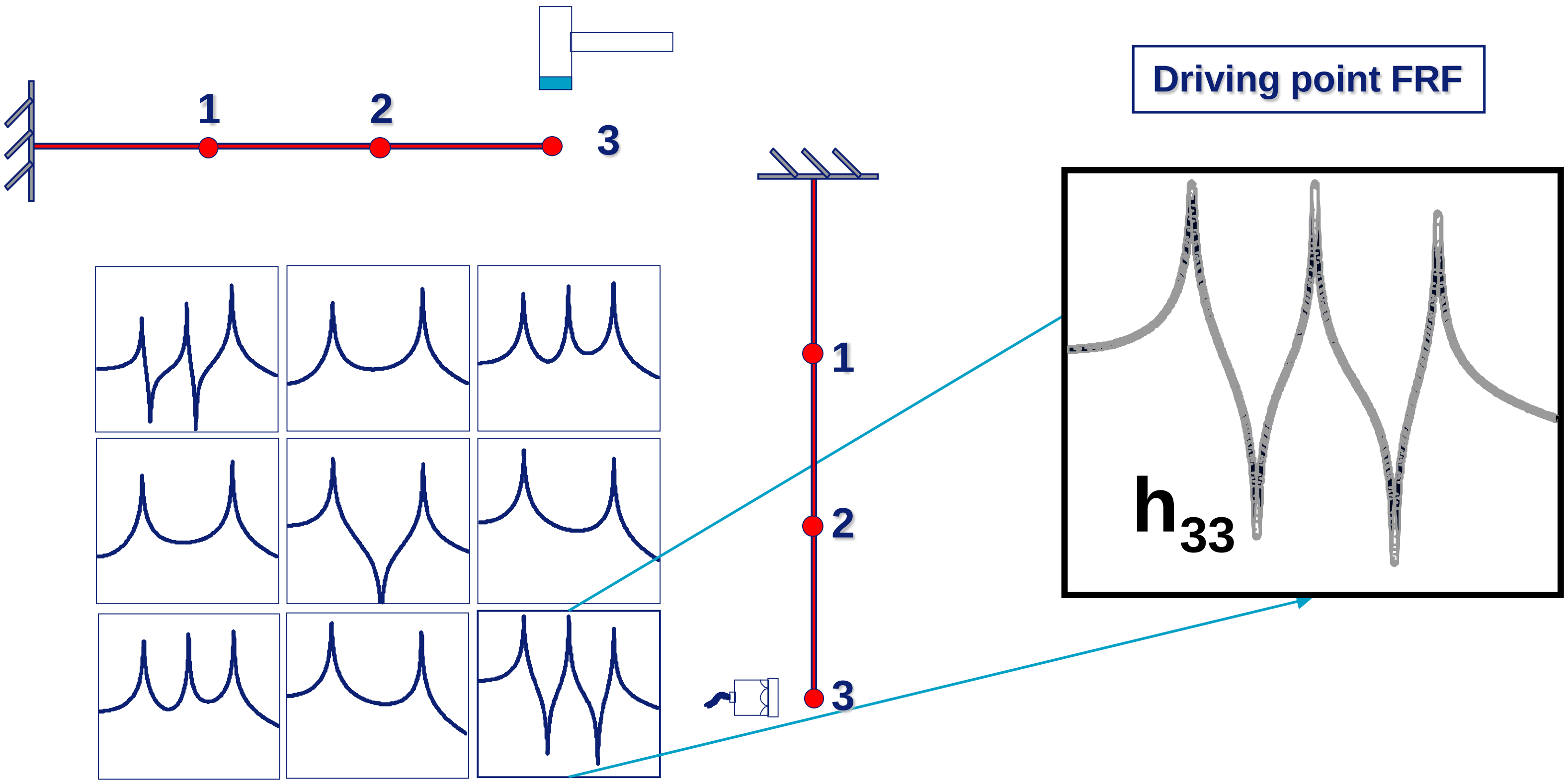


$$H_{ij}(j\omega)=\sum_{k=1}^{n}\frac{A_{ijk}}{j\omega-\lambda_k}+\frac{A_{ijk}^*}{j\omega-\lambda_k^*}$$

Experimental Modal Analysis
Example: Cantilever Beam

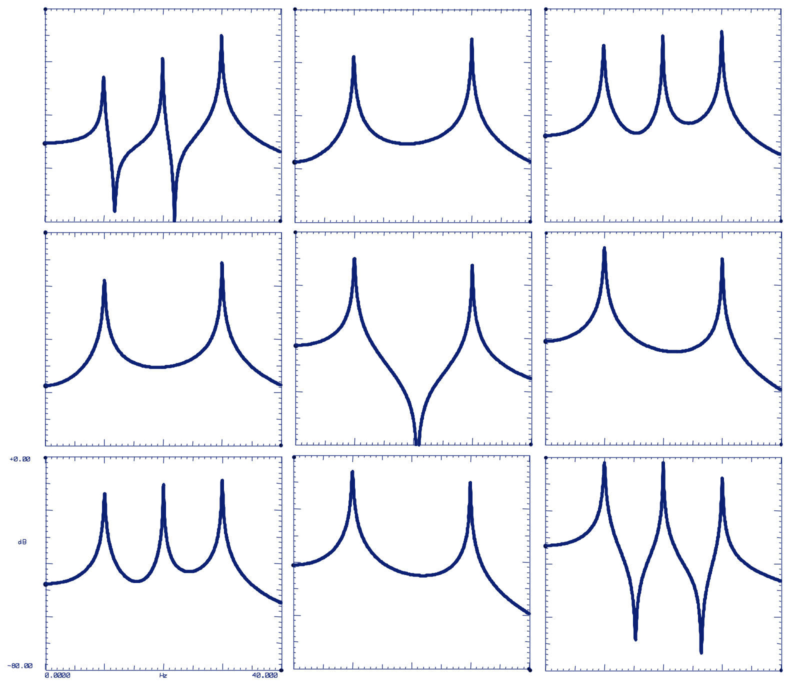


Cantilever Beam
Frequency Response Function Matrix

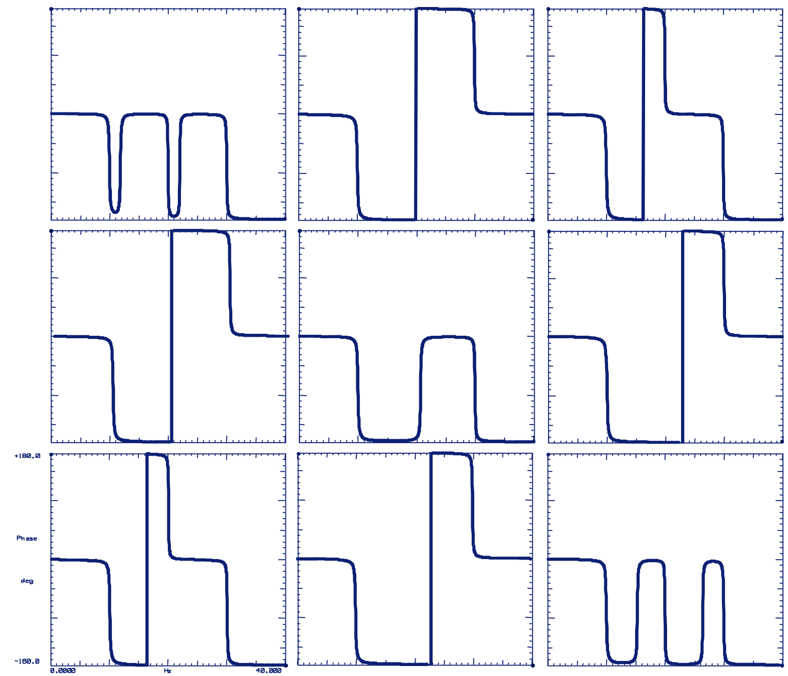


Cantilever Beam
Frequency Response Function Matrix

Log Magnitude

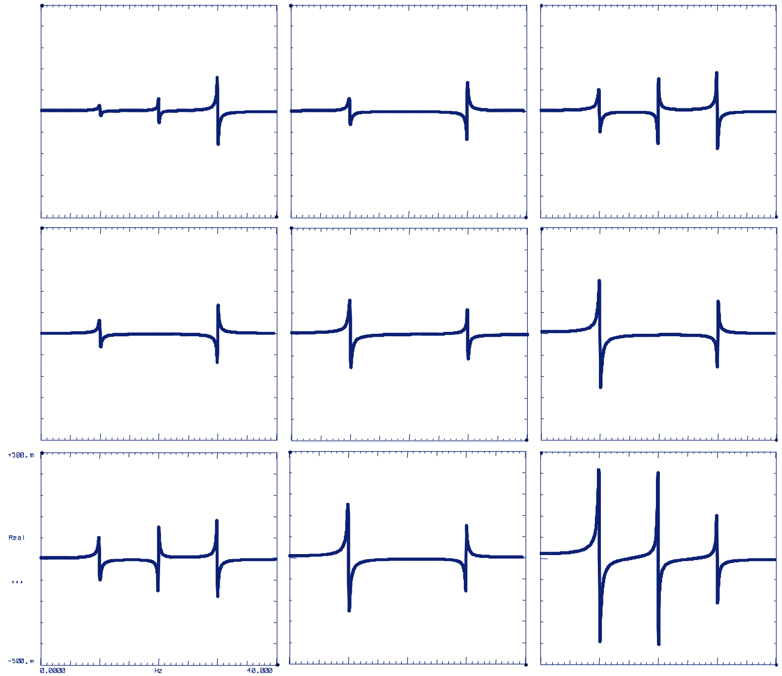


Phase

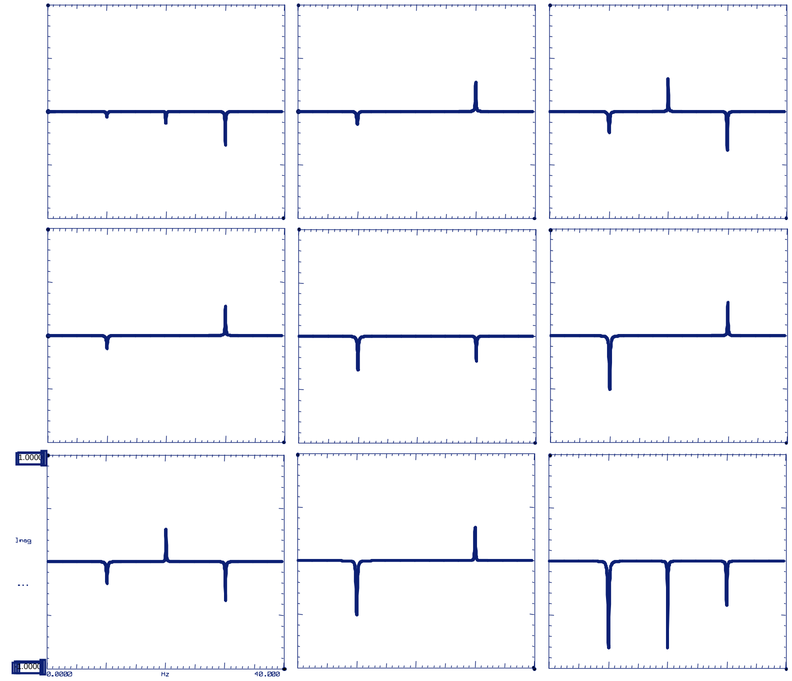


Cantilever Beam
Frequency Response Function Matrix

Real

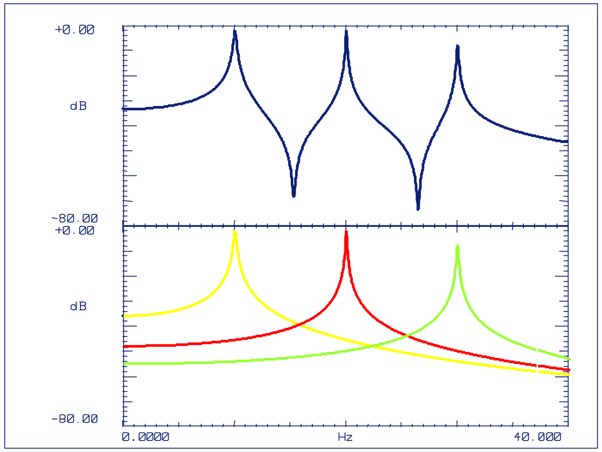


Imaginary

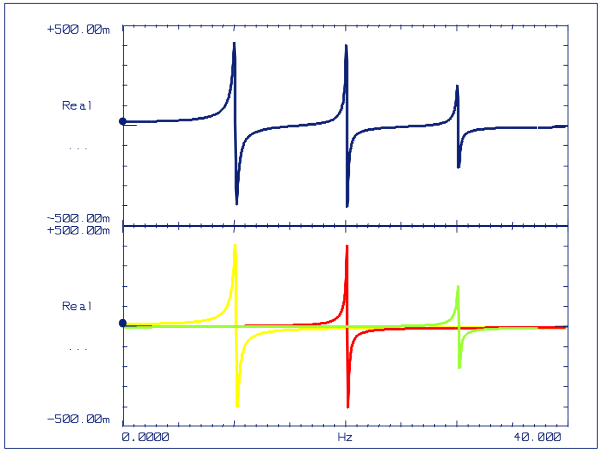


Cantilever Beam
Driving Point Measurements

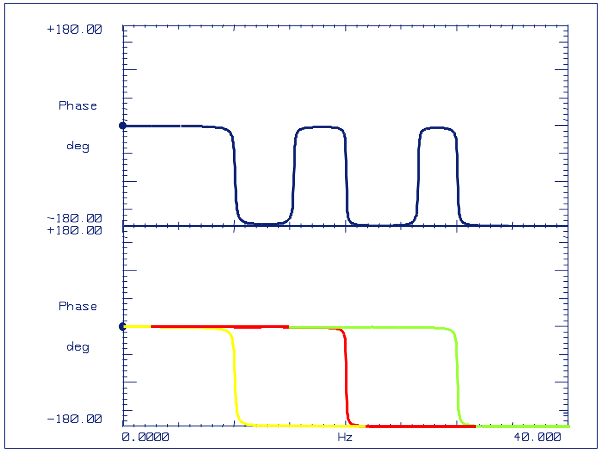
Magnitude



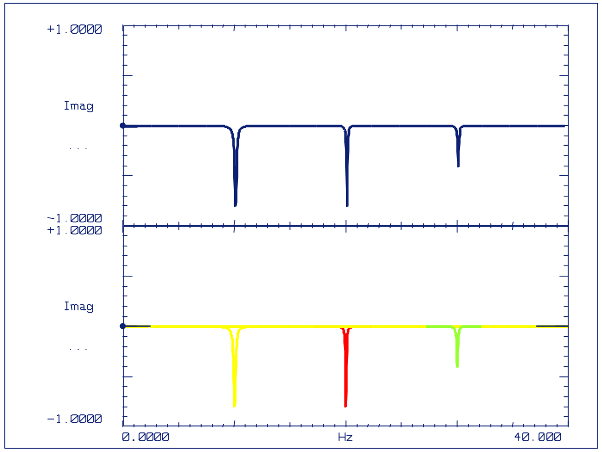
Real



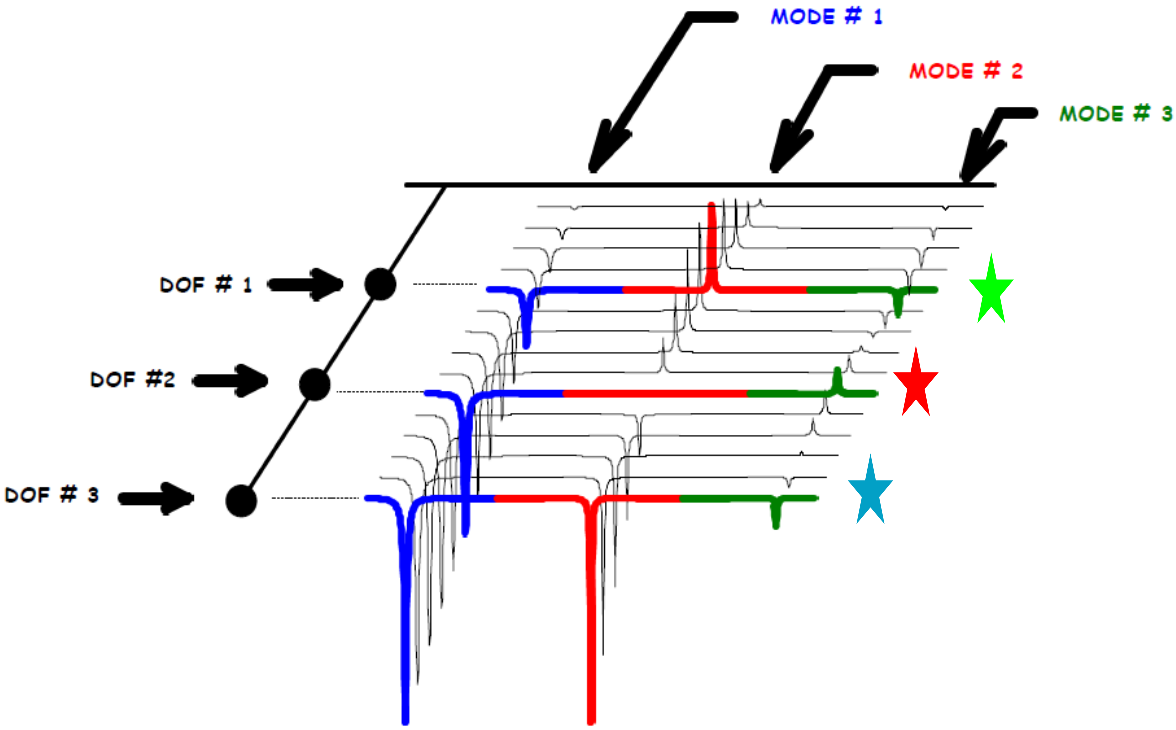
Phase

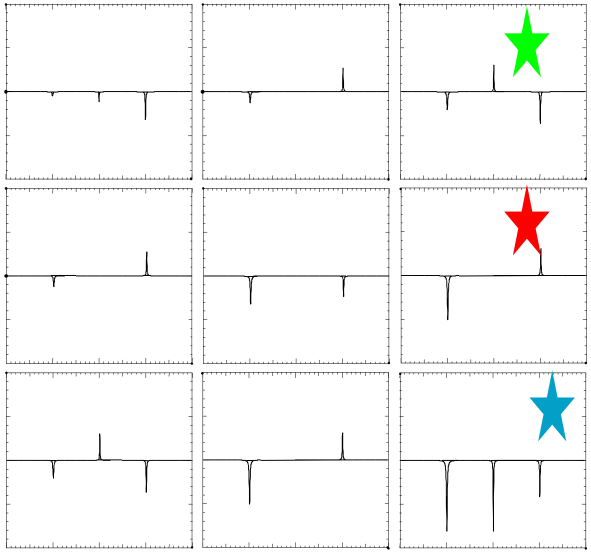


Imaginary



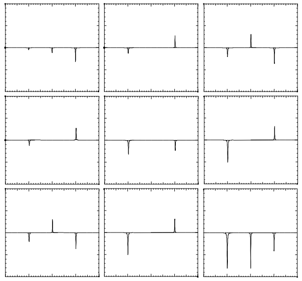
How do I get mode shapes from FRFs

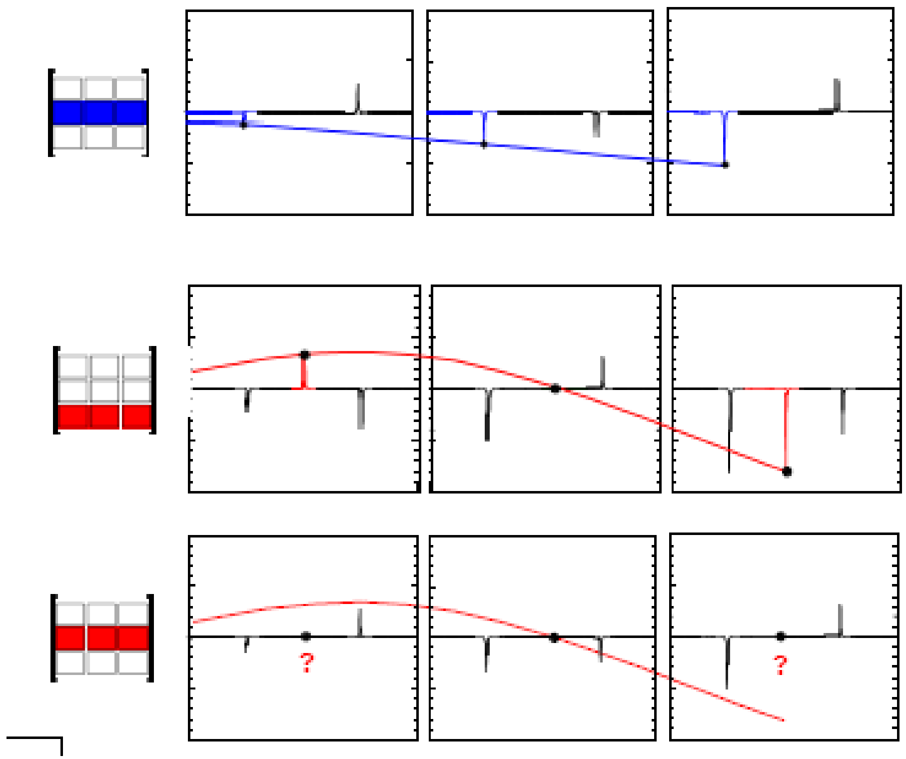




Imaginary part of FRFs

How do I get mode shapes from FRFs



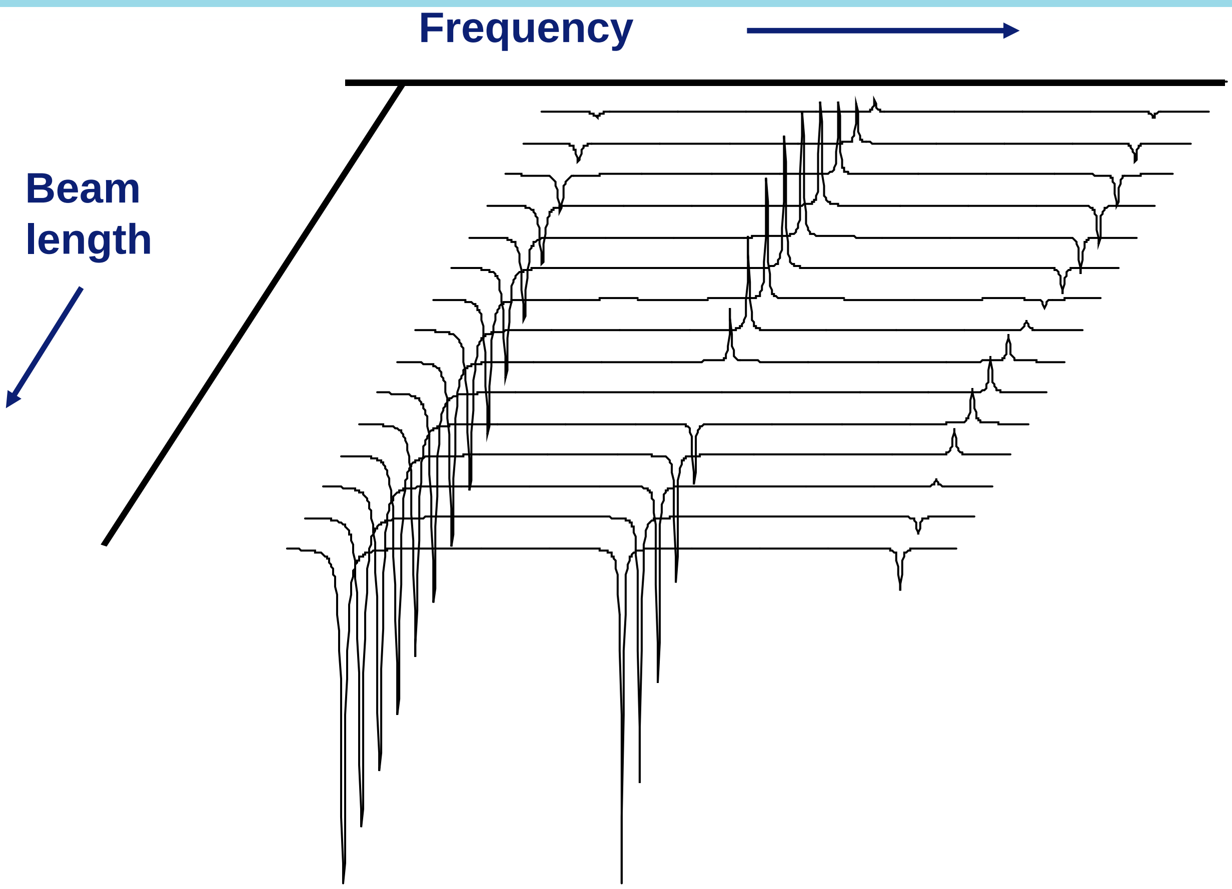


DOF # 2 is a node for the 2^{nd} mode

Any row or column can be used to extract mode shapes

- as long as it is not the node of a mode !

Consider More Modes Along the Beam

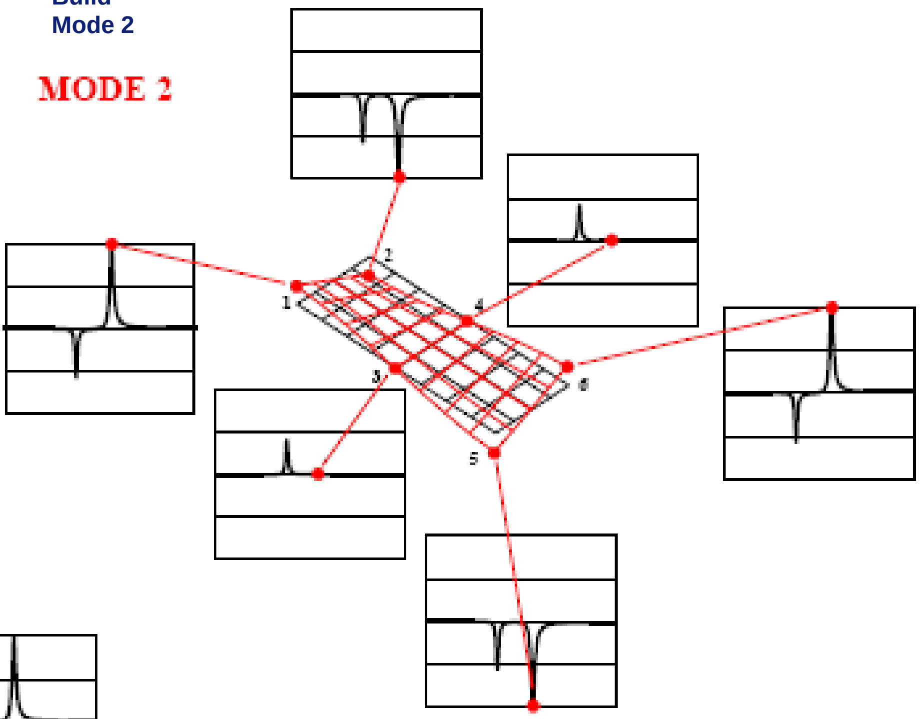


The peak amplitude of the imaginary part of the FRF is a simple method to determine the mode shape of the system

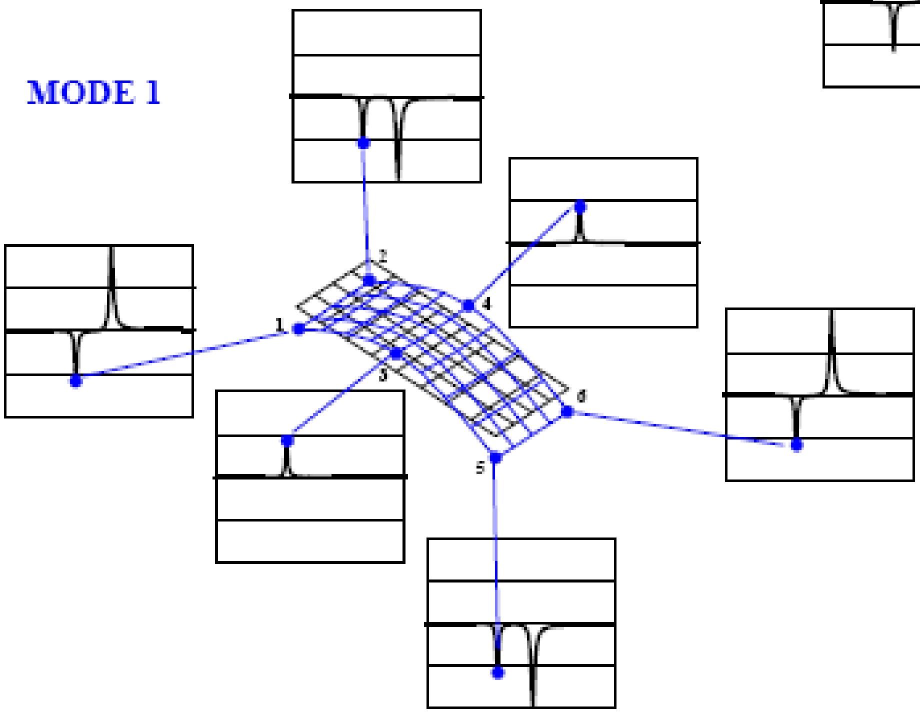
How do I get mode shapes from FRFs

Use the imaginary part of FRF for each Dofs

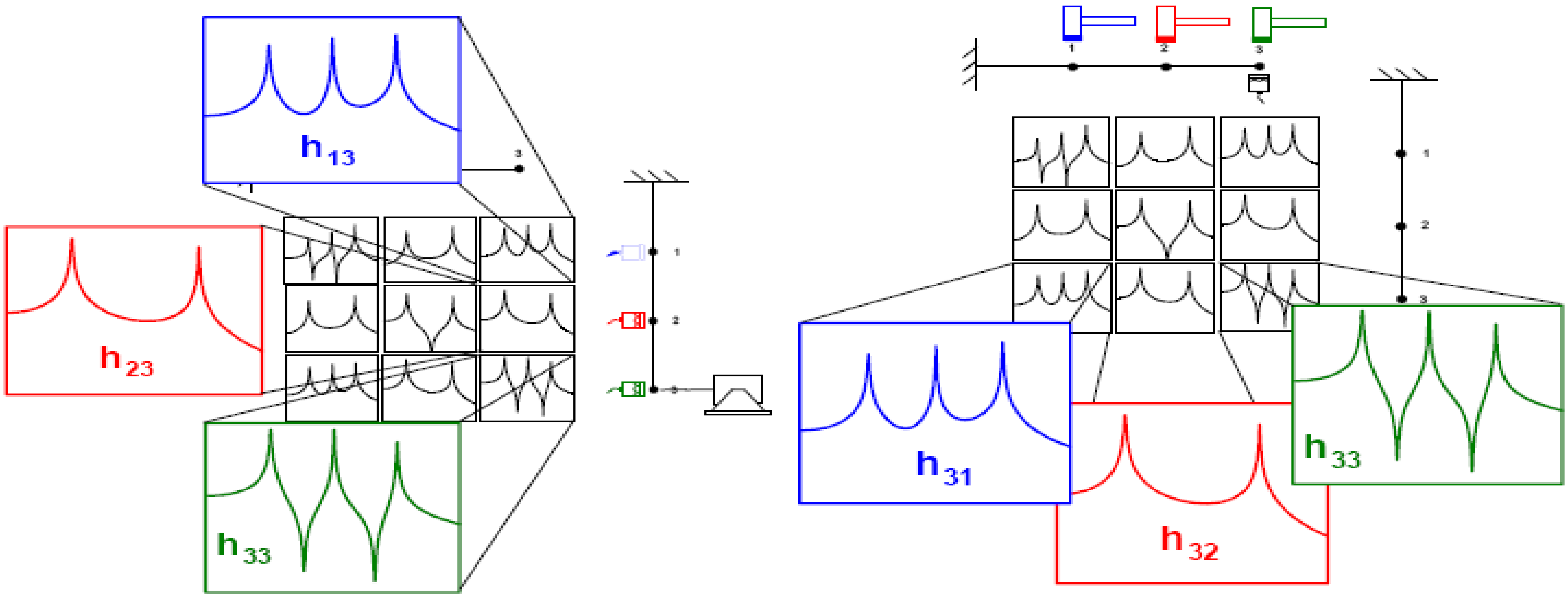
Build Mode 2



Build Mode 1



What's the difference between shaker and impact ?



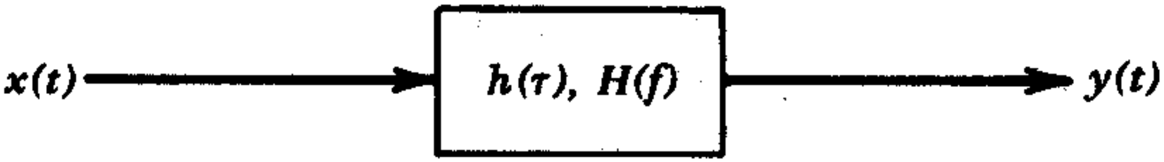
Theoretically - - - NOTHING ! ! !

With the shaker you measure a column of the FRF matrix, with the hammer you measure a line of the FRF matrix

Operative Procedure

- Theoretically in order to measure the FRF $\mathbf{H_{ij}}$:
  - Excite the structure in point *i* with a sinusoidal force of known amplitude and frequency.
  - Measure the induced vibration on point j.
  - Estimate the amplitude of $\mathbf{H_{ij}}$ as a ratio between the amplitude of the response and the amplitude of the force.
  - Estimate the phase of $\mathbf{H_{ij}}$ (φ_{ij}) as the difference the phase of the response and the phase of the excitation.
  - Repeat the operation modifying the frequency of the excitation.
- The response can be displacement, velocity of acceleration in point j.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$



Operative Procedure

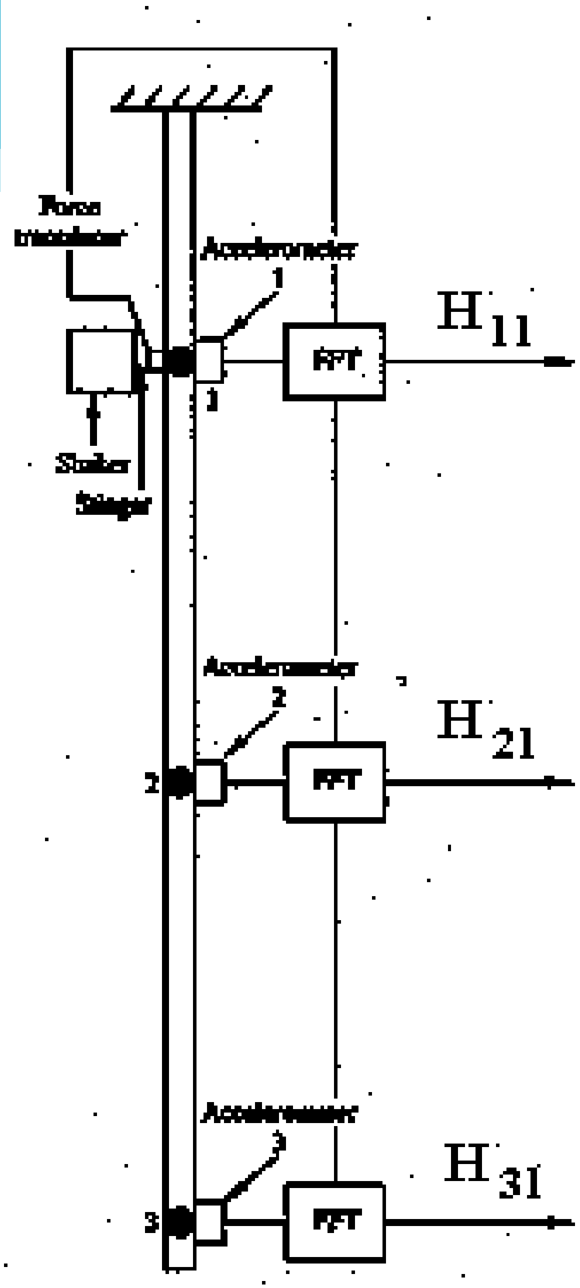
- Actually, it is not necessary to repeat the procedure for each single frequency.
- It should excite in point *i* with an excitation with suitable frequency content in the entire frequency range of interest.
- Measure the excitation in i and the response in j. Estimate the Fourier transform of the two signals and estimate the FRF with proper estimator (H1, H2, Hv, etc)
- Select the boundary conditions of the structure. For free-free EMA, use soft flexible cords in order to introduce in the system very low mode which not create interference with the mode of the structure being tested.
- Select **n** points in the structure. More points means better spatial resolution and better mode shapes. Not select possible node of the structures. Avoid points along symmetric axis.
- Excite in a point (e.g. with shaker) and measure in all the points (e.g. with accelerometers) →roving accelerometer method
- Or, Excite in all the points (e.g. with an hammer) and measure the respionce in a point → roving hammer method
- Select a suitable frequency range (fmin ÷ fmax . The resulting modes will be only the mode within the frequency range of interest.
- Measure the ***n FRFs***.

Operative Procedure

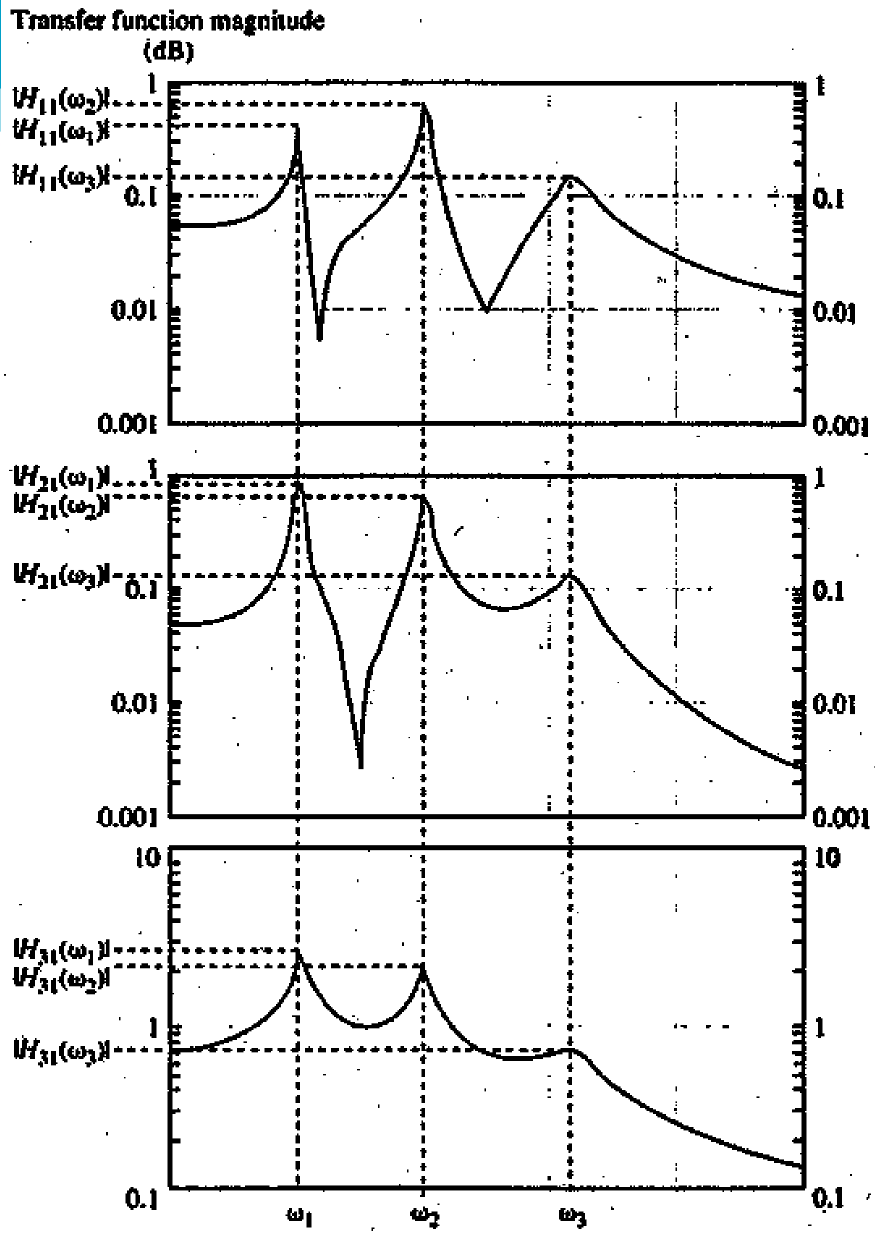
- In each of the **n** FRF , N peaks will be present. They are the **n** natural frequencies in the frequency range (except the possible presence of nodes). If a point corresponds with a node for a certain mode, the corresponding peak will be not visible.
- Estimate the **n** natural frequencies.
- Estimate the **n** modal damping ratio ζ
- Estimate the **n** mode shapes
- Note that the number of determined modes depends only on the frequency range of interest (they are infinite in number)

Operative Procedure

- The figure shows the instrumentation and procedure
  - Excitation in point 1 with shaker
  - Accelerometers in points 1,2,3.
  - Measurement of H11, H21, H31



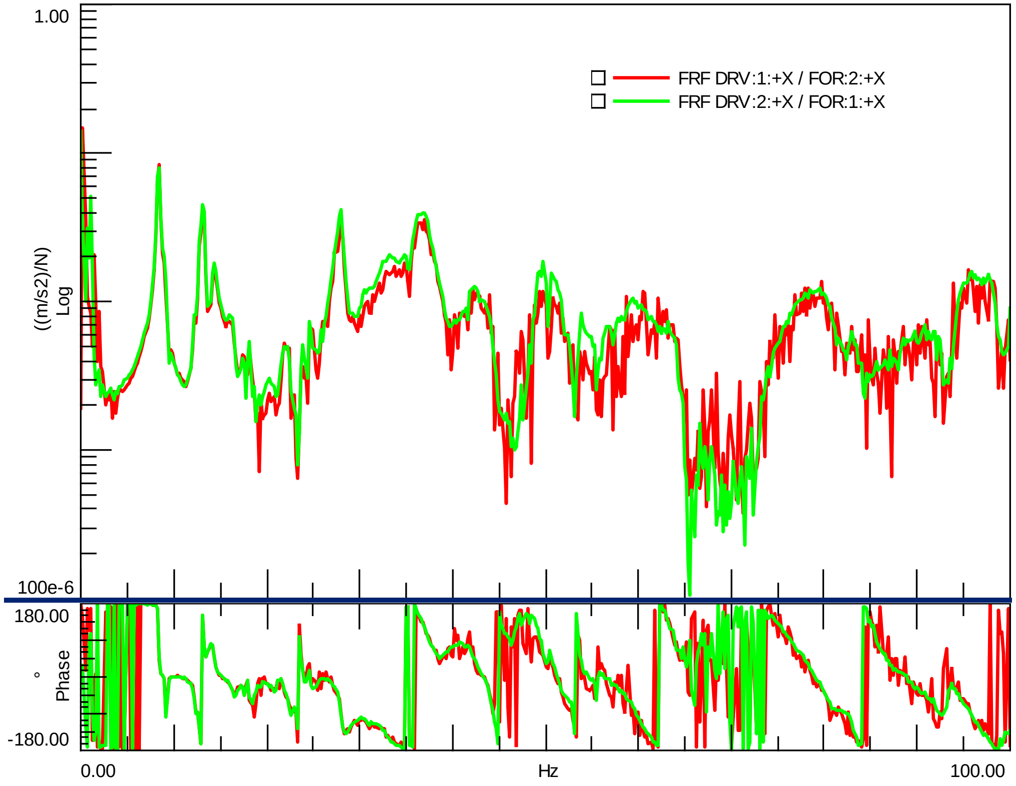
Operative Procedure



- Peaks of FRF define the natural frequencies.
- In correspondence of each natural frequency, the shape of the FRF enable damping determination (half power method)
- Amplitude and phase of the FRF in correspondance of the natural frequencies enable mode shapes (eigenvector)

Reciprocal FRFs

- Example of reciprocal FRF (H12=H21)



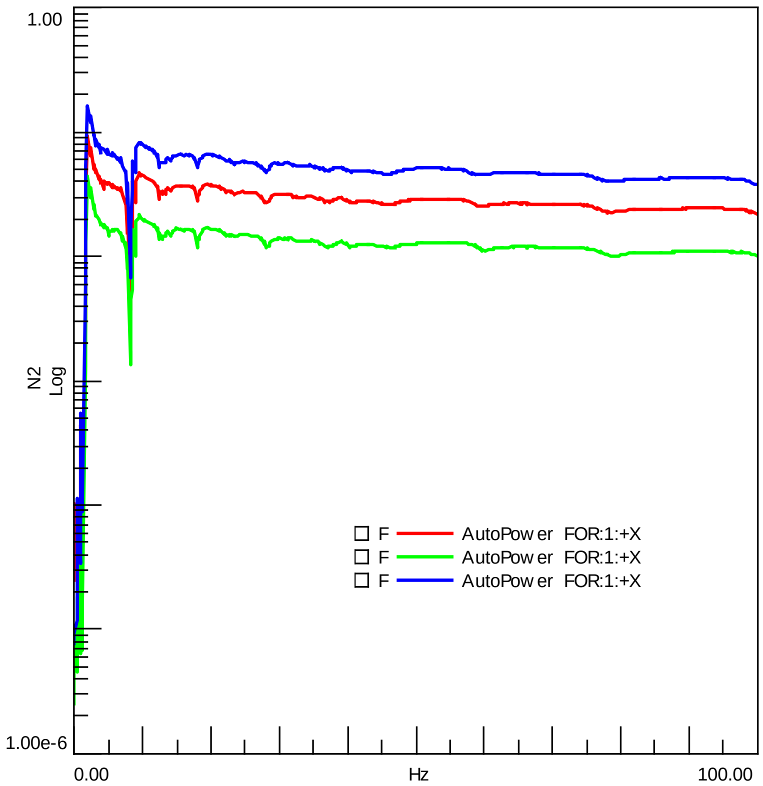
Linearity Check

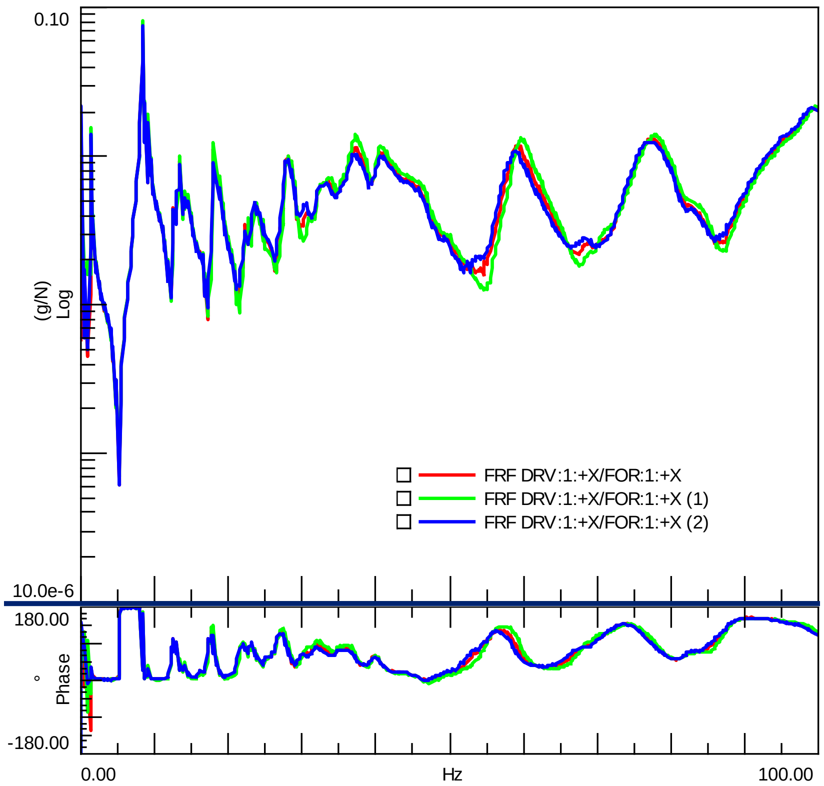
$$X = H^*F$$

$$\alpha X = H^* \alpha F$$

Linearity of FRFs

- 3 different excitation levels



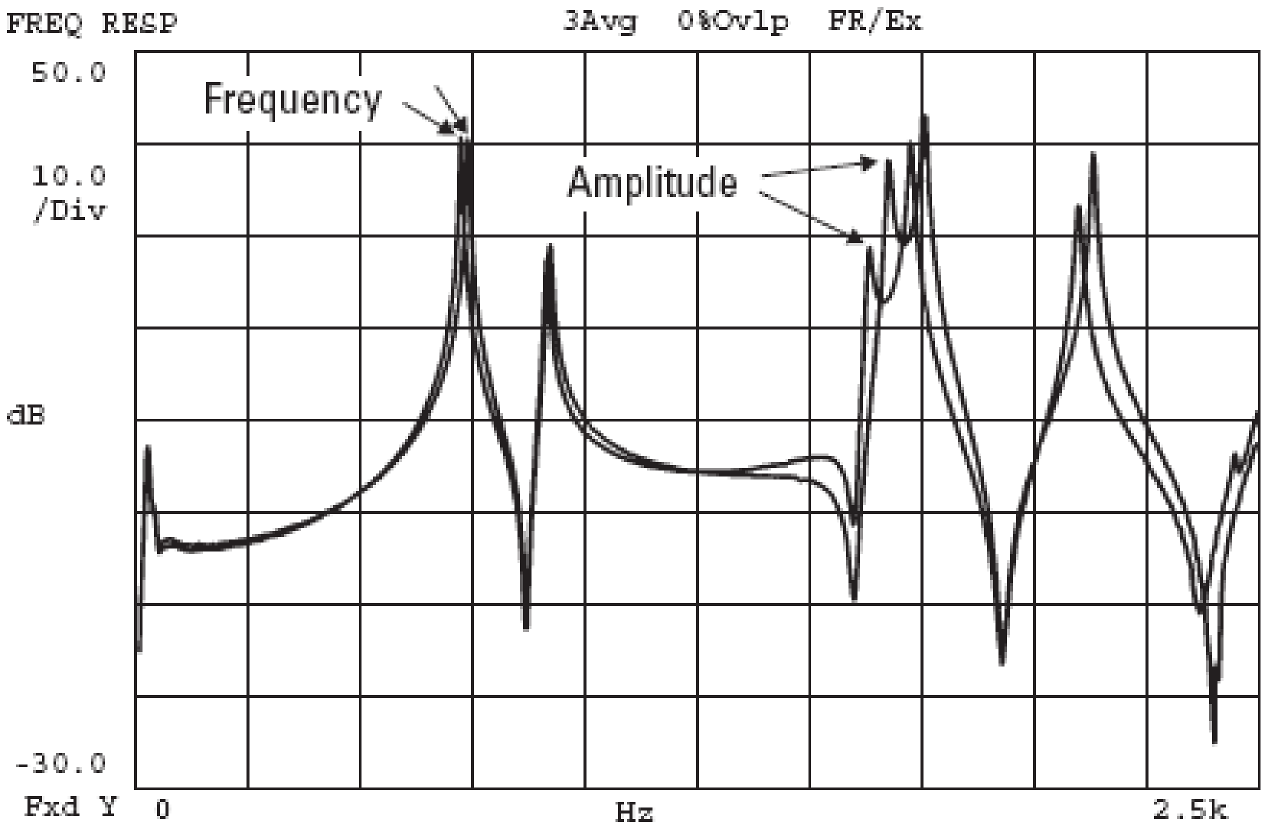


Mass loading

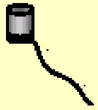
Another important consideration is the effect of mass loading from the accelerometer. This occurs as a result of the mass of the accelerometer being a significant fraction of the effective mass of a particular mode. A simple procedure to determine if this loading is significant can be done as follows:

- Measure a typical frequency response function of the test object using the desired accelerometer.
- Mount another accelerometer (in addition to the first) with the same mass at the same point and repeat the measurement.
- Compare the two measurements and look for frequency shifts and amplitude changes.

If the two measurements differ significantly, as illustrated in Figure 2.19, then mass loading is a problem and an accelerometer with less mass should be used. On very small structures, it may be necessary to measure the response with a non-contacting transducer, such as an acoustical or optical sensor, in order to eliminate any mass loading.

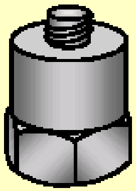


The "Mass Loading" Effect



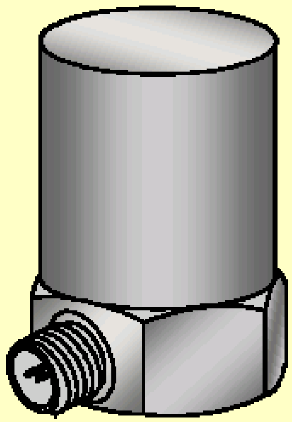
0,1 pC/ms^{-2}

0.65 g ⟹ M > 7 g



10 pC/ms^{-2}

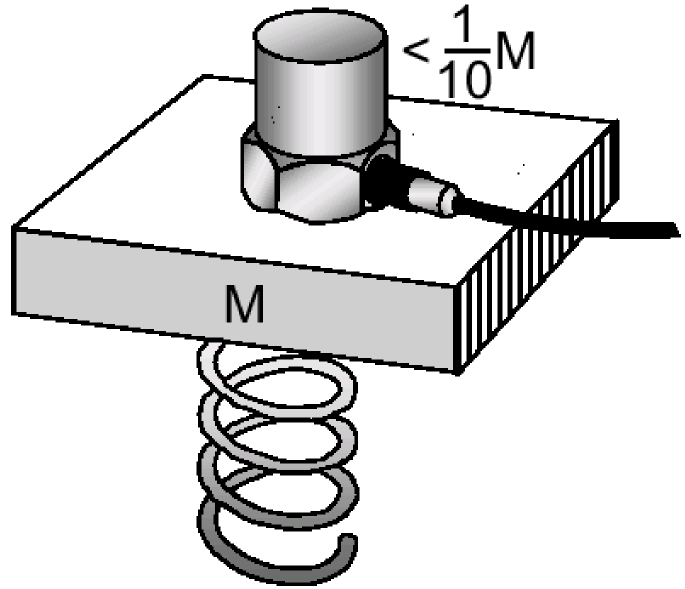
54 g ⟹ M > 600 g



1000 pC/ms^{-2}

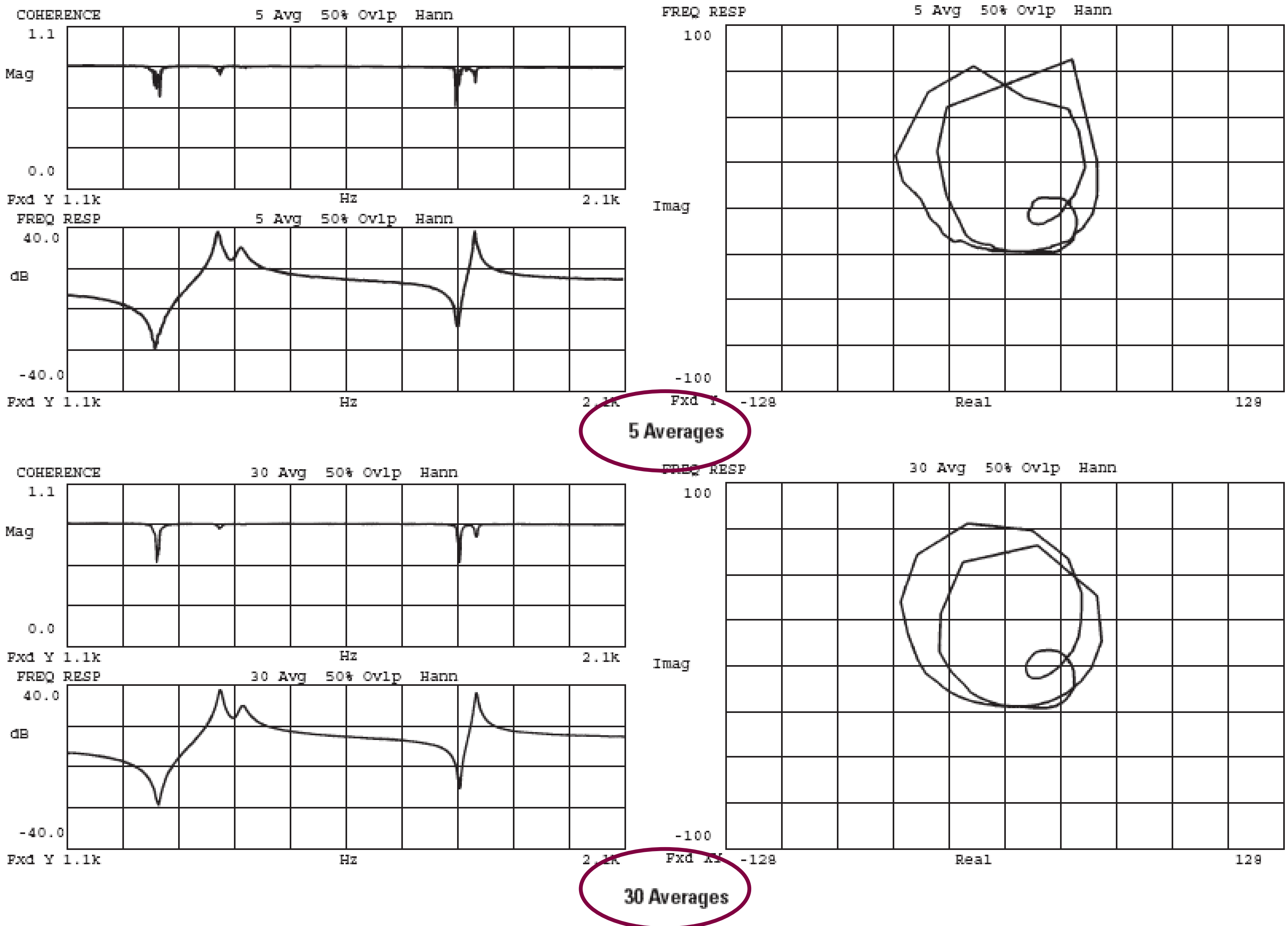
470 g ⟹ M > 5 kg

Dynamic Mass



$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{M + M_{SENSOR}}}$$

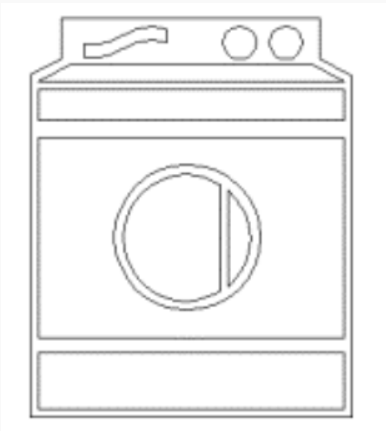
Improving measurements accuracy



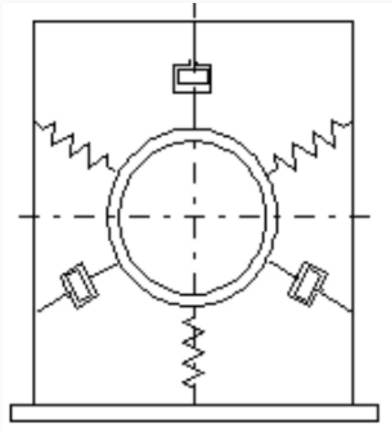
3. SDOF-MDOF systems

INTRODUCTION – MODELING OF CONTINUOS SYSTEMS

Real system



Physical model



Mathematical model

$$m\ddot{x} + c\dot{x} + kx = F$$

INTRODUCTION – MODELING OF CONTINUOS SYSTEMS

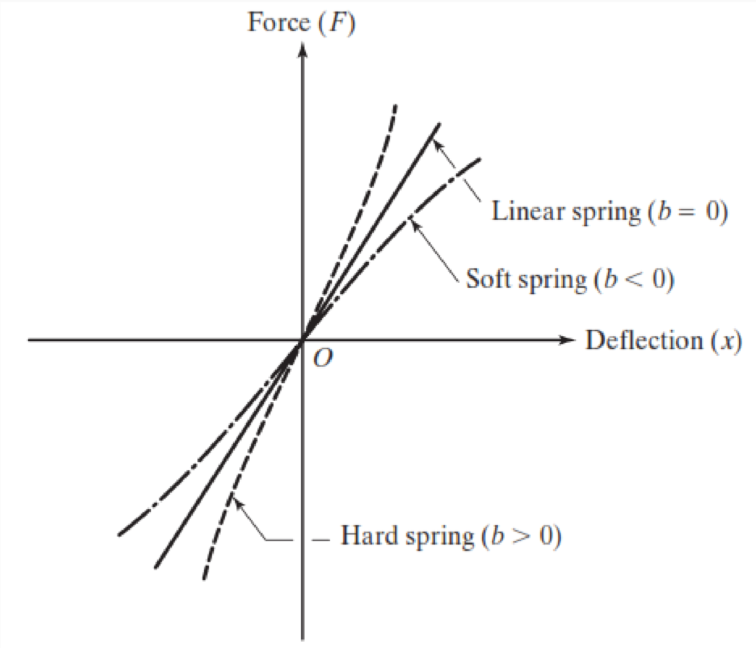
Stiffness

The stiffness of a linear spring is always positive and denotes the force (positive or negative) required to cause a unit deflection (elongation or reduction in length) in the spring.

$$F = kx$$

$F =$ Applied force
$x =$ Elongation or reduction in length
$k =$ Spring stiffness

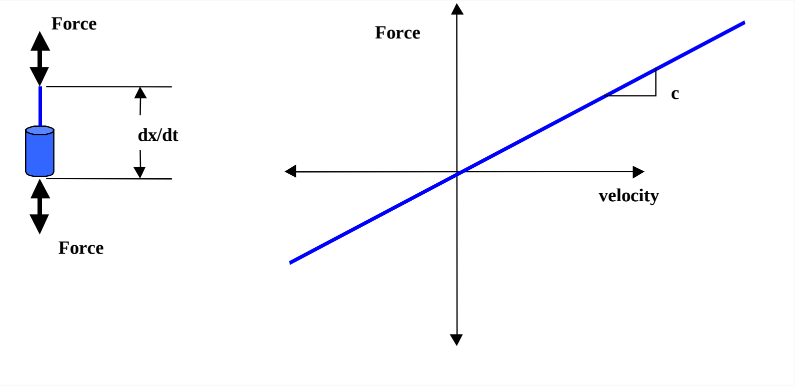
Sometimes practical systems exhibit a nonlinear force-deflection relation



INTRODUCTION – MODELING OF CONTINUOS SYSTEMS

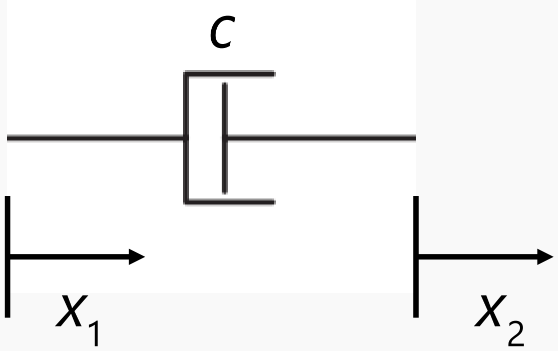
Dissipations could be modelled in several ways:

- Viscous damper
- Non-linear damper
- Coulomb friction
- …

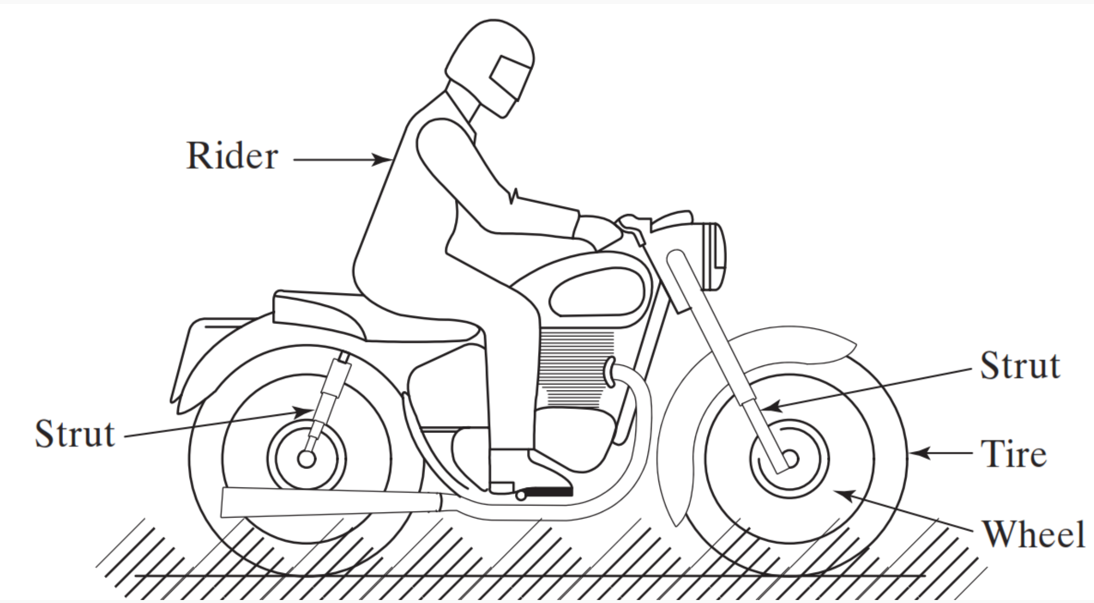


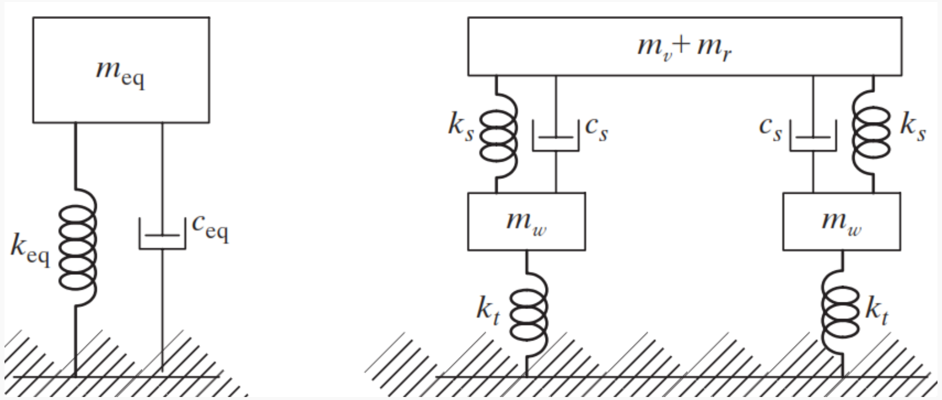
Viscous damping is the most commonly used damping mechanism in vibration analysis. In viscous damping, the damping force is proportional to the velocity of the vibrating body.

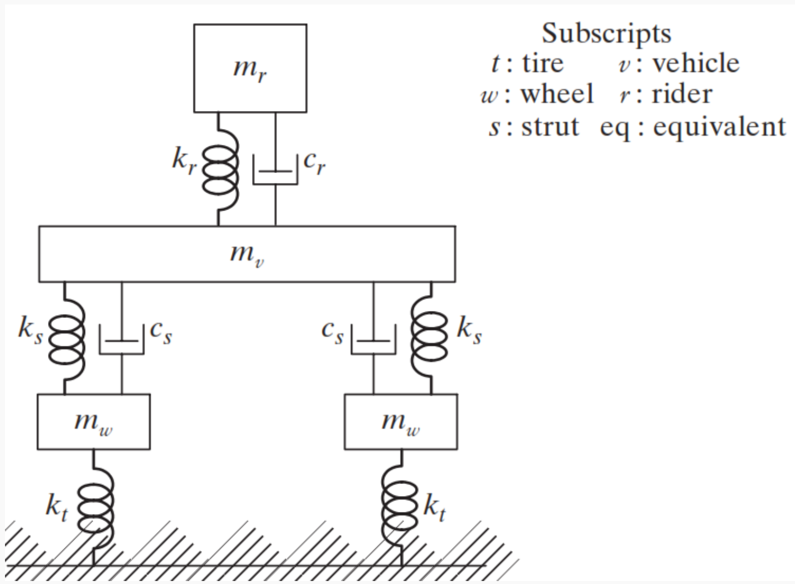
$$F_v = c\left(\dot{x}_2 - \dot{x}_1\right)$$



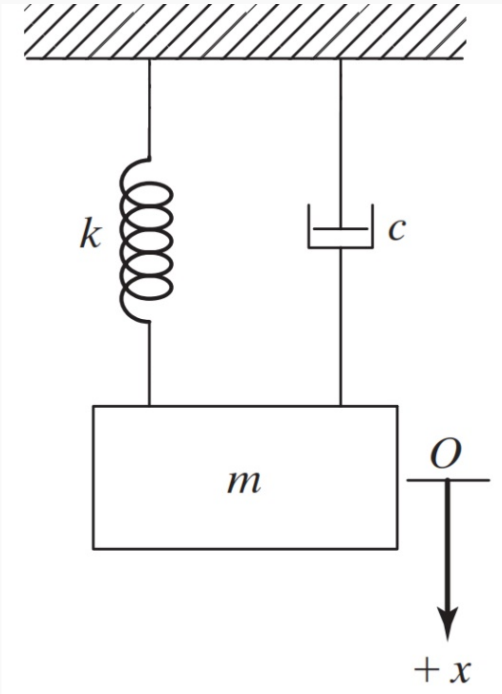
INTRODUCTION – MODELING OF CONTINUOS SYSTEMS

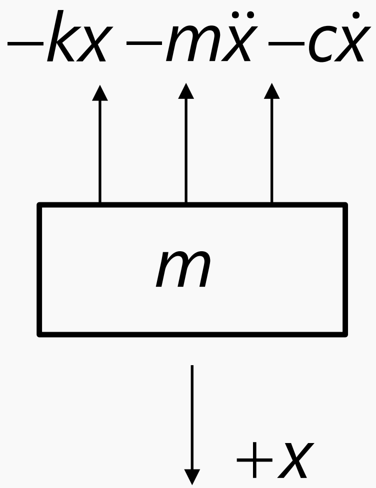






FREE VIBRATION OF SIGLE-DEGREE-OF-FREEDOM (SDOF) SYSTEMS





Inertia force:

$$F = m\ddot{x}$$

Elastic force:

$$F = kx$$

Damping force (viscous damping):

$$F = c\dot{x}$$

D'Alembert's principle:

$$m\ddot{x} + c\dot{x} + kx = 0$$

The solution could be assumed in the form $x(t) = Ce^{st}$

where C and s are undetermined constants

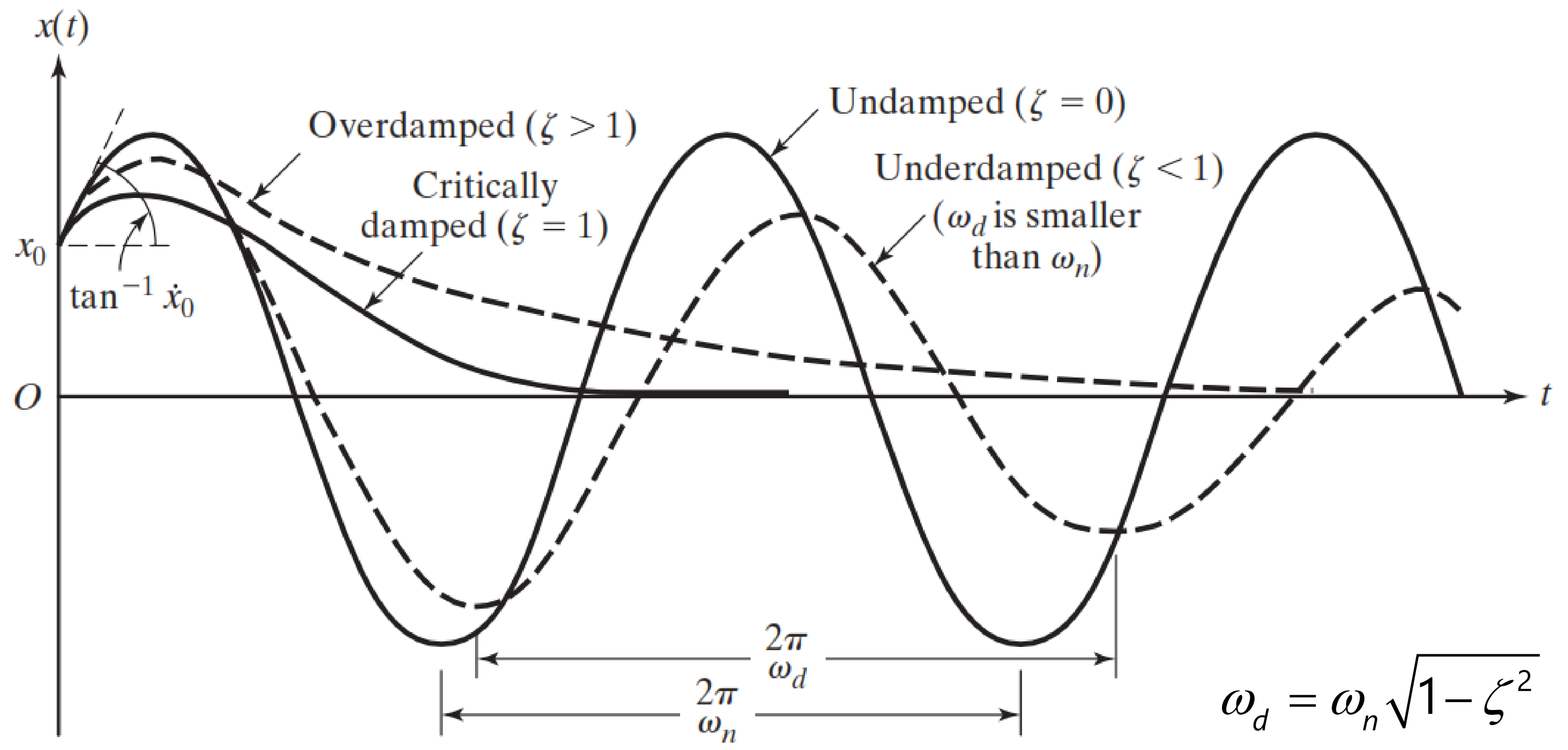
FREE VIBRATION OF SIGLE-DEGREE-OF-FREEDOM (SDOF) SYSTEMS

The solution is

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

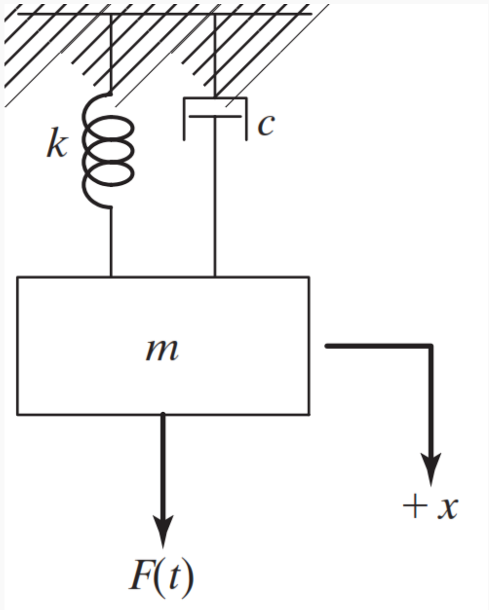
Natural pulsation: $\omega_n = \sqrt{\frac{k}{m}}$

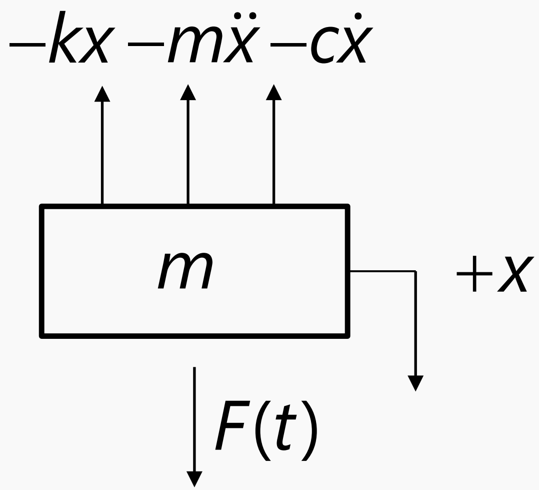
Damping ratio: $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$



$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

FORCED VIBRATION OF SDOF SYSTEMS





where:

$$F(t) = F_0 \cos(\omega t)$$

D'Alembert's principle:

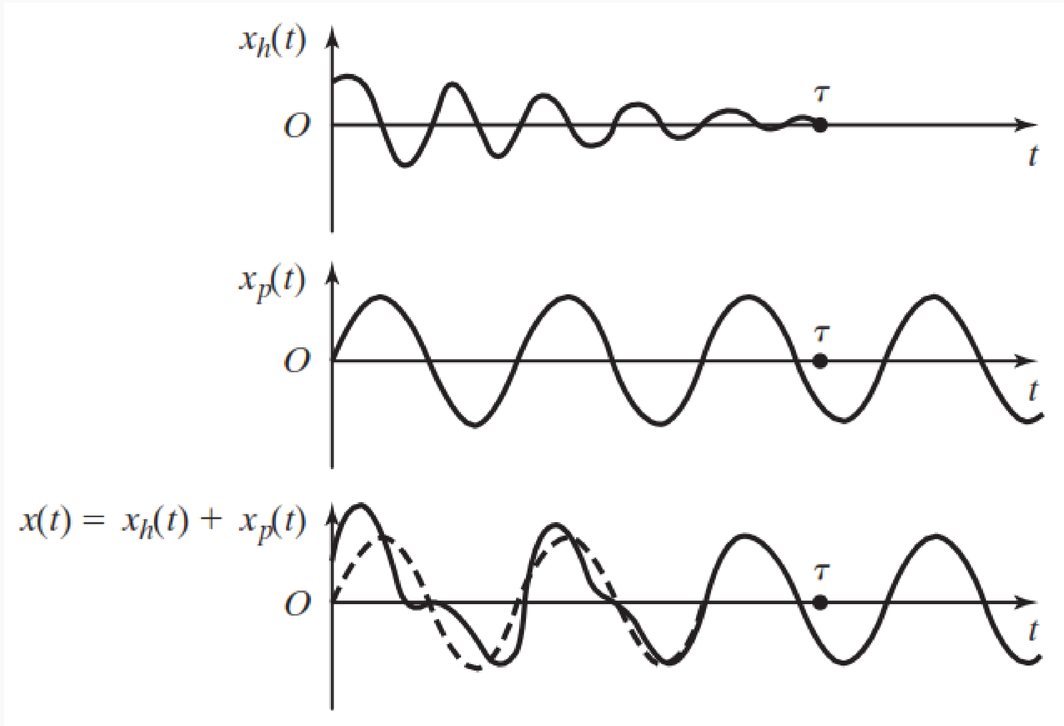
$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Since this equation is nonhomogeneous:

$$x(t) = x_h(t) + x_p(t)$$

We are interest in the steady-state response:

$$x(t) = x_p(t)$$



FORCED VIBRATION OF SDOF SYSTEMS

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F_0 cos(\omega t)$$

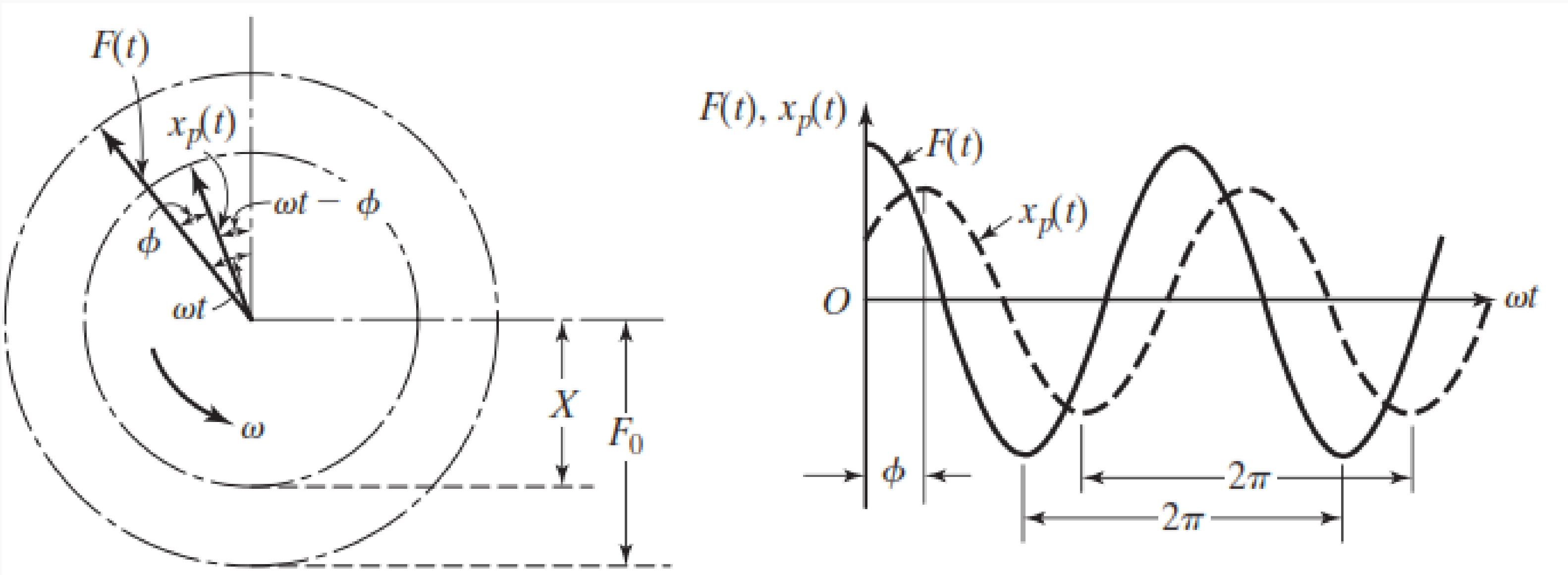
Particular solution:

$$x_p(t) = Xcos(\omega t - \phi)$$

$$x_p(t) = Xcos(\omega t - \phi)$$

Delay of the response wrt the excitation

$$X = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}} \qquad \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$



FORCED VIBRATION OF SDOF SYSTEMS

$$X = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

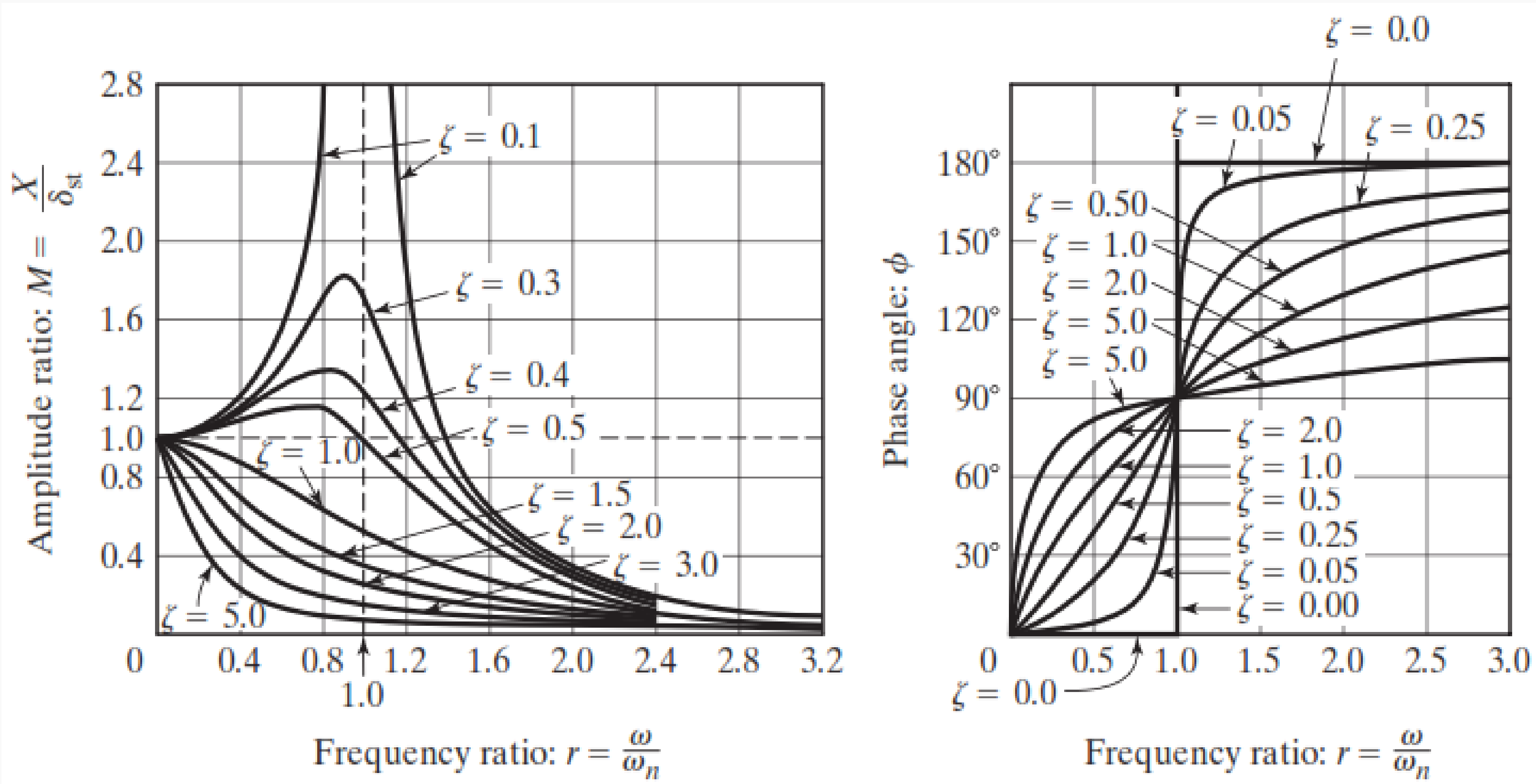
$$\zeta = \frac{c}{2\sqrt{mk}}$$

$$\delta_{st} = \frac{F_0}{k}$$

$$r = \frac{\omega}{\omega_n}$$

$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

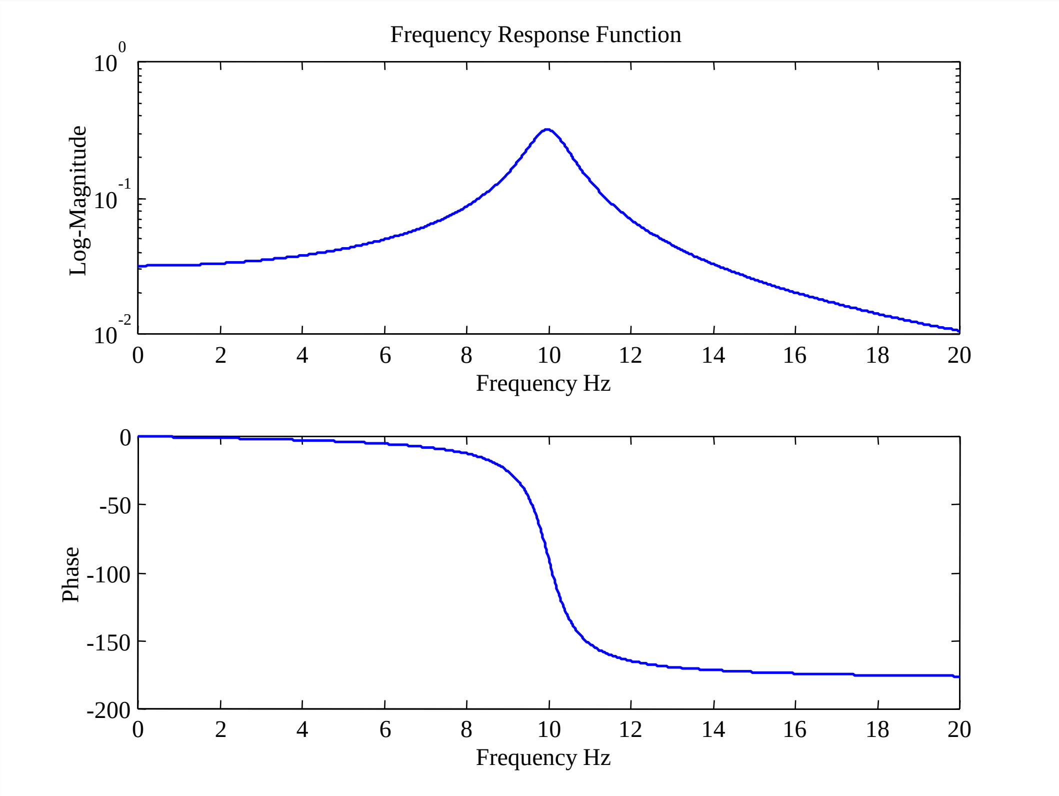


FORCED VIBRATION OF SDOF SYSTEMS

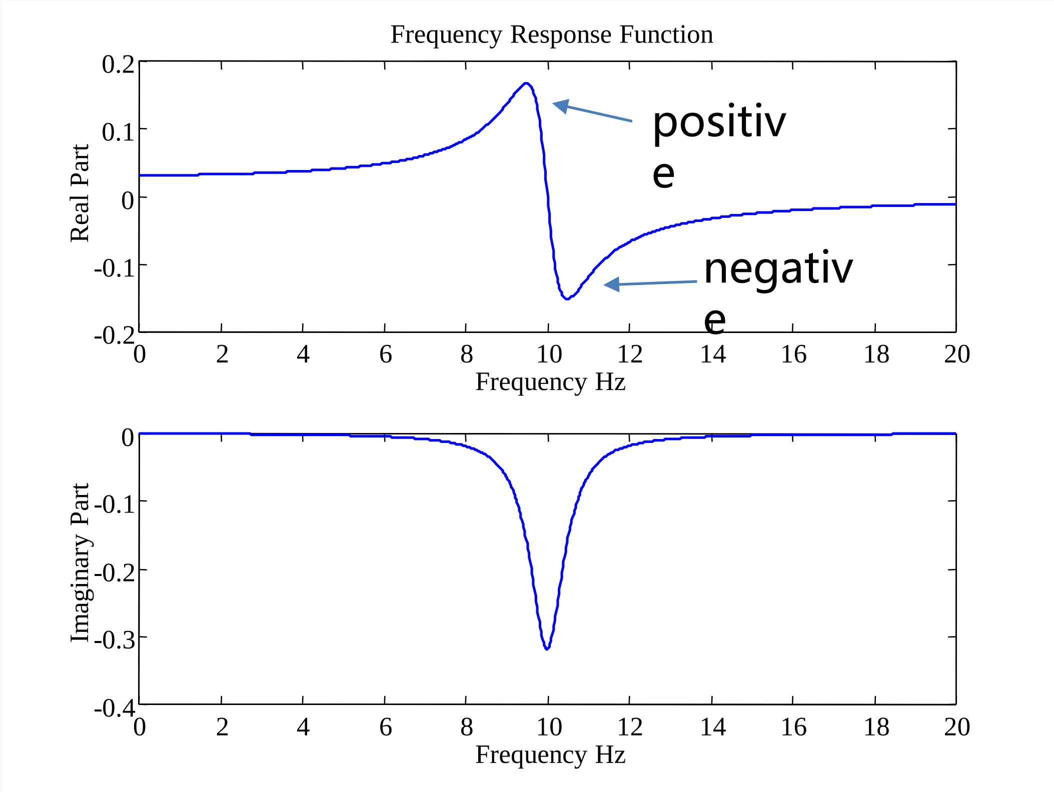
FREQUENCY RESPONSE FUNCTION

$$H(i\omega) = \frac{X}{F}$$

Magnitude & Phase



Real & Imaginary

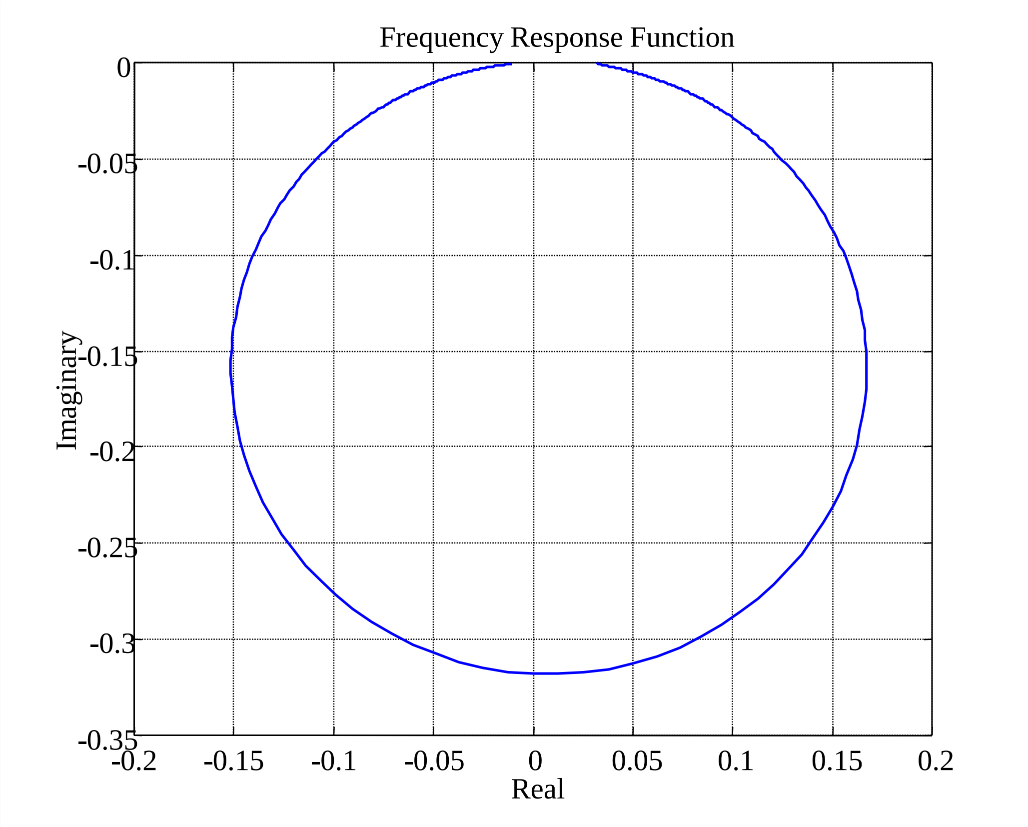


FORCED VIBRATION OF SDOF SYSTEMS

FREQUENCY RESPONSE FUNCTION – Nyquist plot

$$H(\mathrm{i}\omega) = \frac{X}{F}$$

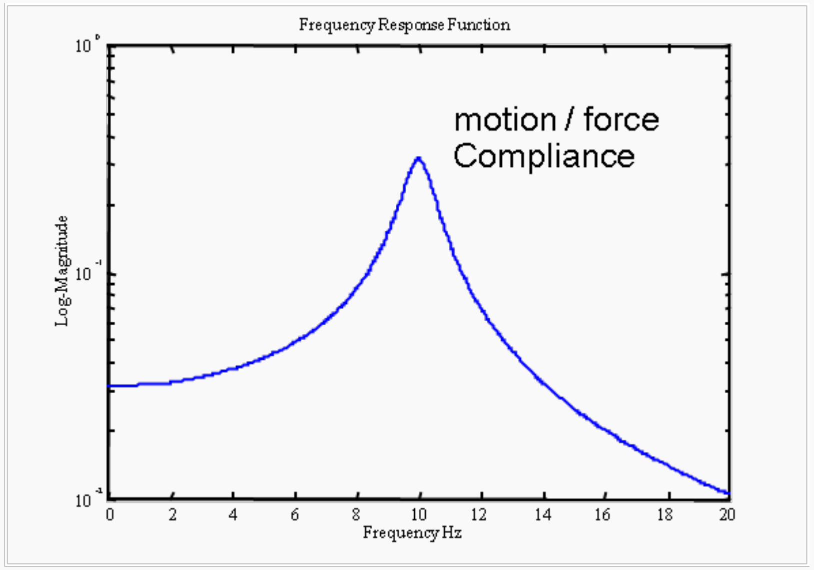
Real vs. Imaginary

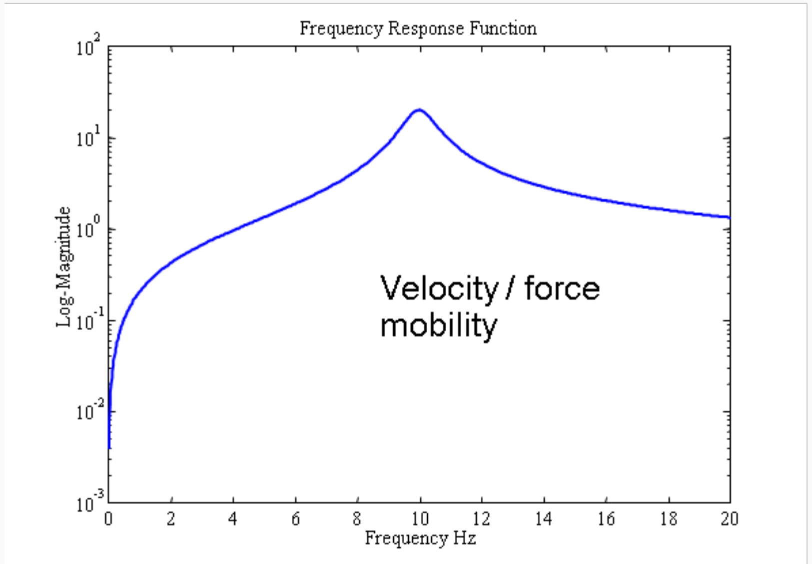


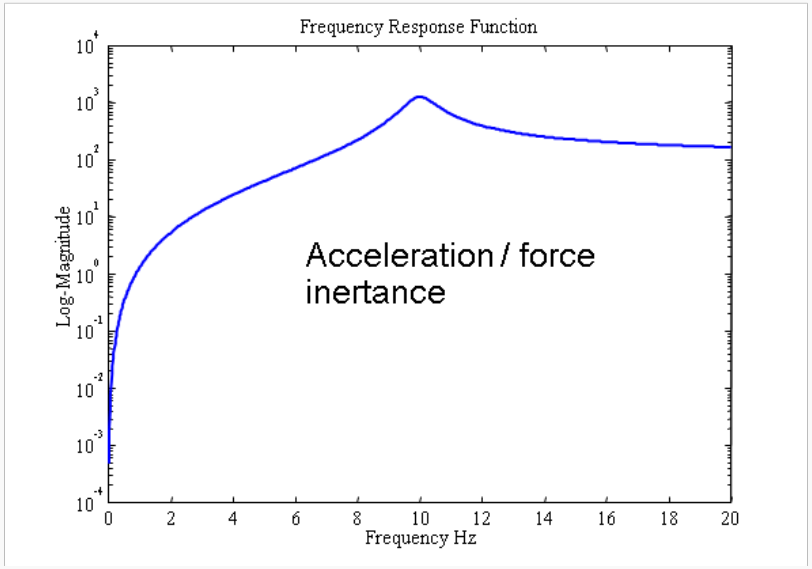
FORCED VIBRATION OF SDOF SYSTEMS

FREQUENCY RESPONSE FUNCTION

- Compliance $X(j\omega)/F(j\omega)$
- Mobility $V(j\omega)/F(j\omega) = j\omega X(j\omega)/F(j\omega)$
- Inertance $A(j\omega)/F(j\omega) = -\omega^2 X(j\omega)/F(j\omega)$

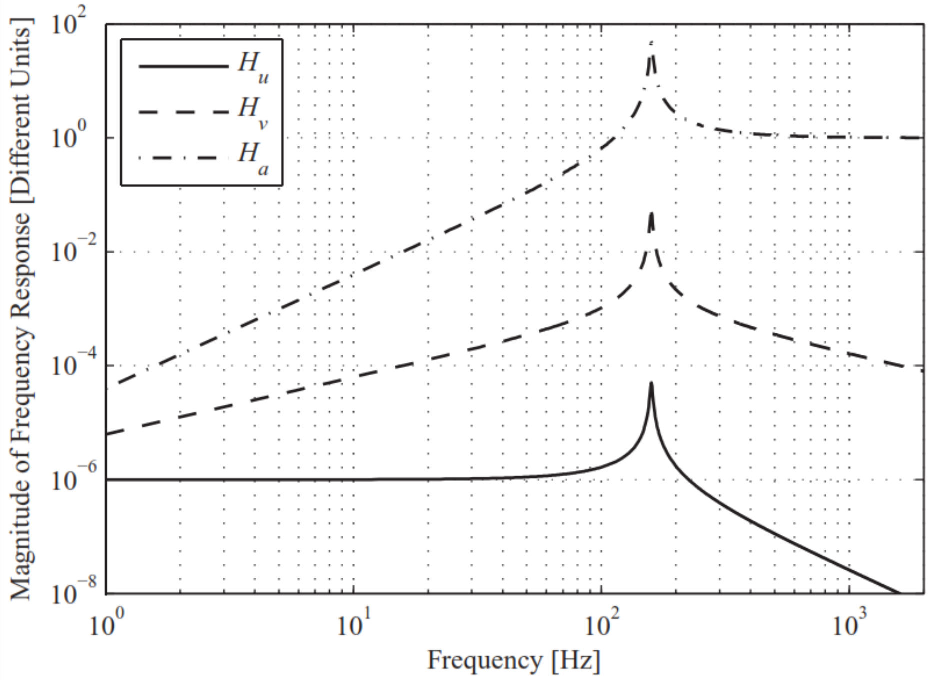






FORCED VIBRATION OF SDOF SYSTEMS

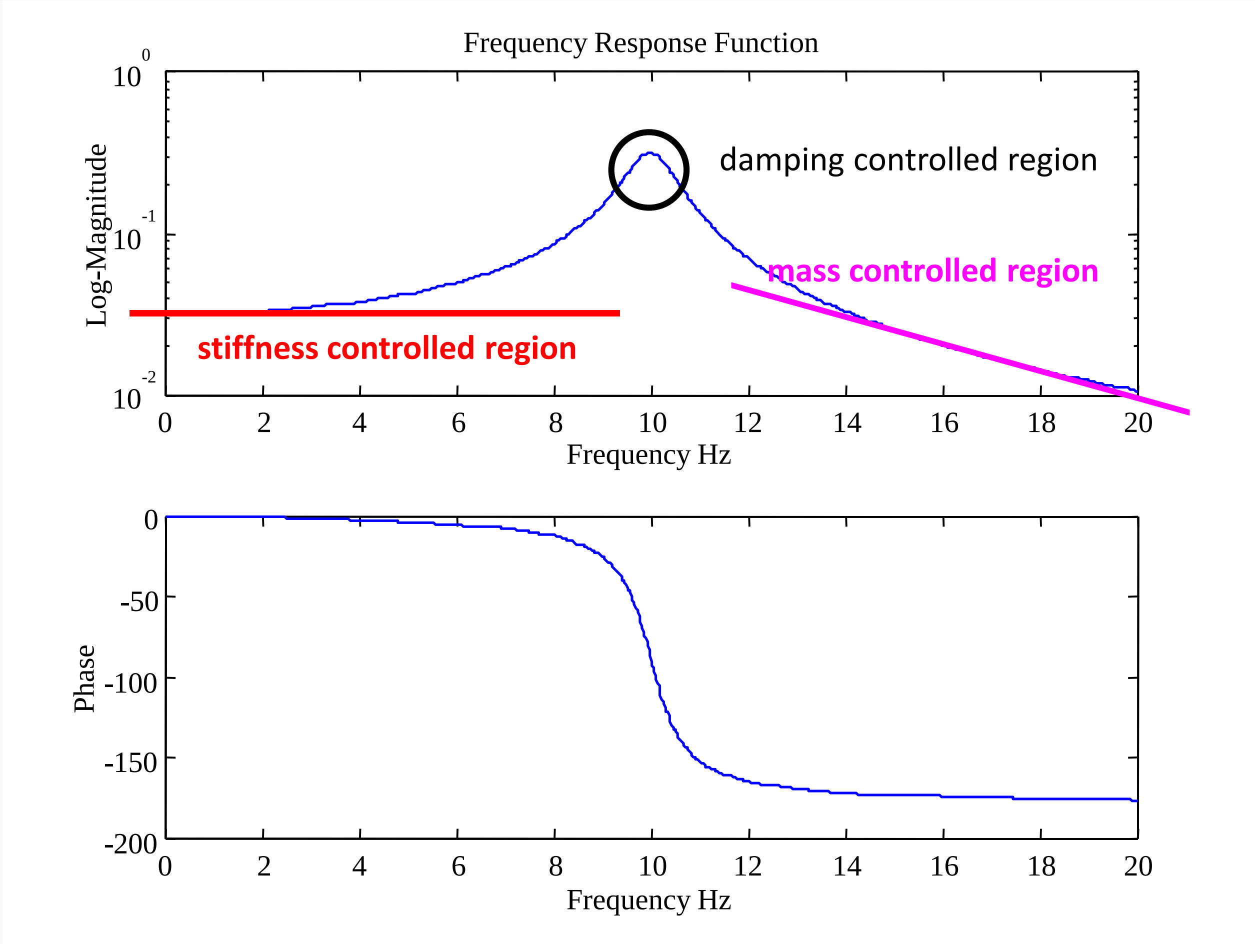
Response quantity, R	R/F		F/R	
Displacement	u/F	**Dynamic flexibility** Receptance Compliance	F/u	**Dynamic stiffness**
Velocity	$v/F = j2\pi f \cdot u/F$	**Mobility** Admittance	$F/v = 1/(j2\pi f) \cdot F/u$	**Mechanical impedance**
Acceleration	$a/F = -(2\pi f)^2 \cdot u/F$	**Accelerance** Inertance	$F/a = -1/(2\pi f)^2 \cdot (F/u)$	**Apparent mass**



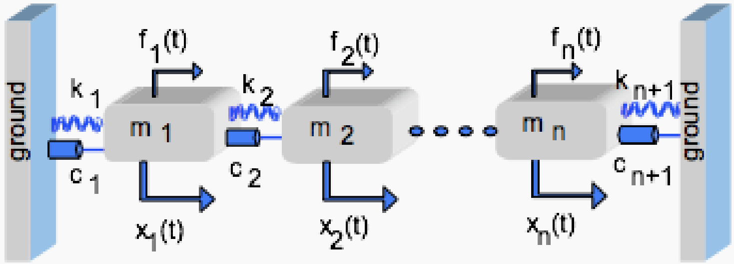
SDOF FRF

- FRF and system parameters

$$X = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$



FORCED VIBRATION OF MDOF SYSTEMS

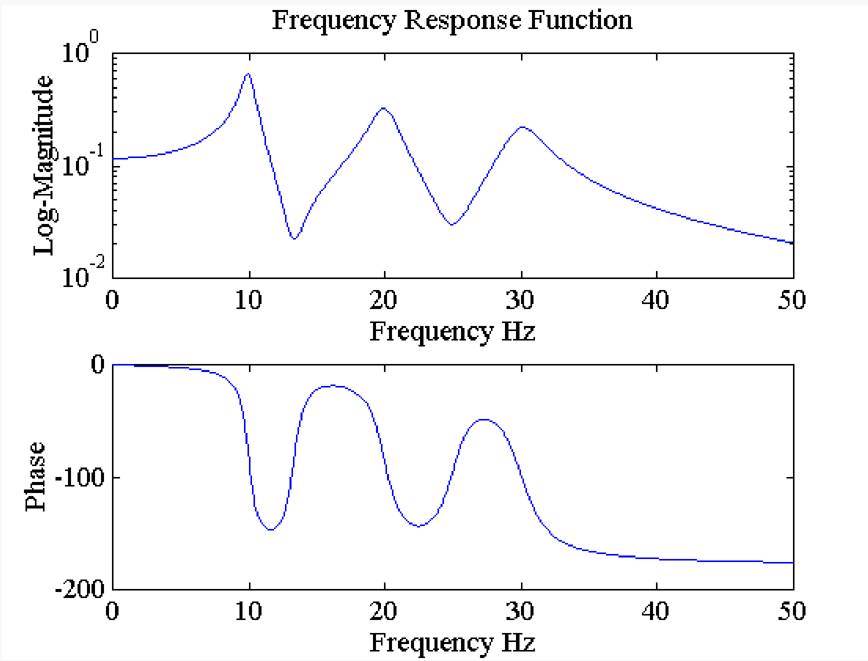


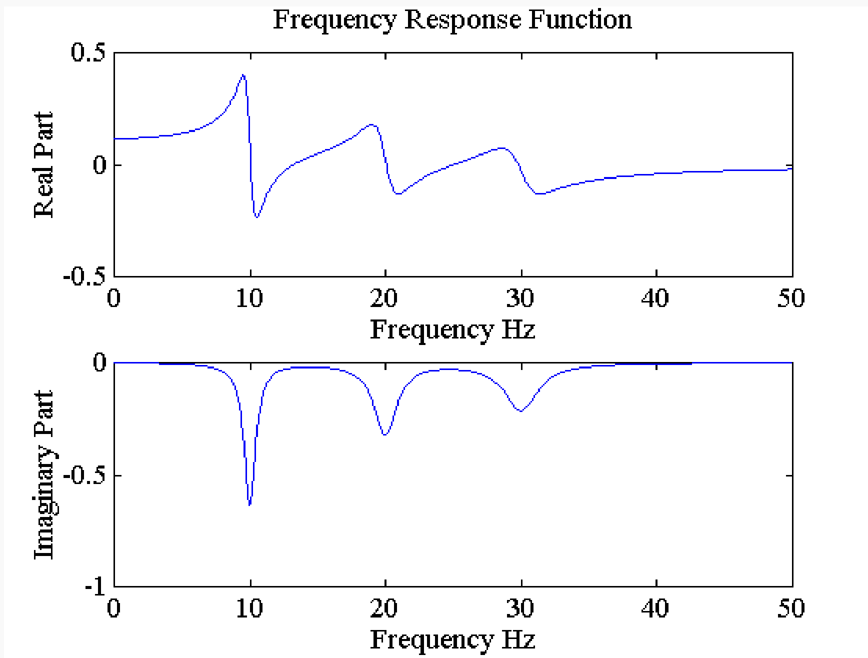
$$\begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2+c_3 & -c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_n+c_{n+1} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_n+k_{n+1} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix}$$

Shorthand notation $M\,\ddot{x}(t) + C\,\dot{x}(t) + K\,x(t) = f(t)$ *Coupled system*

e.g. 3 DOFs

- Magnitude & Phase
- Real & Imaginary





FORCED VIBRATION OF MDOF SYSTEMS

In line with the SDOF system, the FRF can be written as

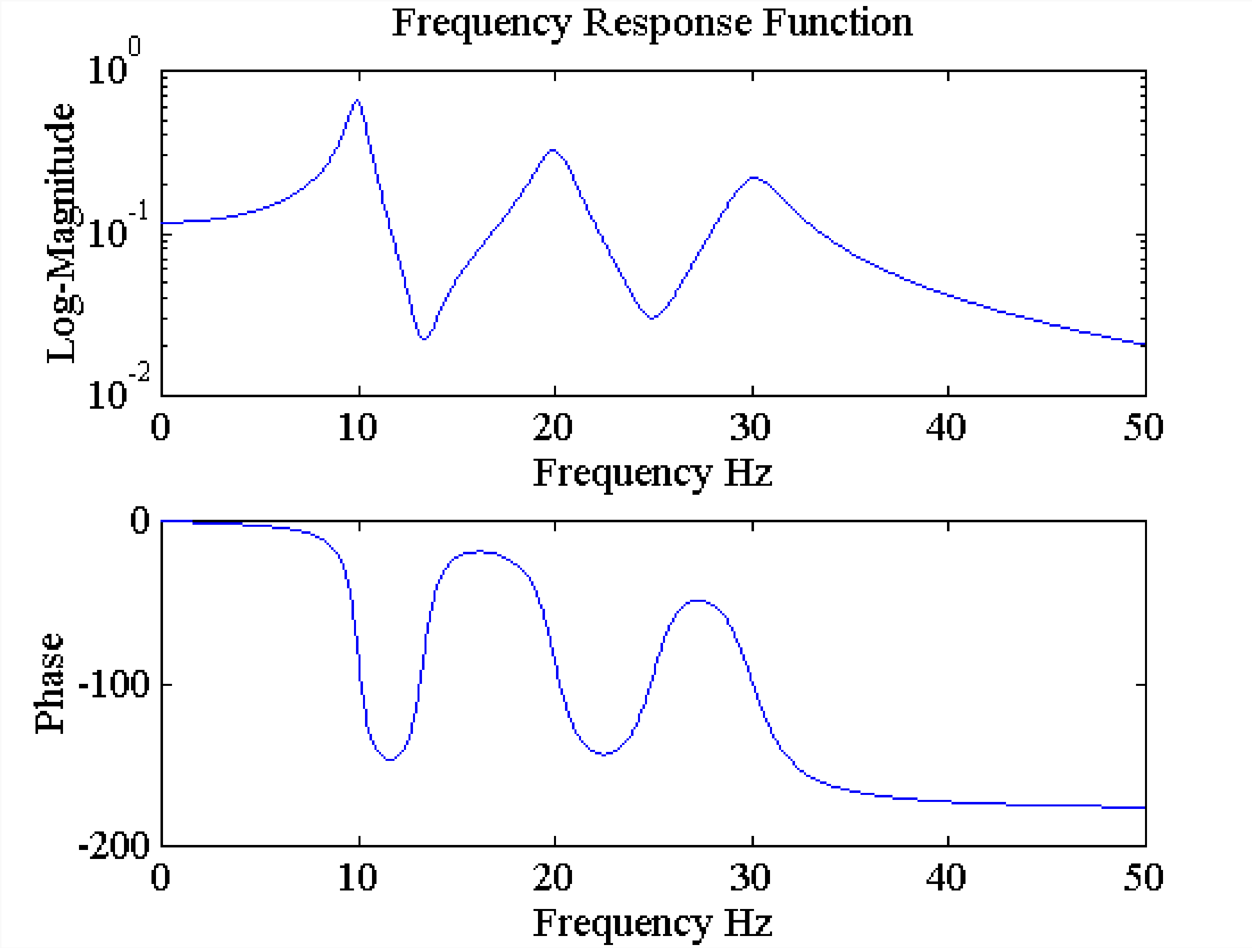
$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{-M\omega^2 + Cj\omega + K}$$

$$(M\ddot{x} + C\dot{x} + Kx = f)$$

As for a SDOF system, the FRF can be written in terms of modal parameters as:

$$|H_{ij}(j\omega)| = \sum_{k=1}^{n} \frac{\psi_{i,k}\psi'_{j,k} / m}{\sqrt{(\omega_{nk}^2 - \omega^2) + (2\zeta_k \omega\omega_{nk})^2}}$$

Magnitude & Phase



FORCED VIBRATION OF MDOF SYSTEMS

$$\left|H_{ij}(j\omega)\right| = \sum_{k=1}^{n} \frac{\psi_{i,k}\psi^{t}_{j,k}/m_k}{\sqrt{\left(\omega_{nk}^2-\omega^2\right)+\left(2\zeta_k\omega\omega_{nk}\right)^2}}$$

With simple mathematics… ;-)

$$H_{ij}(j\omega) = \sum_{k=1}^{n} \frac{A_{ij,k}}{\left(j\omega-\lambda_k\right)} + \frac{A^{*}_{ij,k}}{\left(j\omega-\lambda^{*}_{k}\right)}$$

$$A_{ij,k} = Q_k\psi_{i,k}\psi^{t}_{j,k}$$ → residues

$$\lambda_k = -\zeta_k\omega_{nk} + j\omega_{nk}\sqrt{1-\zeta_k^{\,2}}$$ → poles

Modal parameters

- ω_{nk} Eigenfrequencies
- ξ_k Damping ratios
- ψ_k Mode shapes
- Q_k Modal scaling factors

MDOF Equations of Motion Solution

Proportional damping

- Coupled MDOF equations of motion

$$M\,\ddot{x}(t) + C\,\dot{x}(t) + K\,x(t) = f(t)$$

- Co-ordinate transform: from physical space to modal space

$$x(t) = U\,p(t)$$

Matrix containing all the modes (→ U)

Modal co-ordinates (→ $p(t)$)

$$MU\,\ddot{p}(t) + CU\,\dot{p}(t) + KU\,p(t) = f(t)$$

- Substitution
- Pre-multiplication

$$U^T MU\,\ddot{p}(t) + U^T CU\,\dot{p}(t) + U^T KU\,p(t) = U^T f(t)$$

- Proportional damping

$$U^T CU = \begin{bmatrix} \ddots & & \\ & \bar{c}_i & \\ & & \ddots \end{bmatrix}, \quad \bar{c}_i = 2\zeta_i \omega_i \bar{m}_i$$

- Uncoupled equations of motion (due to the Orthogonality relationship)

$$\begin{bmatrix} \ddots & & \\ & \bar{m}_i & \\ & & \ddots \end{bmatrix} \ddot{p}(t) + \begin{bmatrix} \ddots & & \\ & \bar{c}_i & \\ & & \ddots \end{bmatrix} \dot{p}(t) + \begin{bmatrix} \ddots & & \\ & \bar{k}_i & \\ & & \ddots \end{bmatrix} p(t) = U^T f(t)$$

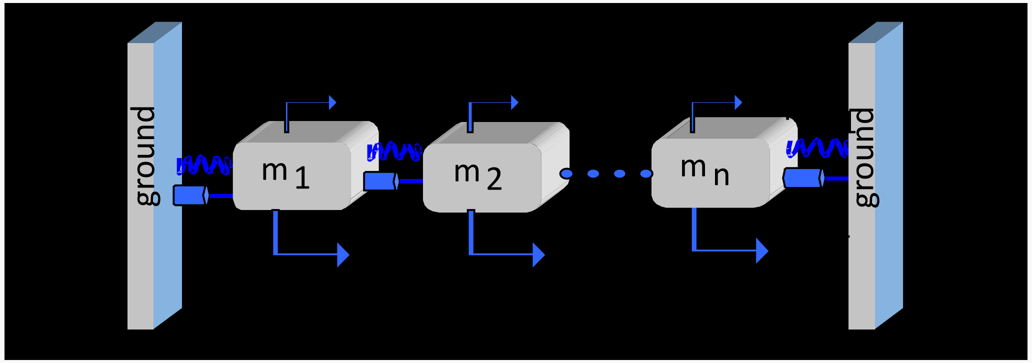
MDOF
Physical Space *vs.* Modal Space

Proportional damping

- Physical space

$$M\,\ddot{x}(t)+C\,\dot{x}(t)+K\,x(t)=f(t)$$

Physical
Mass
Damping
Stiffness



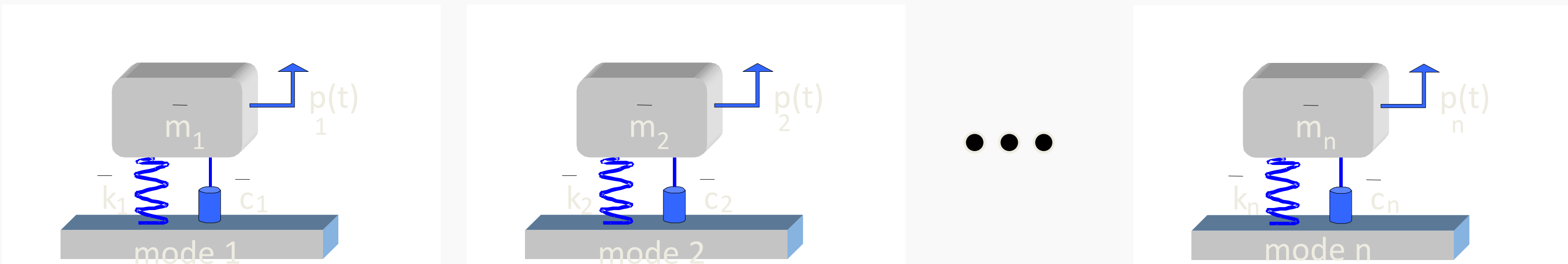
$$x(t)=U\;p(t)$$

Co-ordinate transformation

Modal space

$$\left[\diagdown \bar{m}_i \diagdown\right]\ddot{p}(t)+\left[\diagdown \bar{c}_i \diagdown\right]\dot{p}(t)+\left[\diagdown \bar{k}_i \diagdown\right]p(t)=U^T f(t)$$

Modal
Mass
Damping
Stiffness

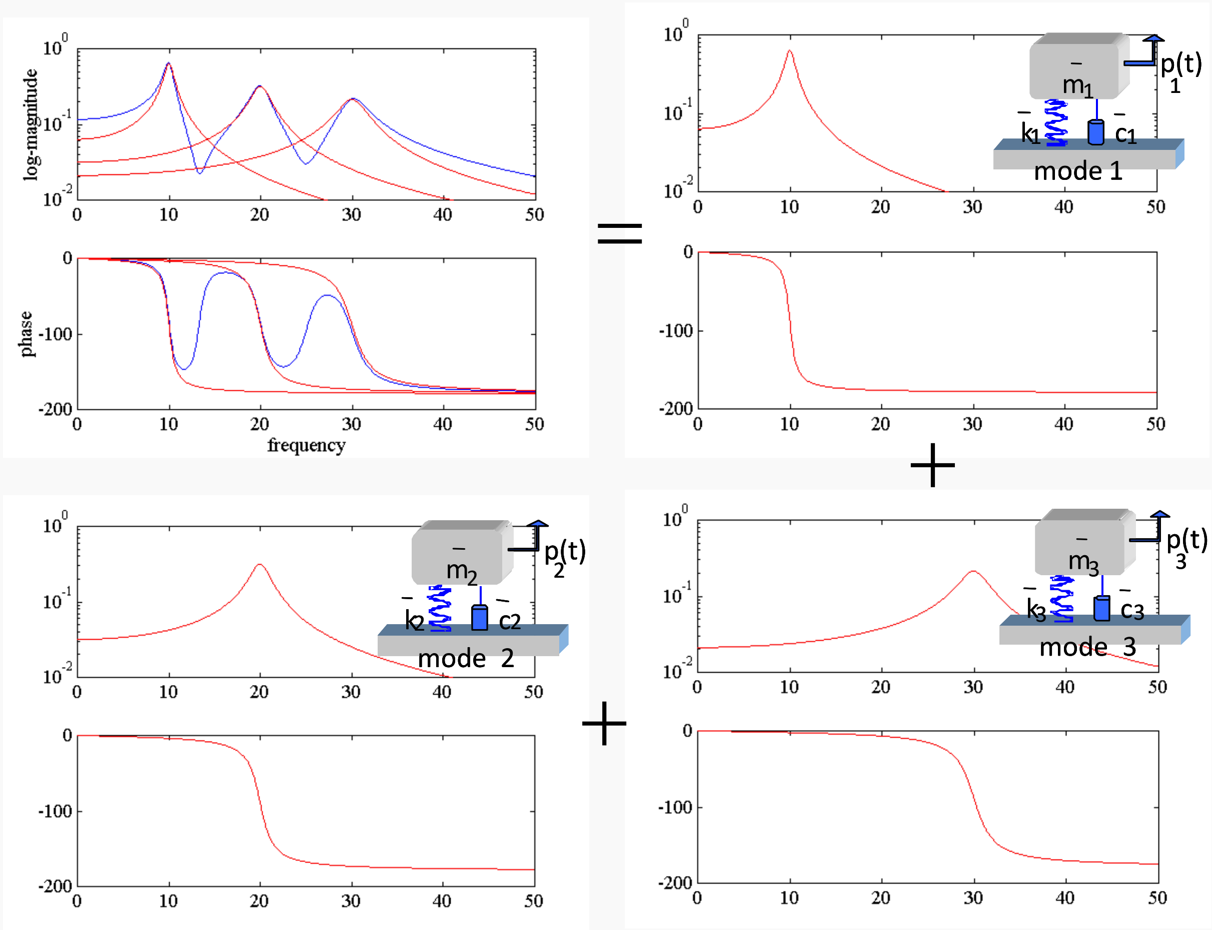


In modal space, the uncoupled equation system corresponds to *n* equations of a SDOF system in modal coordinate *p(t)*. *Thus, a very complicated system (MDOF) can be considered as the sum of n SDOF systems (in modal space)*

FRF
Modal Decomposition

e.g. 3 DOFs

$$H_{pq}(j\omega) = \frac{A_{pq,1}}{j\omega - \lambda_1} + \frac{A^*_{pq,1}}{j\omega - \lambda^*_1} + \frac{A_{pq,2}}{j\omega - \lambda_2} + \frac{A^*_{pq,2}}{j\omega - \lambda^*_2} + \frac{A_{pq,3}}{j\omega - \lambda_3} + \frac{A^*_{pq,3}}{j\omega - \lambda^*_3}$$



The FRF of a 3 DOFs system can be obtained by the sum of 3 SDOF system in modal space.

FORCED VIBRATION OF MDOF SYSTEMS

poles

- Modal parameters
  - Eigenfrequency
    - *Peak in FRF*

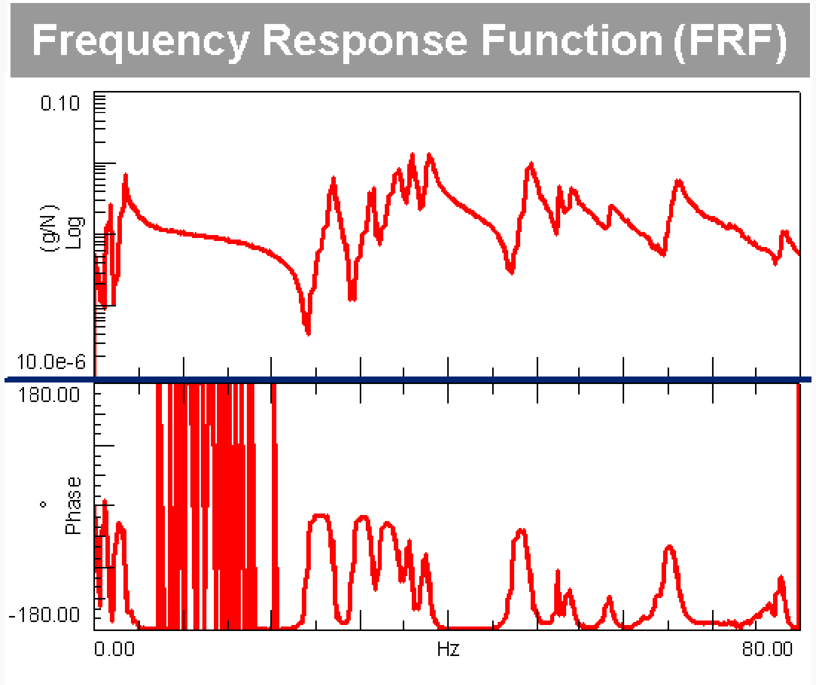
$$\lambda_k, \lambda_k^* = -\xi_k \omega_{nk} \pm j\omega_{nk}\sqrt{1-\xi_k^2}$$

  - Damping ratio
    - *Width of FRF peak*

$$\lambda_k, \lambda_k^* = -\xi_k \omega_{nk} \pm j\omega_{nk}\sqrt{1-\xi_k^2}$$

  - Mode shape
    - *± Deformation at eigenfrequency*

$$A_{ij,k} = Q_k \Psi_{i,k} \Psi^t{}_{j,k}$$



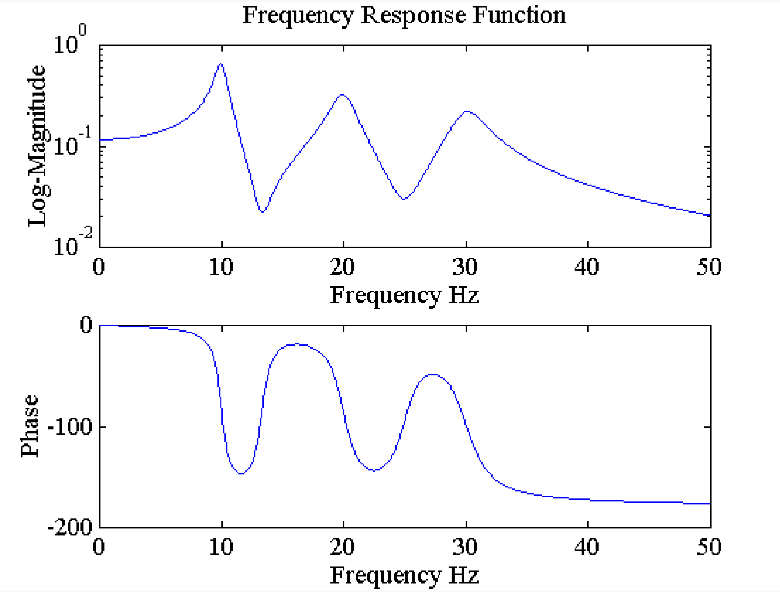
MDOF
Impulse Responses

- FRF

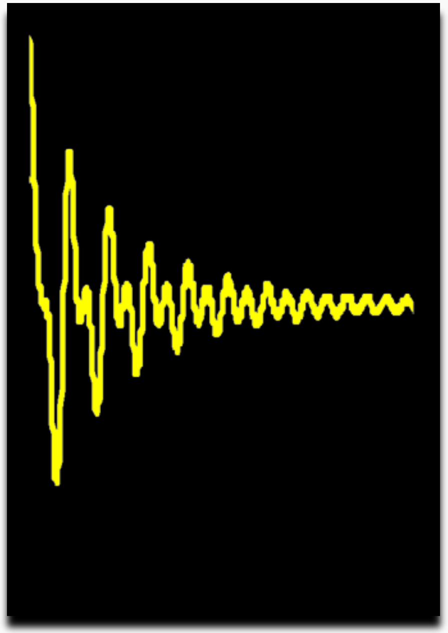
$$H_{pq}(j\omega) = \sum_{k=1}^{n} \frac{A_{pq,k}}{j\omega - \lambda_k} + \frac{A_{pq,k}^*}{j\omega - \lambda_k^*}$$

- Inverse Fourier transform of the FRF is the Impulse Responses

$$h_{pq}(t) = \sum_{k=1}^{n} A_{pq,k} e^{\lambda_k t} + A_{pq,k}^* e^{\lambda_k^* t}$$



IFFT



FRF MATRIX

input

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}$$

response

Reciprocity: $H_{xy} = H_{yx}$

Driving point measurement: H_{xx}

MDOF System Theory - Experimental Implications

physical model

$$M\,\ddot{x}(t) + C\,\dot{x}(t) + K\,x(t) = f(t)$$

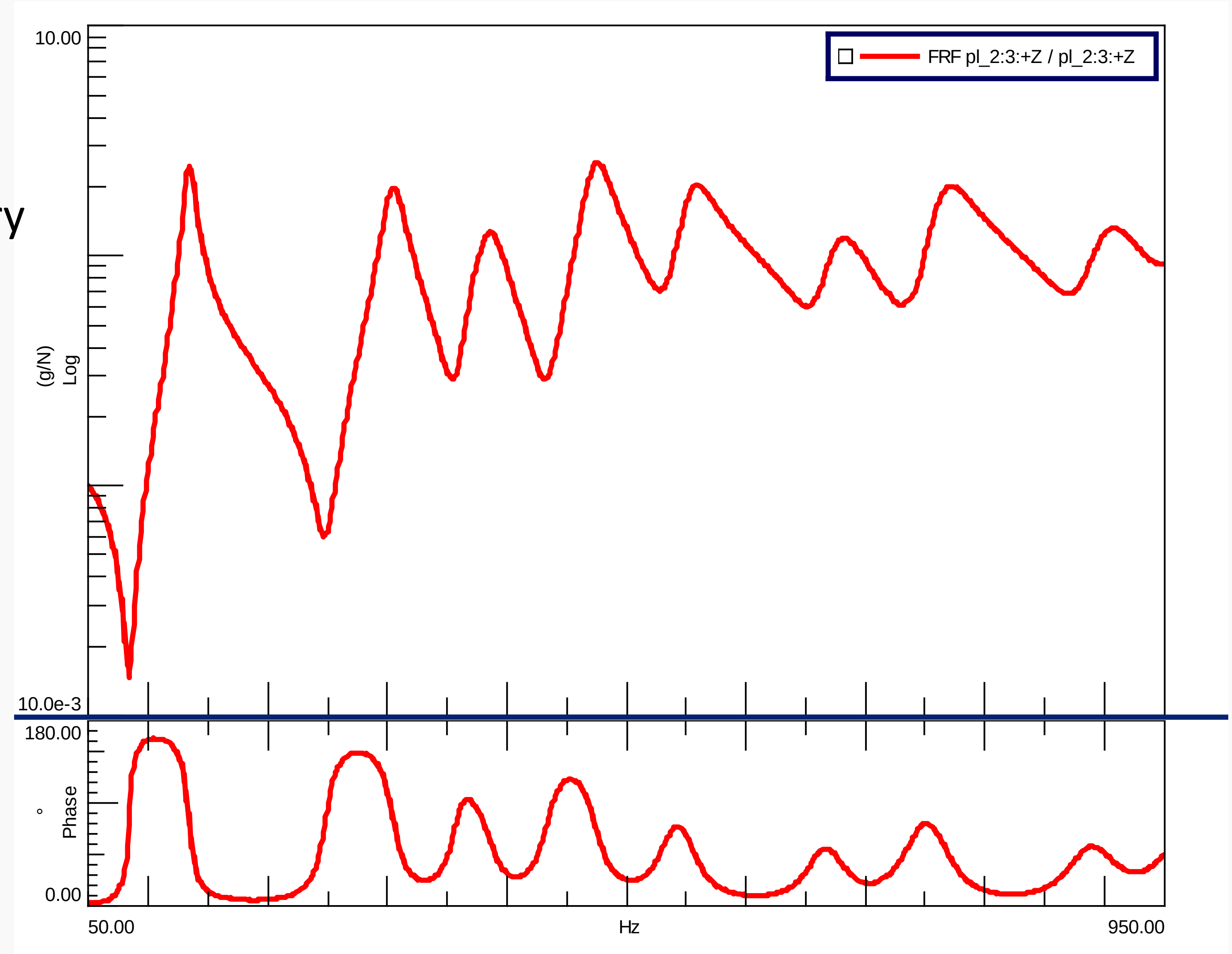
$$H(j\omega) = \sum_{k=1}^{n} \frac{A_k}{j\omega - \lambda_k} + \frac{A_k^*}{j\omega - \lambda_k^*}$$

Modal model

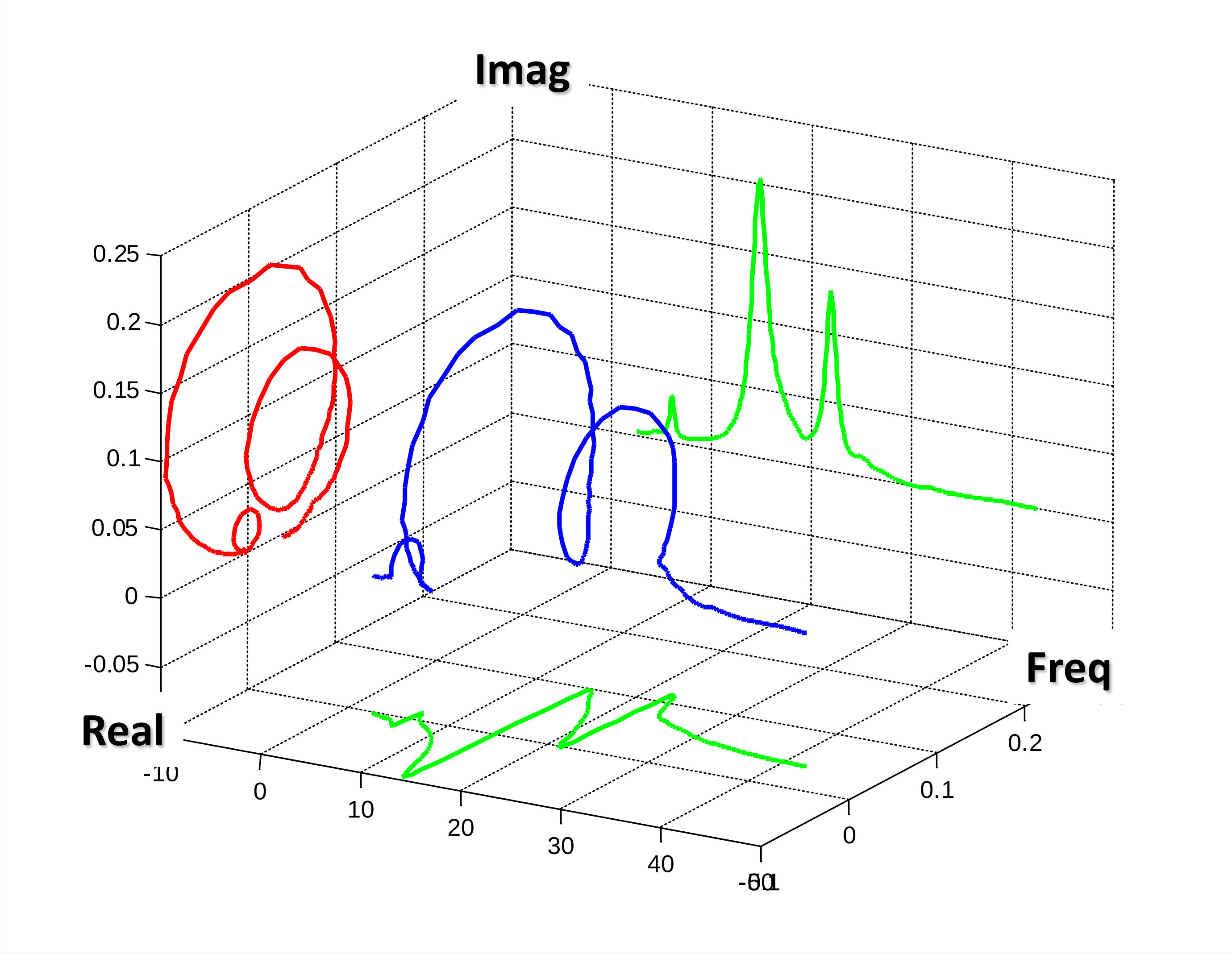
- Experimental conditions
  - –The mass of the structure will remain constant for the duration of the test.
  - –The stiffness of the structure will remain constant for the duration of the test.
  - –The damping in the structure will remain constant for the duration of the test.
  - –The poles of the structure are global properties and as such they remain constant for every FRF measurement, regardless of position on the structure.
  - –It is only necessary to measure one row or column of the FRF matrix to obtain the mode shapes

Driving point FRF's

- anti-resonances occur between every resonance

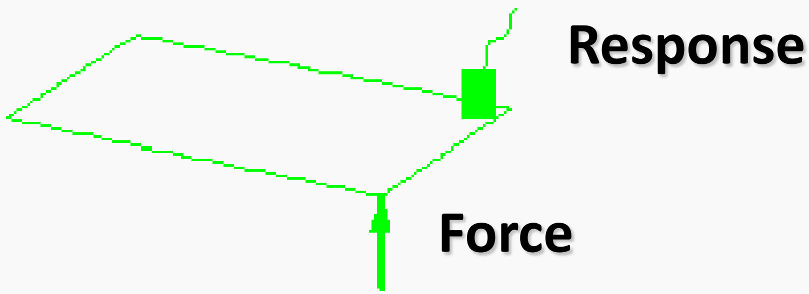


3D FRF & Circle Fit



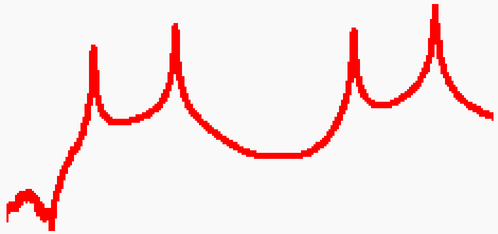
Structural Dynamics

- Apply constant peak force, but with changing rate of oscillation (i.e. change the frequency of the signal)

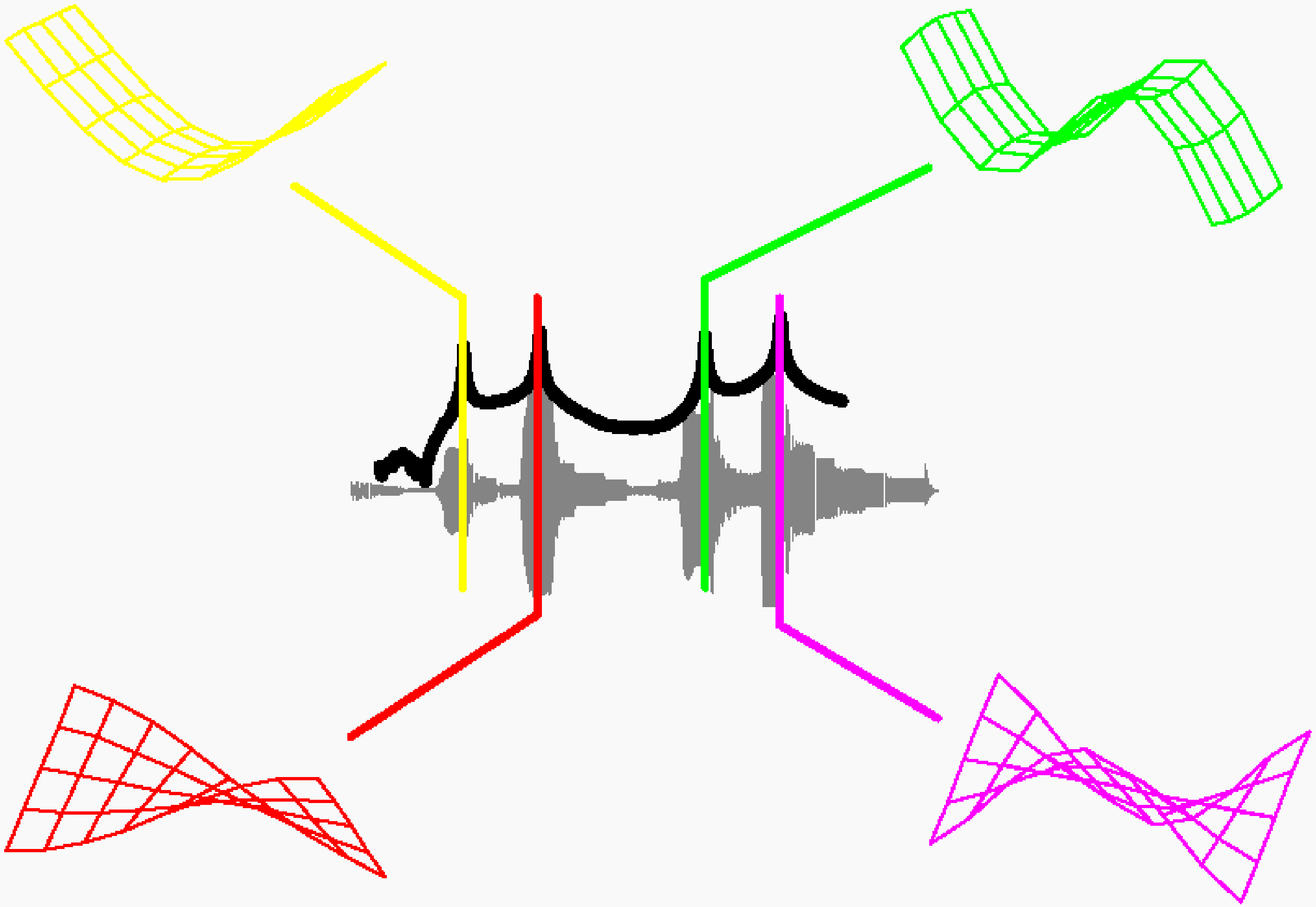


- Response in time domain

- Response in frequency domain

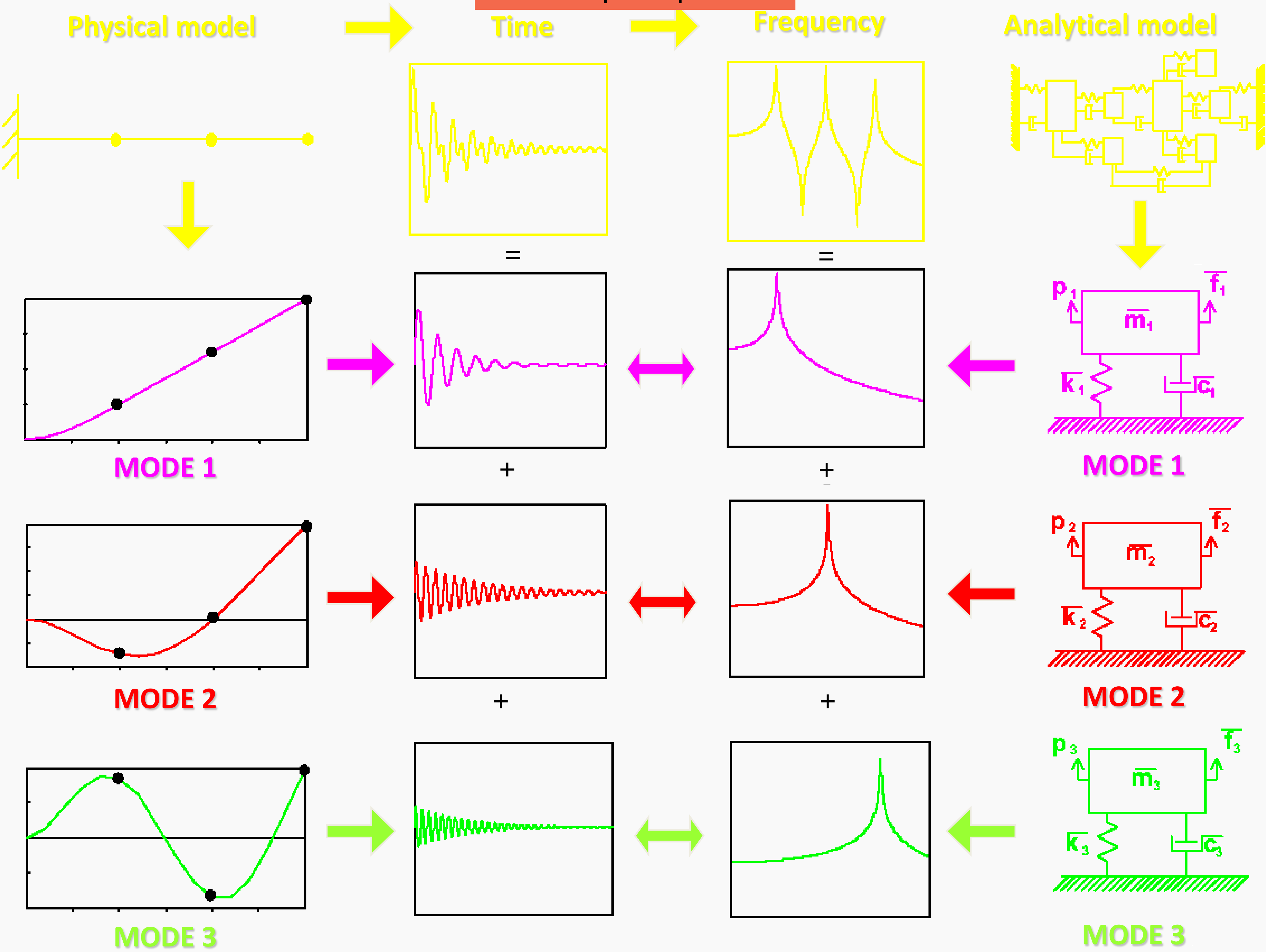


Structural Dynamics



When the response in the time domain is amplified, thus a peak in the frequency domain occurs, due to resonance

The complete picture



DSP overview

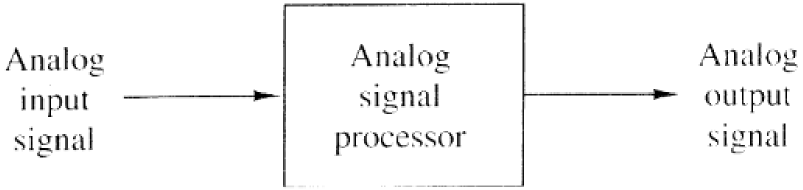
- Analog to digital conversion
- Overview basic digital signal processing concepts:
  - Aliasing
  - Leakage
  - Windowing
  - Fourier transform, Autopower Spectrum, PSD
- Coherence
- Estimation of FRF

Signals - classification

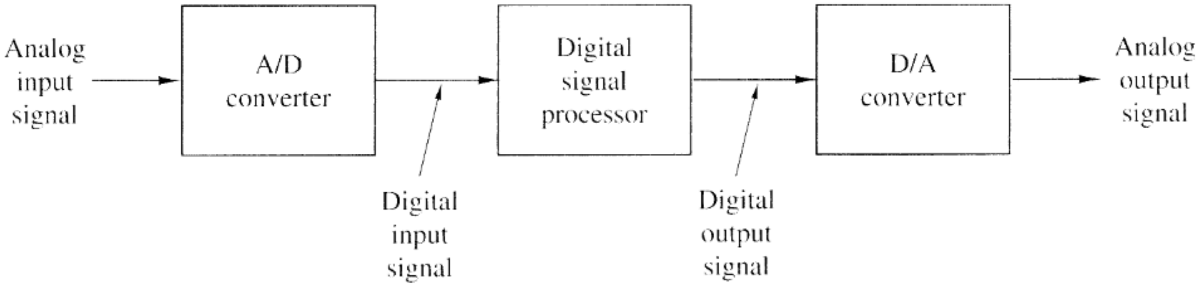
- A **signal** is defined as any physical quantity that varies with time, space, or any other independent variable of variables

- Most of the signals encountered in science and engineering are analog in nature. That is, the signals are functions of a continuous variable and usually take on values in a continuous range. Such signals may be processed directly by appropriate analog systems.

Analog signal processing:



Digital signal processing:



Signals- classification

Continuous-Time
&
Continuous-Value

} **Analog signal**

Discrete-Time
&
Discrete-Value

} **Digital signal**

Signals – the concept of frequency

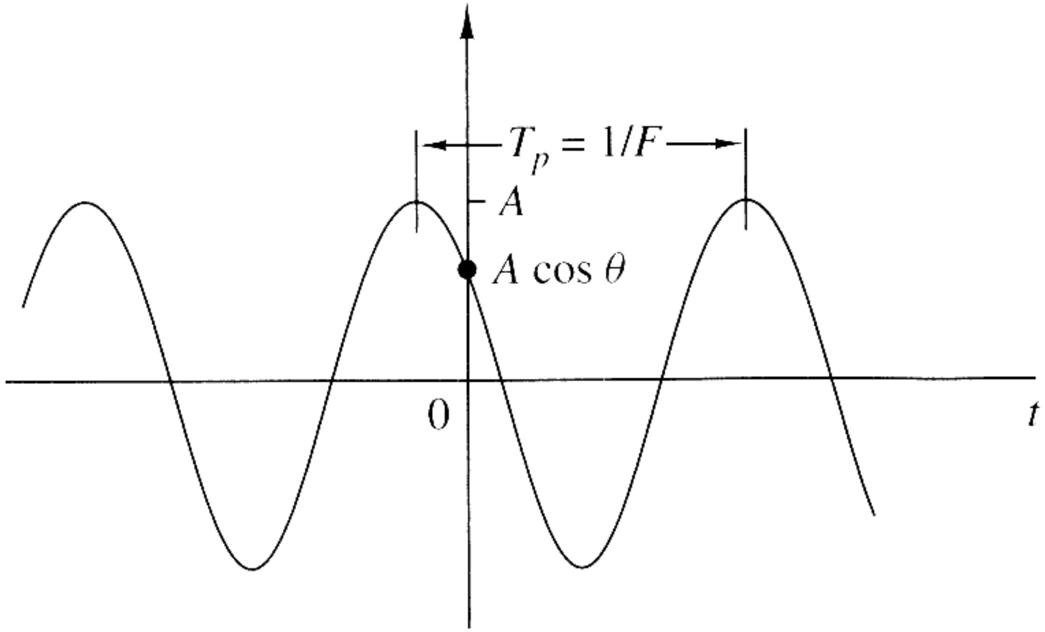
An analog signal

$$x_a(t) = A\cos(2\pi F t + \theta) \qquad -\infty < t < +\infty$$

Amplitude

Frequency [Hz] $F = \dfrac{\Omega}{2\pi}$

Phase [rad]



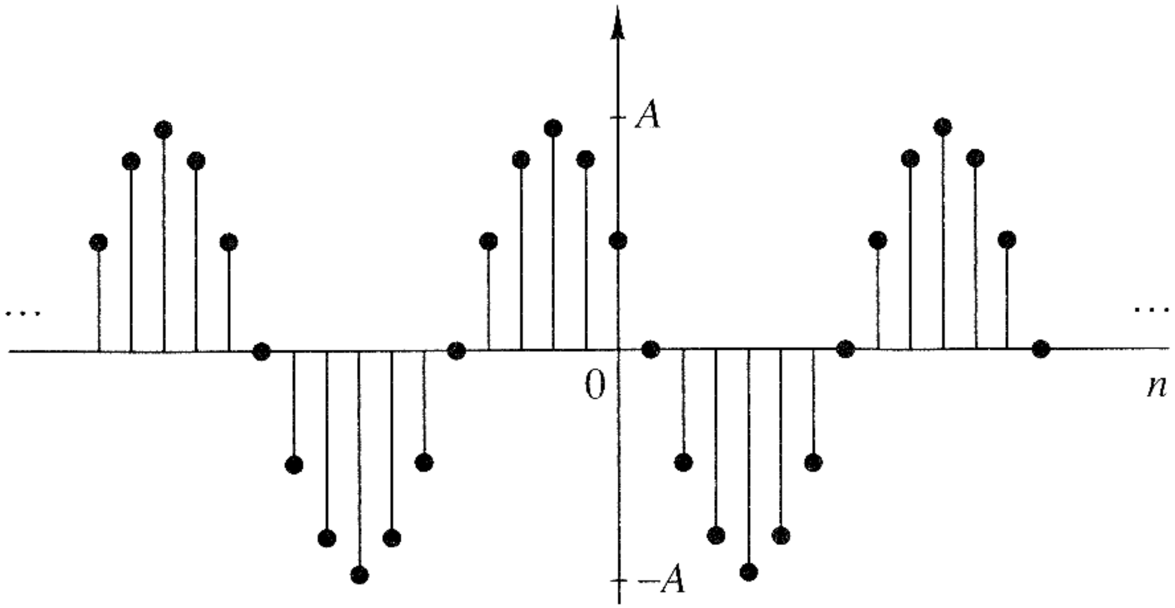
$$x_a(t+T) = x_a(t)$$

Increasing the frequency results in an increase in the rate of oscillation of the signal !!!!

Signals – the concept of frequency

Discrete-time signal

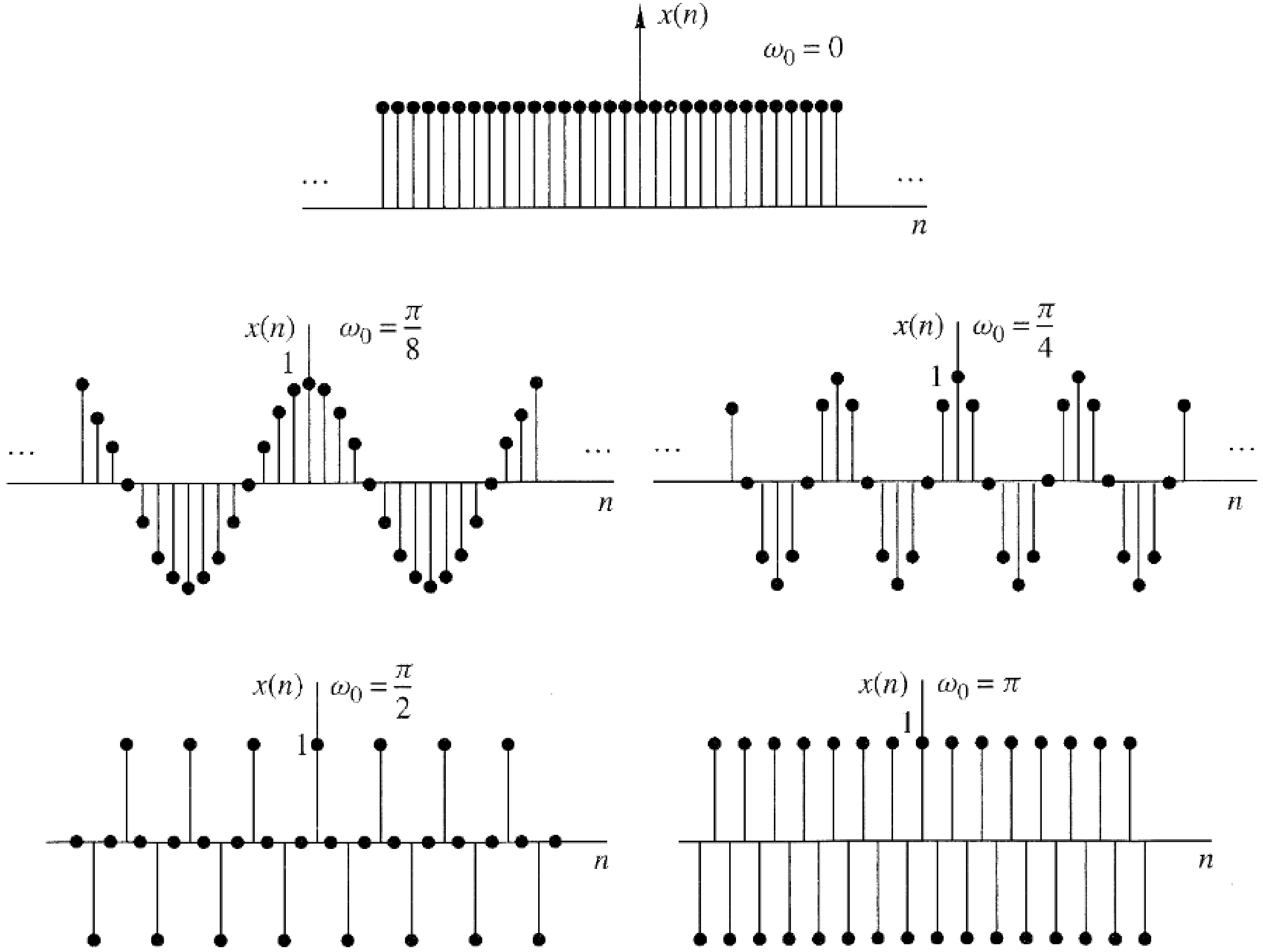
$$t \to n \qquad \longrightarrow \qquad x(n) = A\cos(2\pi fn + \theta) \qquad -\infty < n < \infty$$



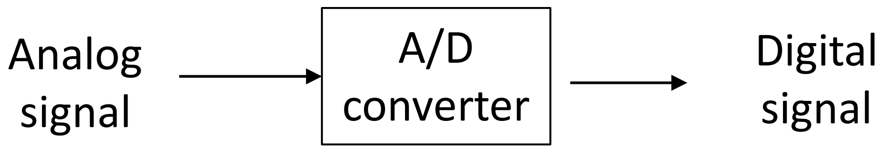
Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical

$$\cos(2\pi(f_0 + 2\pi)\text{n} + \theta) = cos(2\pi f_0 n + 4\pi n + \theta)$$
$$= \cos(2\pi f_0 n + \theta)$$

Signals – the concept of frequency



Signals – sampling of analog signals



The A/D converter perform two operations:

1) Sampling:

Continuous-time → Discrete-time

2) Quantization:

Continuous-valued → Discrete-valued

The operation of sampling and quantization can be performed in either order but, in practice, sampling is always performed before quantization

ANALISI IN FREQUENZA

Serie di Fourier
Come è noto, una funzione *x(t)* periodica di periodo T si può rappresentare mediante la serie di Fourier:

$$x(t) = X_0 + X_1 \cos(2\pi\ f_1 t + \varphi_1) + X_2 \cos(2\pi 2 f_1 t + \varphi_2) + \ldots + X_1 \cos(2\pi\ n f_1 t + \varphi_n),$$

ovvero:

$$x(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi\ n f_1 t + \varphi_n)$$

dove:

f_1 è la frequenza fondamentale (frequenza dell'armonica fondamentale, che ha ampiezza X_1)
X_0 è il valore medio di *x(t)*
X_n è l'ampiezza della *n*–esima armonica, di frequenza nf_1
φ_n è la fase della *n*–esima armonica

Abbiamo riportato la notazione più usata, cioè quella solo in coseno ma, naturalmente, si può trovarla anche solo in seno o in seno e coseno.

Se si ha una funzione periodica, effettuarne l'analisi di Fourier significa ricavare le ampiezze X_n e le fasi φ_n. Si può pensare di compiere l'analisi di Fourier con un filtro che abbia la caratteristica di lasciar passare solo le componenti comprese tra una certa frequenza *f** e la *f** più un certo incremento. Ricordiamo che il filtro è un circuito elettronico (dato che il segnale è elettrico). In figura è rappresentato un filtro ideale; in realtà è presente una certa dispersione.

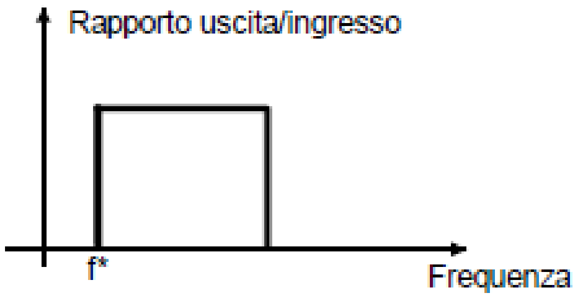


Fig. 8.2 – Filtro ideale

Trasformata di Fourier
Per una funzione *x(t) non periodica,* con la condizione che l'integrale da –∞ a +∞ del valore assoluto di *x(t)* sia una quantità finita, al posto della serie si definisce la *Trasformata di Fourier:*

$$X(f) = \mathbf{F}\{x(t)\} = \int_0^{\infty} x(t)\, e^{-i2\pi f t} dt$$

La trasformata di Fourier è una funzione complessa, per cui si rappresenta con la parte reale e la parte immaginaria:

$$X(f) = \Re[X(f)] + i\Im[X(f)]$$

oppure mediante modulo e fase: $X(f) = |X(f)| e^{i\Phi(f)}$

in cui: $|X(f)| = \sqrt{\Re[X(f)]^2 + \Im[X(f)]^2}$ $\qquad tg[\Phi(f)] = \dfrac{\Im[X(f)]}{\Re[X(f)]}$

La *X(f)* si rappresenta graficamente mediante gli andamenti della parte reale e di quella immaginaria, o di ampiezza e fase in funzione della frequenza.

In realtà, però, il segnale che si ha a disposizione non permette, a rigore, di calcolare la trasformata di Fourier. Infatti ciò che si possiede è un segnale rilevato da un certo istante iniziale fino ad un tempo T* finito.
Le conseguenze sono che:

- $X(f) = \mathbf{F}\{x(t)\} = \lim_{T \to \infty} \int_0^{T} x(t)\, e^{-i2\pi f t} dt$ può non esistere
- se si elabora questo segnale calcolandone la trasformata di Fourier, è come se si considerasse il segnale "prolungato" da –∞ a +∞ prima e dopo l'intervallo di acquisizione T*. Cioè, è come se il segnale si ripetesse periodicamente, con periodo T*, per *t* da –∞ a +∞.

Si deve perciò calcolare in realtà: $X(f, T^*) = \mathbf{F}\{x(t)\} = \int_0^{T^*} x(t)\, e^{-i2\pi f t} dt$

chiamata *Trasformata Finita di Fourier.*

In questo modo la funzione che si considera non è più non periodica, ma "periodica" di periodo T*, definita da −∞ a +∞.

Se si riportano le ampiezze in funzione delle frequenze, si ottiene uno spettro discontinuo, appunto per il fatto che la funzione viene trattata come periodica di periodo T*.

Lo spettro ha una *risoluzione* (distanza tra due linee contigue): $\Delta f=1/T^*$

È importante sottolineare che la frequenza Δf non è (in generale) una frequenza del segnale, ma dipende solo dal tempo di acquisizione T*. Non è detto che tale frequenza, o qualcuno dei suoi multipli, siano effettivamente presenti nel segnale.

Supponiamo, ad esempio, di avere una struttura che vibra: essa avrà una certa frequenza f_I del primo modo, f_{II} del secondo modo e così via. Se si rileva il segnale mettendo il trasduttore sulla struttura, tali frequenze saranno presenti nel segnale. Se si rileva il segnale per un tempo T*, nello spettro compaiono componenti alle frequenze pari ad un multiplo intero della frequenza fondamentale $\Delta f=1/T^*$. Di regola succederà che f_I e f_{II} non siano dei multipli di Δf: nello spettro si trova allora solo un "addensamento" attorno a tali valori.

In corrispondenza delle componenti f_I e f_{II}, che non si ritrovano perché hanno una frequenza che non esiste sullo spettro discreto, compaiono allora delle componenti a frequenze vicine (vedi figura 8.3), la cui energia totale coincide con quella delle componenti f_I e f_{II}.

Questo fenomeno è detto *leakage* (dispersione): poiché si rileva la funzione in un tempo T* finito, cioè guardando il segnale attraverso una finestra rettangolare, le frequenze effettivamente presenti si "disperdono" nelle frequenze prossime ad esse, ma sempre multiple di $\Delta f=1/T^*$.

Per diminuire la dispersione si utilizzano *finestre* di forma diversa; uno dei tipi più usati è la finestra Hanning, che ha la proprietà di annullare il segnale all'inizio e alla fine dell'acquisizione, per cui si elimina la discontinuità che altrimenti si avrebbe all'inizio del periodo. Utilizzando le finestre si ottengono degli spettri più vicini alla realtà rispetto alla finestra rettangolare, che dà spettri più dispersi.

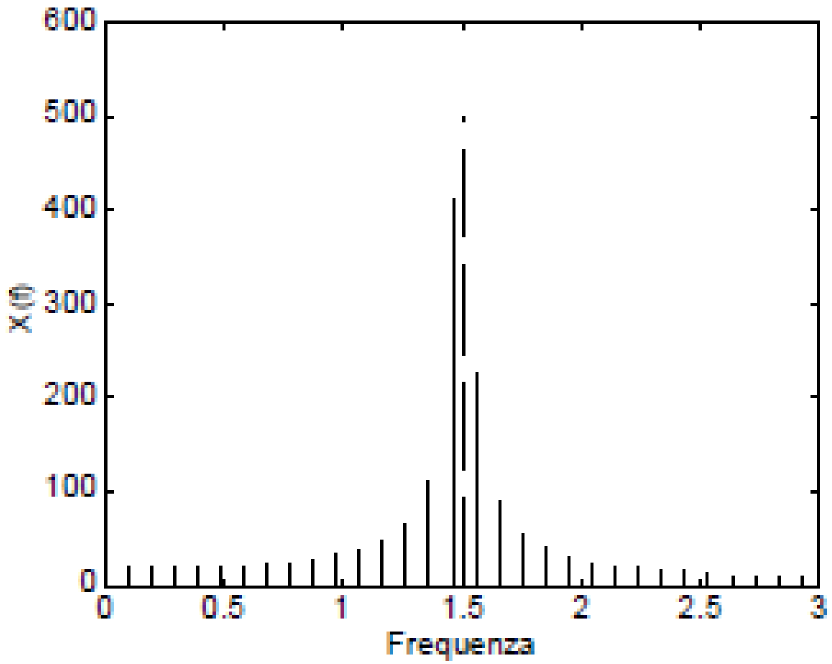
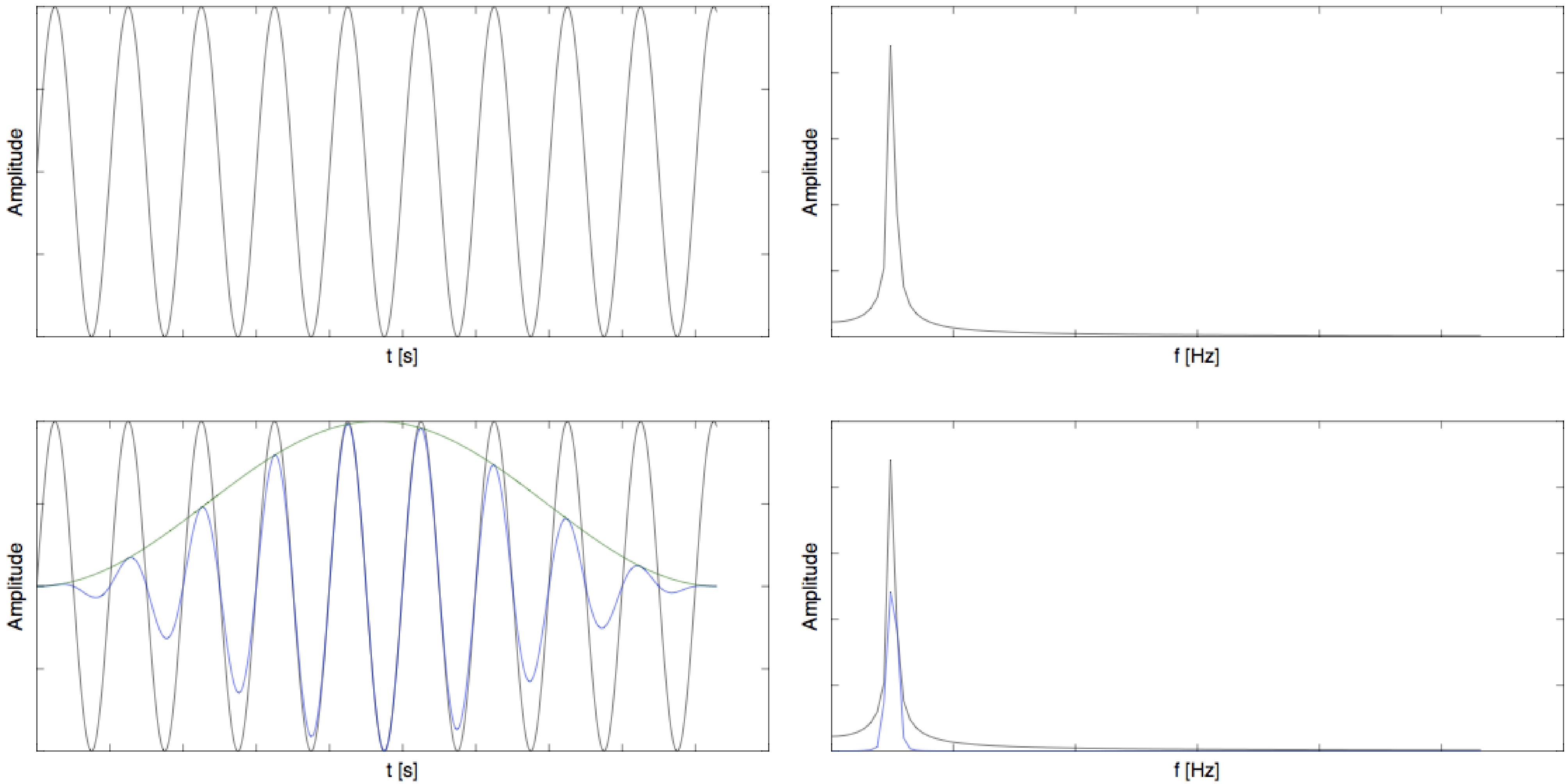


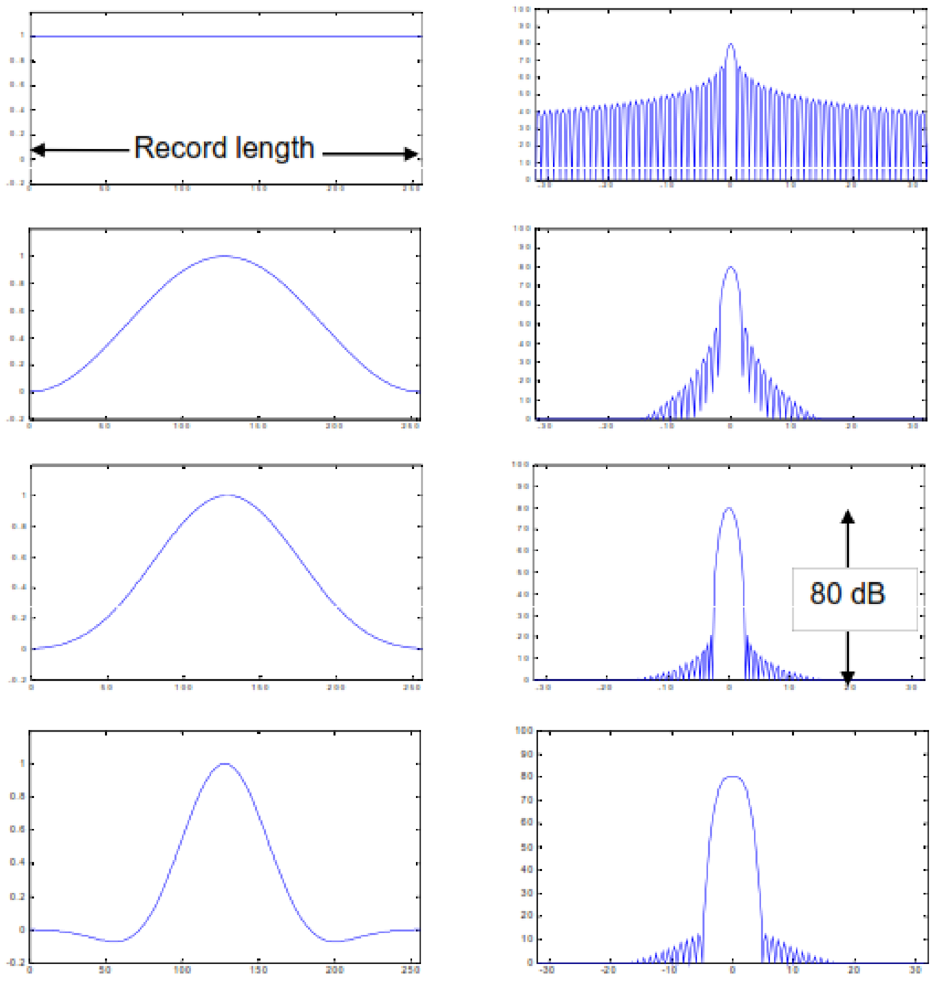
Fig. 8.3 – Dispersione

Effect of time truncation - Leakage



Leakage - Windows

Nel tempo   In frequenza



Rectangular – ie do nothing

Hanning – good general purpose window

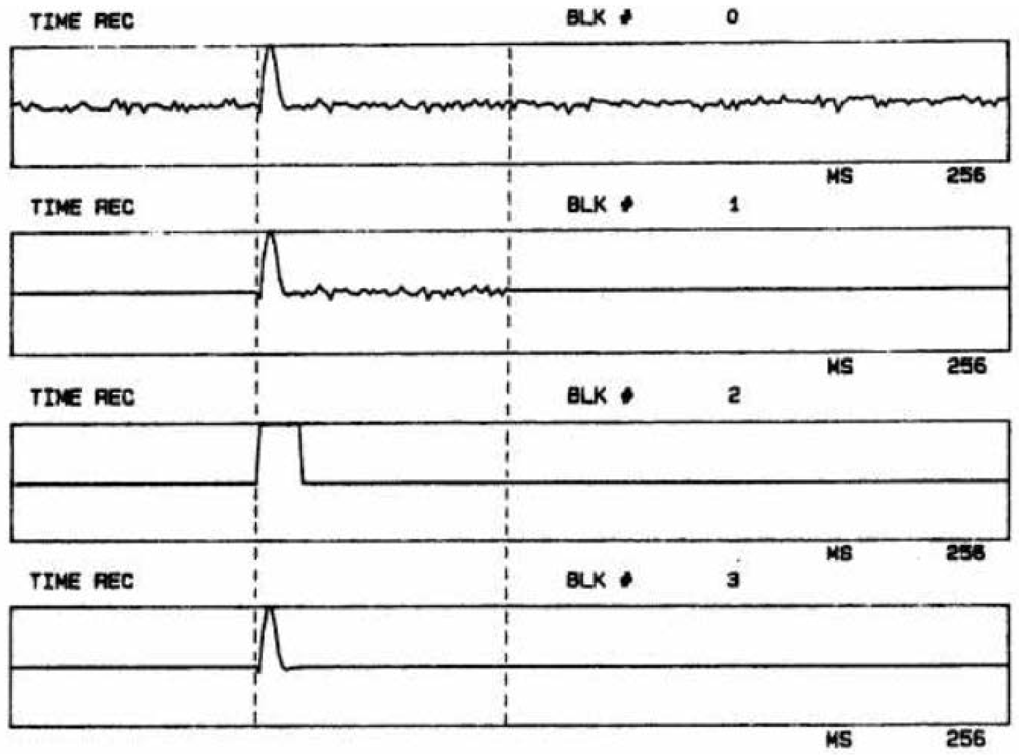
Kaiser-Bessel – gives best separation of adjacent components

Flat top – zero picket fence correction – good for calibration and sinusoidal components

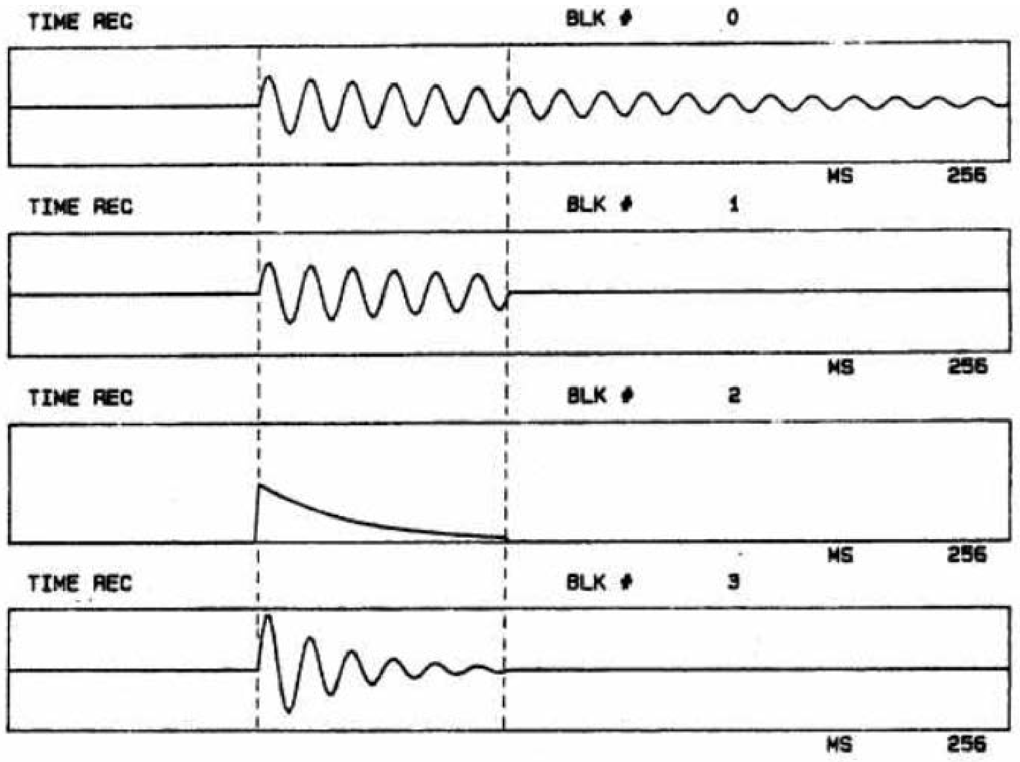
Windows - Force/Exponential for Impact Testing

Special windows are used for impact testing:

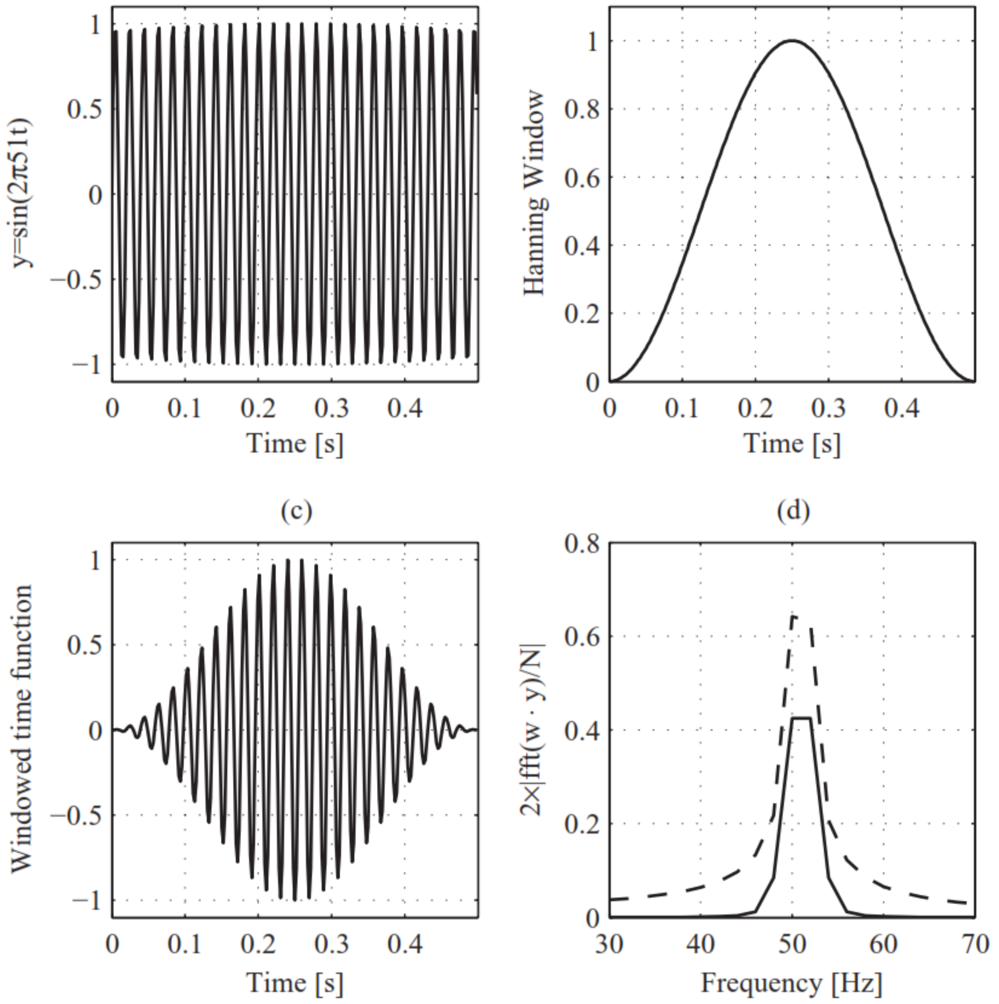
Force window



Exponential window



Windows – Amplitude correction



finestra

$$X(\mathrm{k}) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,\mathrm{w(n)}\,e^{-j2\pi\frac{k}{N}n}$$

Amplitude correction factor:

$$A_w = \frac{N}{\sum_{n=0}^{N-1} w(n)}$$

$$X_w(k) = \frac{A_w}{N}\sum_{n=0}^{N-1} x(n)w(n)\mathrm{e}^{-\mathrm{j}2\pi kn/N}$$

CAMPIONAMENTO

È possibile analizzare il segnale con un computer se è presente nella catena di misura un convertitore A/D che lo trasformi in una serie di numeri. L'operazione viene chiamata *campionamento:* ad intervalli regolari di tempo il convertitore legge il valore istantaneo del segnale.

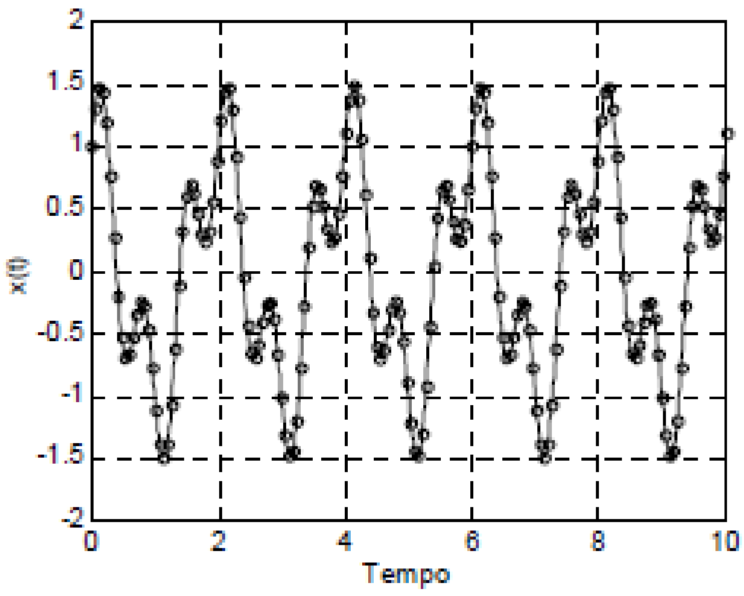


Fig. 8.4 – Campionamento

All'uscita dal convertitore A/D non si ha più un segnale continuo ma un segnale discreto.

L'intervallo di tempo Δt_c tra due acquisizioni successive è detto *intervallo di campionamento;* il suo inverso $f_c = 1/\Delta t_c$ è detto *frequenza di campionamento.*

Il campionamento permette un'analisi del segnale veloce e sofisticata, ma occorre che la f_c sia adeguata per non alterare il segnale.

ALIASING

Supponiamo che il segnale sia sinusoidale: effettuandone il campionamento con una f_c troppo bassa, il segnale viene interpretato come un segnale a frequenza più bassa. Qualsiasi analisi successiva dà allora risultati errati, perché è fatta su un segnale diverso da quello effettivo.

Questo fenomeno è detto *aliasing* (alterazione). Per evitare l'aliasing deve essere soddisfatto il *Teorema di Shannon* o del campionamento, secondo il quale deve essere:

$f_c \geq 2 f_{max}$ essendo f_{max} la più alta frequenza contenuta nel segnale.

Dato che non si conosce a priori il contenuto in frequenza del segnale da analizzare, affinché sia soddisfatta tale condizione bisogna usare un *filtro antialiasing* (AA), che è un filtro passa-basso che lascia passare solo le componenti con frequenza inferiore alla frequenza massima di interesse f_{max}. La frequenza di campionamento dovrà essere non inferiore a 2 f_{max}. Solitamente si assume f_c = 2.5 f_{max}.

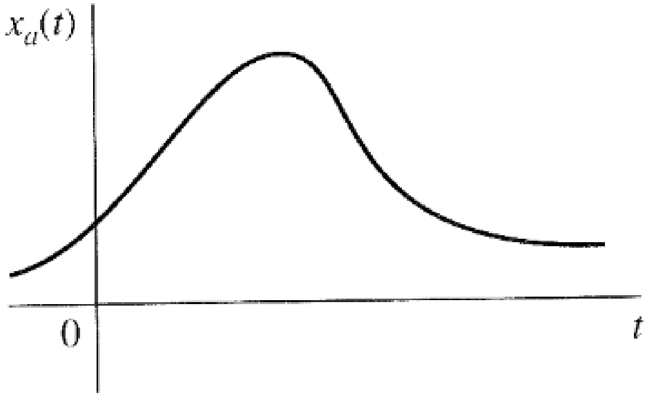
Valgono le seguenti relazioni:

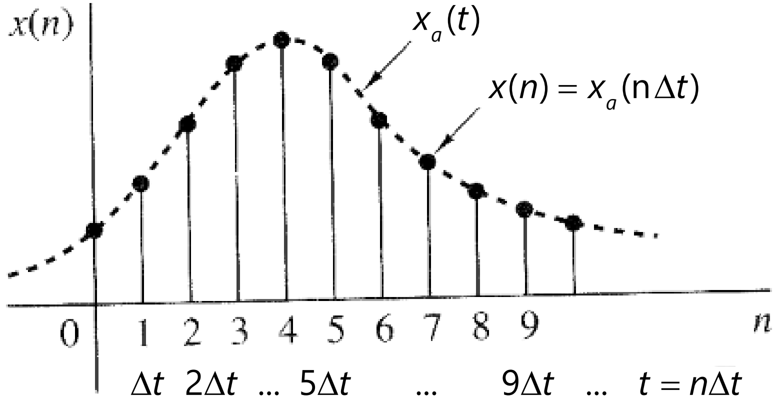
$$T^* = N \cdot \Delta t_c = N \frac{1}{f_c} = \frac{1}{\Delta f}$$

in cui:

Δf è la risoluzione dello spettro
f_c è la frequenza di campionamento, che in pratica vale 2.5 volte la massima frequenza di interesse
T^* è il tempo di acquisizione
Δt_c è l'intervallo di campionamento
N è il numero di campioni

Signals – Sampling Of Analog Signals





$$x_a(t) = A cos(2\pi F t + \theta) \longrightarrow x_a(n\Delta t) = x(n) = A cos(2\pi F n \Delta t + \theta)$$

$$F_s = \frac{1}{\Delta t}$$

$$= A\cos\left(2\pi \frac{F}{F_s} n + \theta\right)$$

Signals – Sampling of analog signals

Periodic sampling of a continuous-time signal implies a mapping of the infinite frequency range of the variable F into a finite frequency range of the variable f, where the maximum frequency value inside the analog signal is

$$F_{max} = \frac{F_s}{2}$$

Sampling theorem:

If the highest frequency contained in an anlog signal $x_a(t)$ is F_{max} and the signal is sampled at rate $F_s > 2F_{max}$, then $x_a(t)$ can be exactly recovered from its sample values

→ $\frac{F_s}{2}$ is called the Nyquist frequency or folding frequency

$$x_a(n\Delta t) = A\cos\left(2\pi \frac{F}{F_s} n + \theta\right)$$

Signals – sampling of analog signals

E.g.:

$x_1(t) = cos(2\pi 10t)$

$x_2(t) = cos(2\pi 50t)$

Sample frequency: $F_s = 40Hz$ ⟶

$$x_1(\text{n}) = cos\left(2\pi \frac{10}{40} n\right) = \cos\left(\frac{\pi}{2} n\right)$$

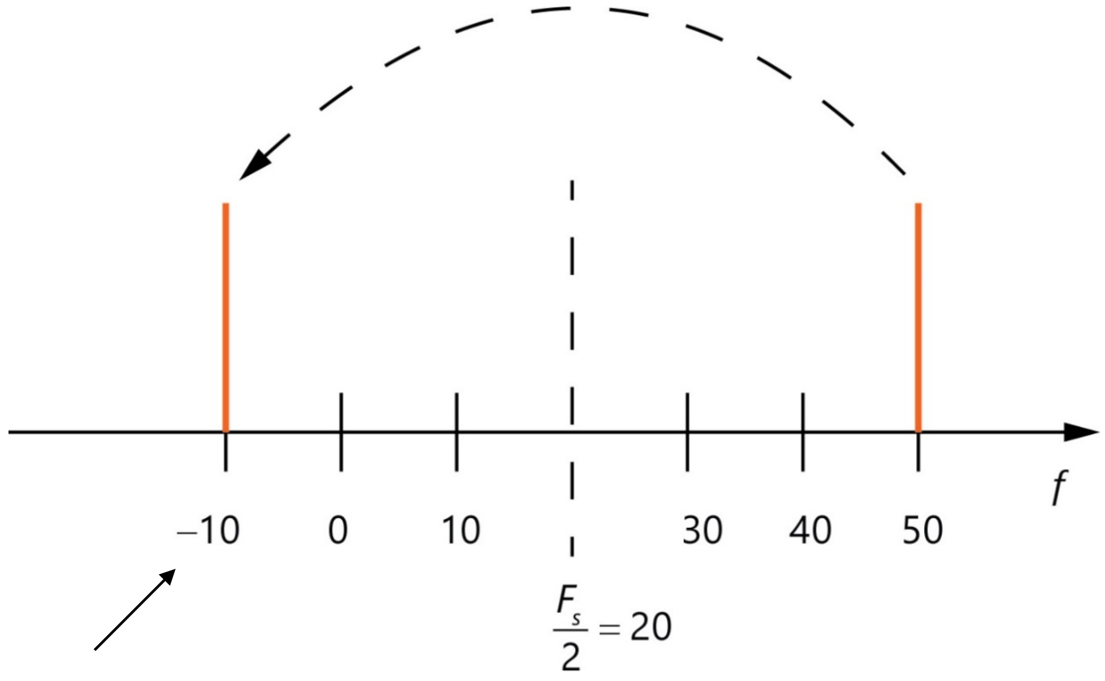
$$x_2(\text{n}) = cos\left(2\pi \frac{50}{40} n\right)$$

$$= \cos\left(\pi \frac{5}{2} n\right)$$

$$= \cos\left(2\pi n + \frac{\pi}{2} n\right)$$

$$= \cos\left(\frac{\pi}{2} n\right)$$

alias



$\cos(\theta) = cos(-\theta)$

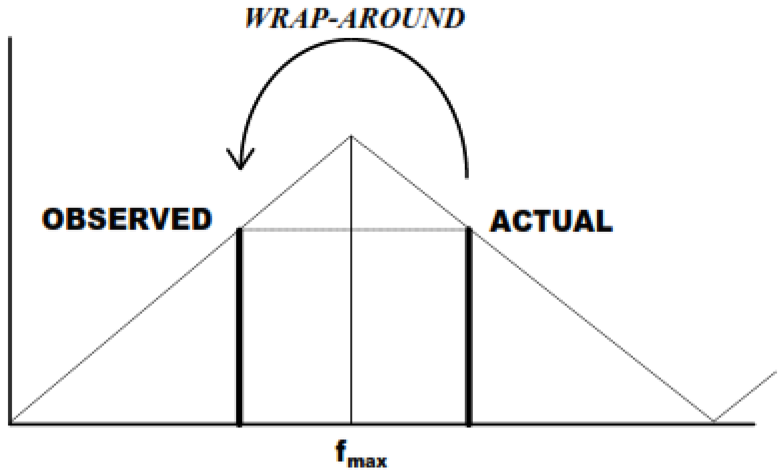
Signals – sampling of analog signals

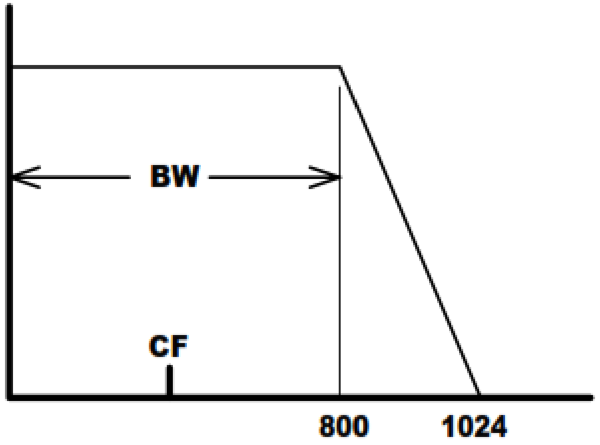
Anti-aliasing filter

Most good analysers have anti-aliasing filters which protect against aliasing.

These are low pass filters that typically have a roll off rate and are not ideal.

Usually only 80% of the anti-aliasing filter range is used to provide additional protection against aliasing

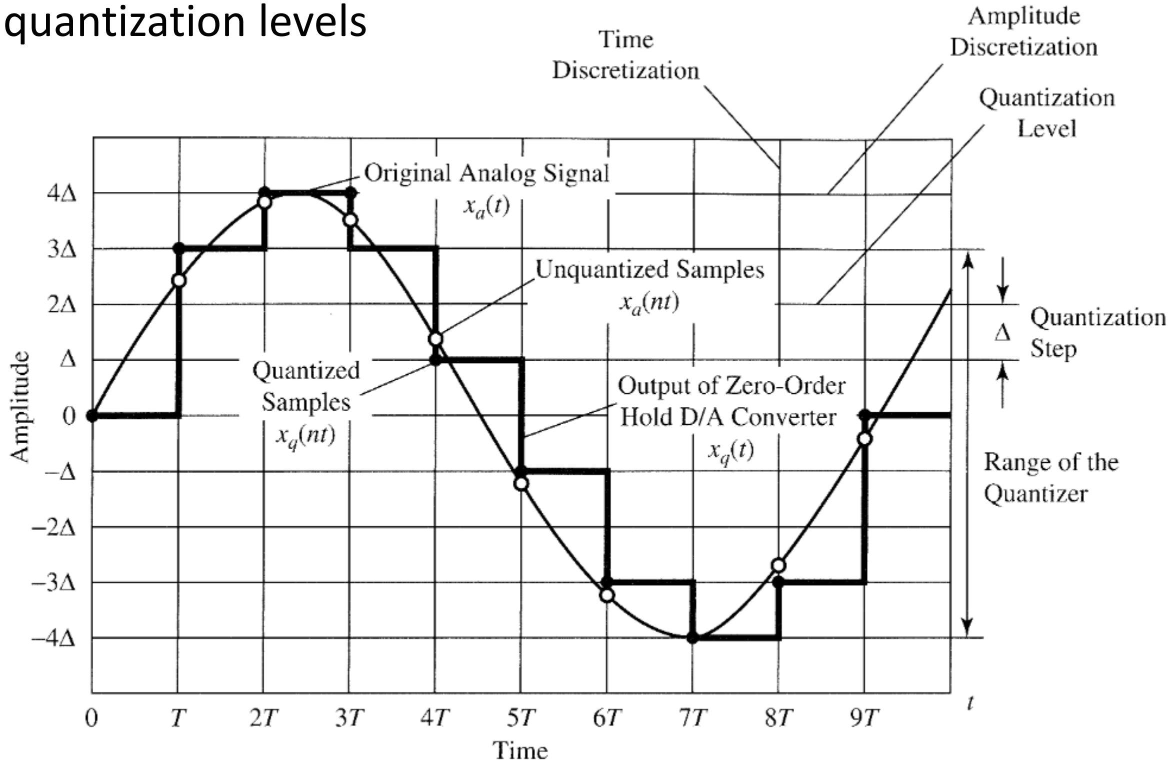




Signals – sampling of analog signals

Quantization

If x_{max} and $x_{\min}$ represent the maximum and the minimum values of the signal and L is the number of quantization levels



$$\Delta = \frac{x_{max} - x_{min}}{L-1}$$

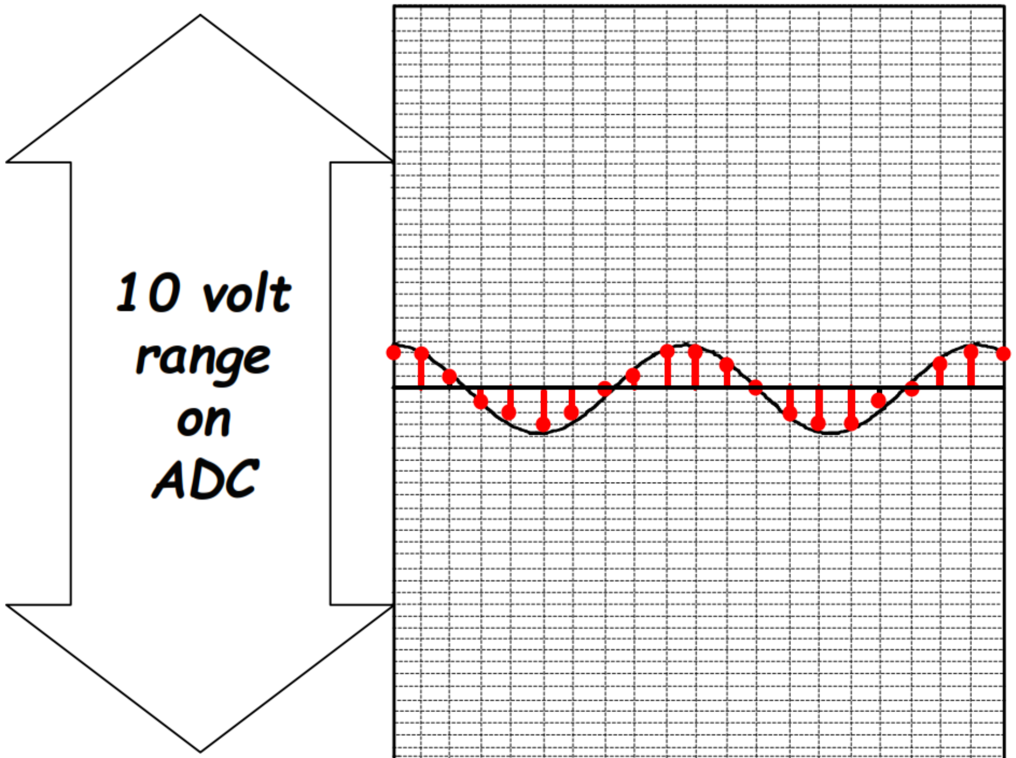
where $L = 2^b$

b is the number of bits of the A/D converter

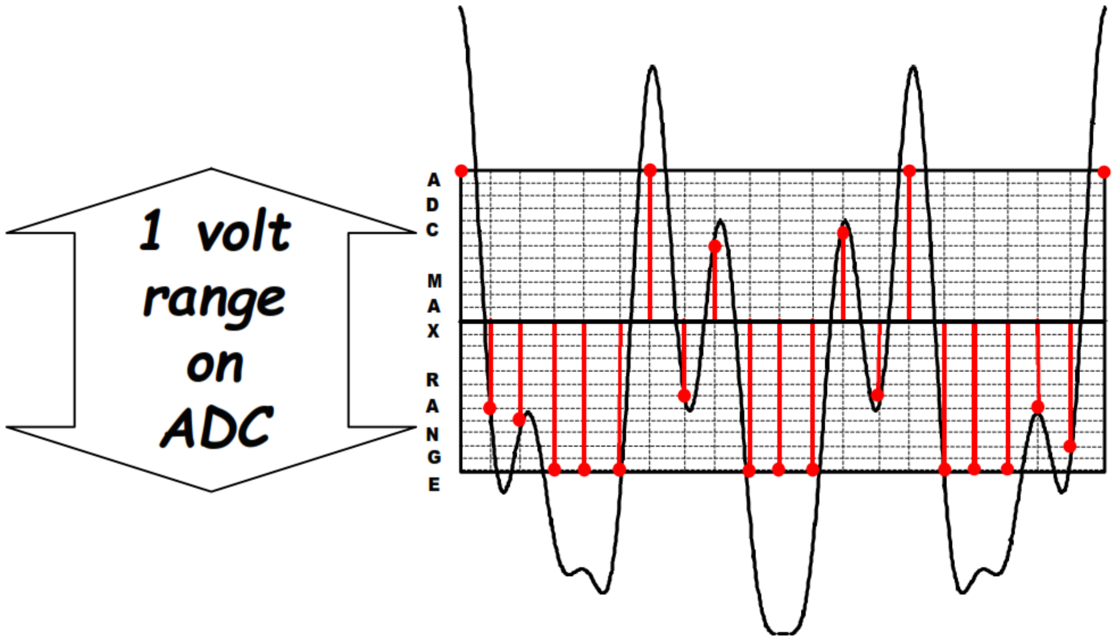
With 24 bits L = 16777216

Signals – sampling of analog signals

0.5 V signal



1.5V signal



TRASFORMATA DISCRETA DI FOURIER

Ritornando all'analisi di Fourier, nel caso di un segnale campionato si parla di *trasformata discreta di Fourier* (DFT), perché l'analisi viene effettuata su una funzione discreta (segnale campionato):

$$X(k\,\Delta f) = \Delta t_c \sum_{n=0}^{N-1} x_n e^{-i2\pi k \frac{n}{N}}$$

dove:

x_n	è il generico valore n–esimo di $x(t)$, cioè $x(t) = x(n\,\Delta t_c)$
$X(k\,\Delta f)$	rappresenta il termine *k-esimo* dello spettro di $x(t)$
N	è il numero di campioni, cioè il numero di valori di $x(t)$ rilevati a intervalli regolari Δt_c
k	è l'ordine dell'armonica, che va da 0 a $(N\text{-}1)/2$.

Se il numero di campioni elaborati è una potenza di 2, il calcolo viene effettuato con algoritmi chiamati FFT (*Fast Fourier Transform*), che velocizzano l'operazione (sono da 100 a 200 più veloci della procedura normale) e consentono di avere la trasformata di Fourier in tempo reale.

Scelta dei parametri: esempio

Individuata la frequenza di interesse (f_{utile}) si sceglie la FREQUENZA DI CAMPIONAMENTO.
Per evitare il fenomeno dell'*aliasing* deve essere: $f_{camp} \geq 2\ f_{utile}$ (teorema di Shannon)
Nella pratica: $f_{camp} \geq 2.5\ f_{utile}$ (per tener conto dell'imperfezione del filtro anti-aliasing)

La RISOLUZIONE dello spettro è pari a: $\Delta f = 1/T^*$ T^* = PERIODO DI ACQUISIZIONE
Δf = distanza tra due linee spettrali
Se Δf risulta "piccolo" si parla di elevata RISOLUZIONE dello spettro.

Scelte la frequenza di campionamento e la risoluzione, il NUMERO DI CAMPIONI risulta:
$N = T^* / \Delta t$ $\Delta t = 1 / f_{camp}$ Δt = TEMPO DI CAMPIONAMENTO
$N = T^*\ f_{camp}$ Generalmente il numero N di campioni deve essere potenza di 2 (per poter usare l'algoritmo veloce FFT).

Esempio
Si vogliano rilevare le frequenze proprie di un sistema libero-libero nel range 0-3000 Hz.
Da uno studio preliminare (eseguito utilizzando, per esempio, il metodo degli elementi finiti) tali frequenze risultano essere le seguenti:

$f_1 = 800$ Hz $f_2 = 1300$ Hz $f_3 = 1500$ Hz $f_4 = 2300$ Hz $f_5 = 2315$ Hz $f_6 = 2800$ Hz

Fissiamo innanzitutto la frequenza di campionamento. Per evitare il fenomeno dell'*aliasing* deve essere:

$f_{camp} \geq 2.5\ f_{utile}$ $f_{utile} = 3000$ Hz ==> $f_{camp} \geq 2.5 * 3000 = 7500$ Hz

La distanza tra due linee spettrali adiacenti della Trasformata Finita di Fourier è pari a:

$\Delta f = 1/T^*$ T^* = periodo di acquisizione

Dal momento che la quarta e la quinta frequenza differiscono tra loro di 15 Hz, è necessario che la risoluzione dello spettro sia piuttosto alta (Δf sufficientemente piccolo).

Imponiamo che sia: $\Delta f = 2$ Hz ==> $T^* = 1/\Delta f = 0.5$ s

A questo punto calcoliamo il numero di campioni che sono contenuti in 0.5 secondi:

$N = T^* / \Delta t$ $\Delta t = 1/f_{camp}$ $N = T^*\ f_{camp} = 3750$

Perché possa essere eseguita la FFT (Fast Fourier Transform) il numero N di campioni deve necessariamente essere potenza di 2.
Scegliamo quindi il primo numero potenza di 2 superiore a 3750: N = 4096

A questo punto abbiamo due possibilità:

Manteniamo la risoluzione $\Delta f = 2$ Hz
Numero di campioni N = 4096
Durata dell'acquisizione $T^* = 0.5$ s
Frequenza di campionamento
$f_{camp} = N / T = 8192$ Hz
Frequenza utile
$f_{utile} = f_{camp} / 2.5 = 3276.8$ Hz

Manteniamo la frequenza di campionamento
$f_{camp} = 7500$ Hz

Numero di campioni N = 4096
Durata dell'acquisizione $T^* = N / f_{camp} = 0.546$ s
Risoluzione dello spettro $\Delta f = 1 / T^* = 1.831$ Hz
Frequenza utile $f_{utile} = f_{camp} / 2.5 = 3000$ Hz

Energy and Power

For a given signal x(t) the instantaneous power is defined as:

$$p(t) = x(t)x^*(t) = \left|x(t)\right|^2$$

Mean Energy

$$E = \int_{-\infty}^{+\infty} \left|x(t)\right|^2 dt$$

Mean Power

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} \left|x(t)\right|^2 dt$$

Parseval

$$\int_{-\infty}^{+\infty} \left|x(t)\right|^2 dt = \int_{-\infty}^{+\infty} \left|\mathrm{X(f)}\right|^2 df$$

Mean Energy

$$E = \int_{-\infty}^{+\infty} \left|\mathrm{X(f)}\right|^2 df$$

Mean Power

$$P = \int_{-\infty}^{+\infty} \frac{\left|\mathrm{X(f)}\right|^2}{T} df$$

Energy and Power

$$X(k\,\Delta f) = \Delta t_c \sum_{n=0}^{N-1} x_n e^{-i2\pi k\frac{n}{N}}$$

Energy Spectral Density (or Autopower)

$$|X(f)|^2 = X(f)X^*(f) \qquad \left[EU^2s^2\right] = \left[\frac{EU^2s}{Hz}\right]$$

Power Spectral Density (PSD)

$$\frac{|X(f)|^2}{T} \qquad \left[\frac{EU^2s^2}{s}\right] = \left[EU^2s\right] = \left[\frac{EU^2}{Hz}\right]$$

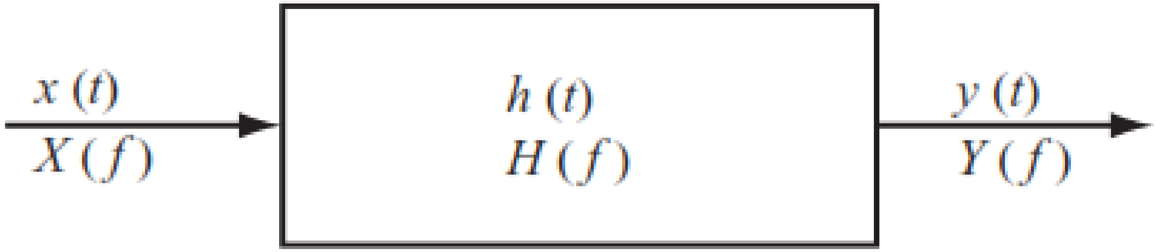
Energy and Power

Integral Fourier Transform

$$PSD = \frac{\Delta t}{N}\left|\sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{k}{N} n}\right|^2 \qquad \left[\frac{EU^2}{Hz}\right]$$

FRF estimation

Linear time-invariant system with input x(t) and output y(t) is a system in which the coefficients of the system differential equation do not change with time. In practice, this means that the system parameters, for example mass, damping and stiffness, do not change during the time we study (measure) the system.

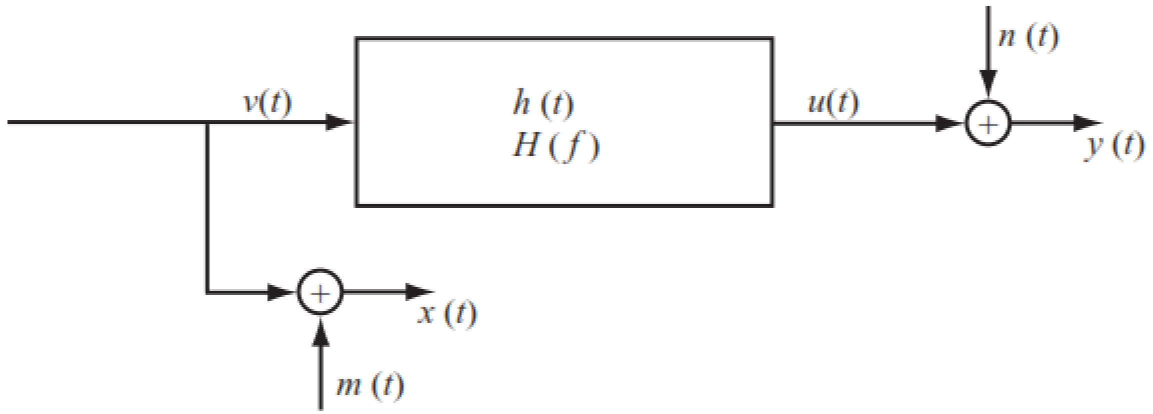


In the time domain: $y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\mathrm{u})\mathrm{h}(\mathrm{t}-\mathrm{u})\,\mathrm{du}$ h(t) is the impulse respense function

In the frequency domain: $Y(f) = H(f)X(\mathrm{f})$ H(t) is the frequency respense function

FRF estimation

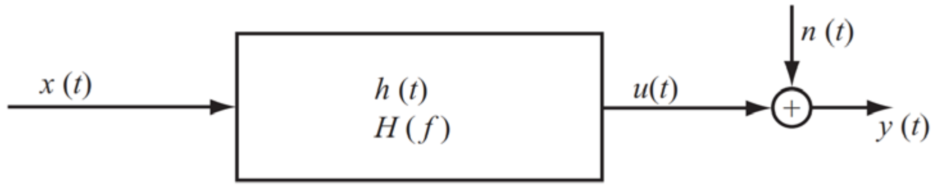
In real life system we have no hope of being able to measure x(t) and y(t) without at least one of these signals being contaminated by extraneous noise from the sensor and input electronics in the measurement system.



However, we can assume that the contaminating noise is uncorrelated with the input and output signals.

FRF estimation – H_1 estimator

We thus assume that we can measure the input *x(t)* without any extraneous noise, but that the output contains noise.



In the frequency domain we have: $Y(f) = X(f)H(f) + N(f)$

$$X^*(f)Y(f) = X^*(f)X(f)H(f) + X^*(f)N(f) \longrightarrow G_{yx}(f) = G_{xx}(f)H(f) + G_{nx}(f)$$

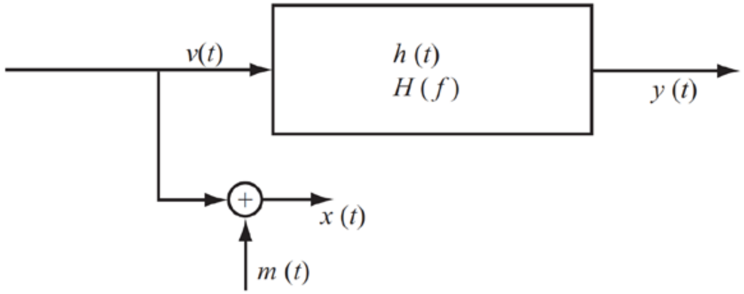
The cross-spectral density between the input and the noise, approches zero when we average, since these signals are uncorrelated

$$H_1(f) = \frac{G_{yx}(f)}{G_{xx}(f)}$$

- Under estimation of H(f)
- Good fit in anti-resonances

FRF estimation – H_2 estimator

We now assume noise in the input signal *x(t).*



In the frequency domain we have: $Y(f) = [X(f) - M(f)]H(f)$

$$Y^*(f)Y(f) = Y^*(f)[X(f) - M(f)]H(f) \longrightarrow G_{yy}(f) = [G_{xy}(f) - G_{my}(f)]H(f)$$

The cross-spectral density between the output and the noise, approches zero when we average, since these signals are uncorrelated

$$H_2(f) = \frac{G_{yy}(f)}{G_{xy}(f)}$$

- Over estimation of H(f)
- Bad fit in anti-resonances

FRF estimation – Coherence function

- In case of existing input noise, not included in the assumption for the H1 estimator, the magnitude of the estimate will be less than or equal to the true value of H.
- In case of existing output noise, not included in the assumption for the H2 estimator, the magnitude of the estimate will be greater than or equal to the true value of H

$$\longrightarrow \quad \left|H_1(f)\right| \le \left|H(f)\right| \le \left|H_2(f)\right|$$

The coherence function is defined as:

$$\gamma_{xy}^2(f) = \frac{H_1(f)}{H_2(f)} = \frac{\left|G_{xy}(f)\right|^2}{G_{xx}(f)G_{yy}(f)}$$

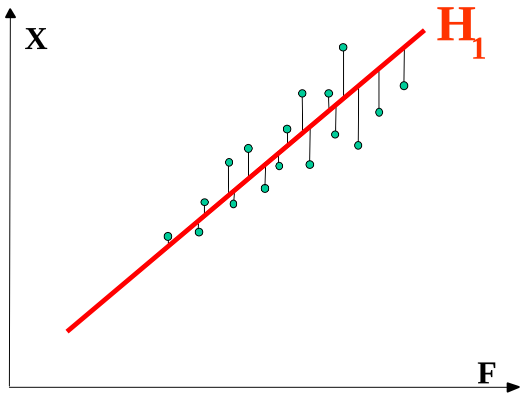
$$\longrightarrow \quad 0 \le \gamma_{xy}^2(f) \le 1$$

The coherence function is not 1 when:

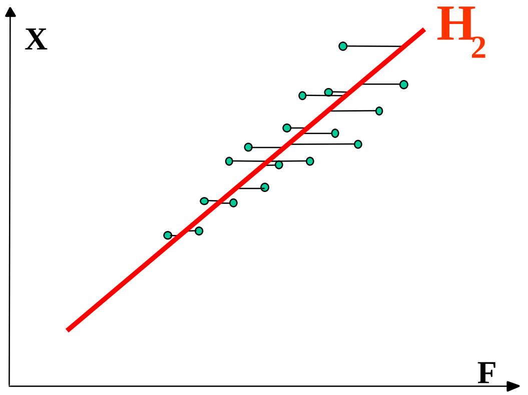
- Noises are not negligible compared with the measured signals
- The system is nonlinear or not time invariant
- Estimation errors due to insufficient frequency resolution

FRF estimation – Graphical interpretation

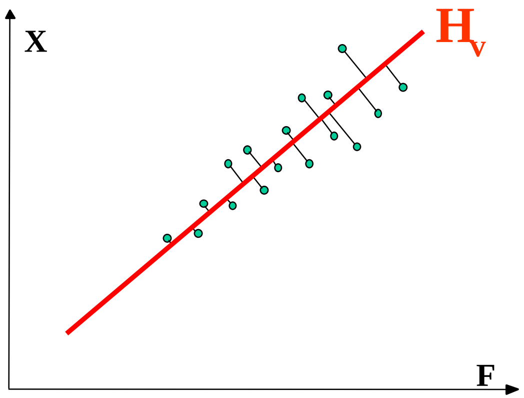
At a single frequency ω:



Least squares estimation minimizing the output noise

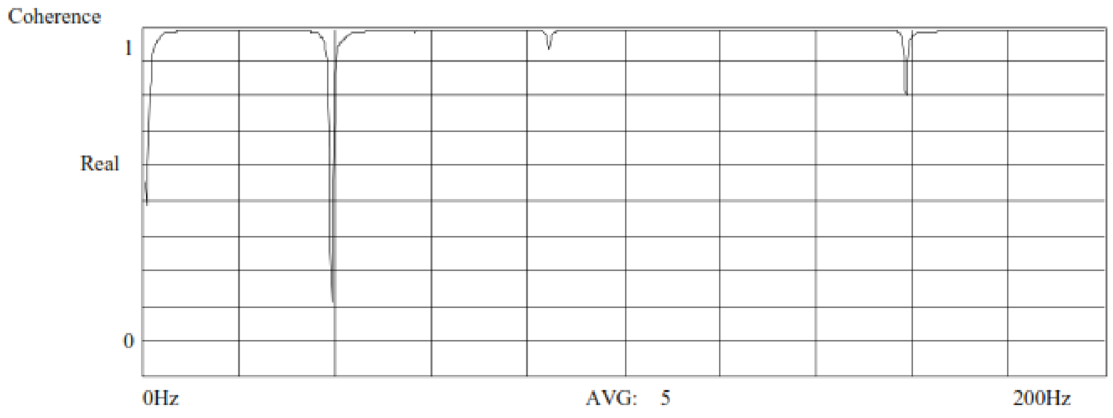


Least squares estimation minimizing the input noise

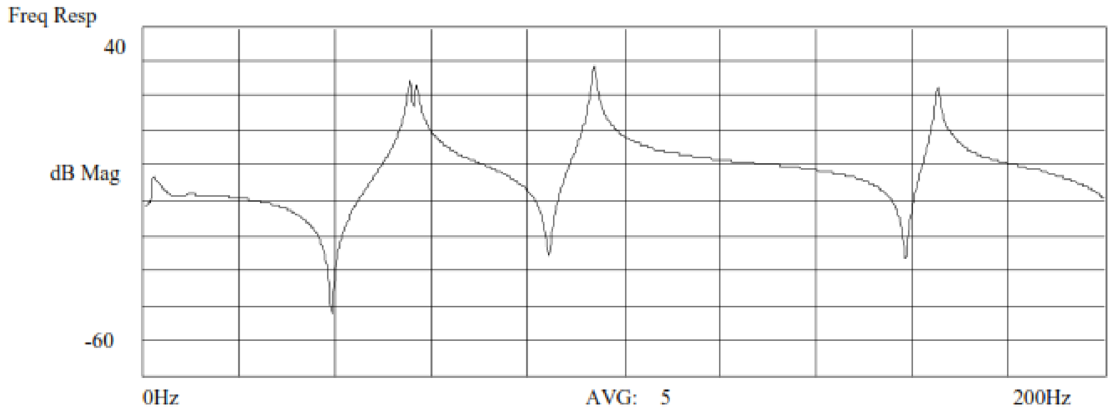


Least squares estimation minimizing both the input and output noises

FRF estimation



COHERENCE

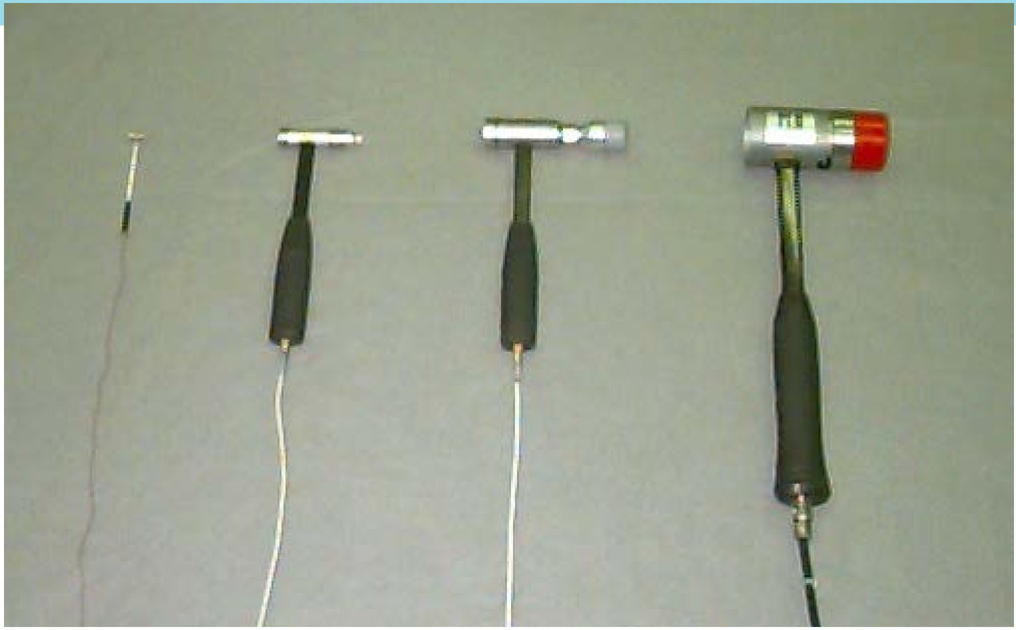


FREQUENCY RESPONSE FUNCTION

5. Measurement overview: excitation impact, excitation shaker, trigger, pre-trigger, windowing for modal analysis

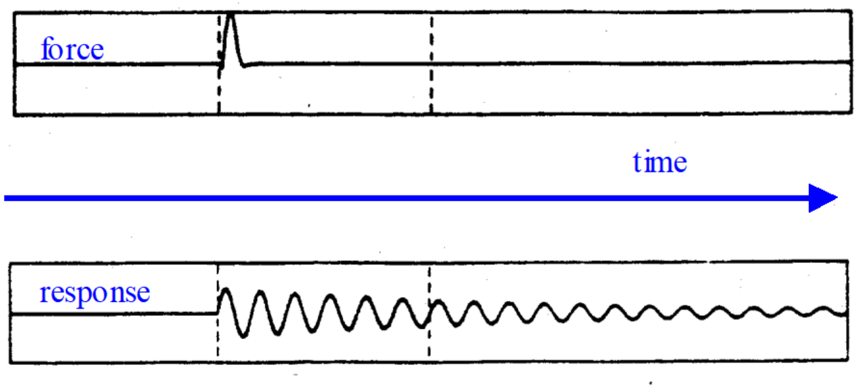
Impact Hammer

ADVANTAGES
-Easy setup
-Fast measurement time
-Minimum of equipment
-Low cost (relatively…2k€)
DISADVANTAGES
-Poor rms to peak levels
-Poor for nonlinear structures
-Force/response windows needed
-Pre-trigger delay needed
-Double impacts may occur

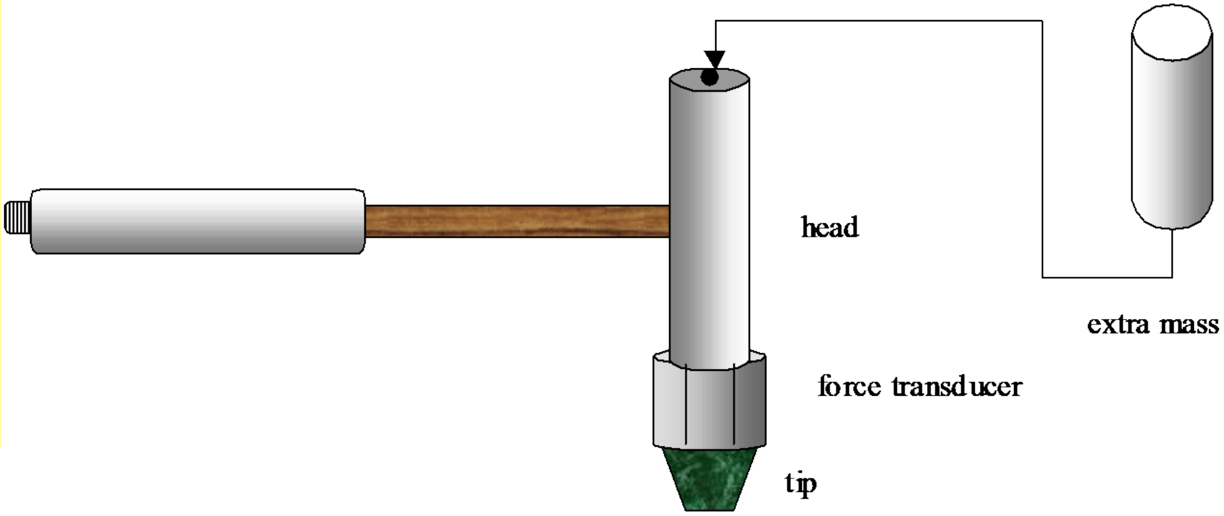


Impact Hammer

An impulsive excitation is very short in the time window, usually lasting less than 5% of the sample interval.
The frequency content of the input spectrum is broadband, flat, and proportional to the duration of the pulse.



Impact excitation is typically provided by a specially designed "impact hammer". This device is equipped with a force transducer, a removable mass, and interchangeable tips.
The removable mass changes the energy level capacity of the hammer.
The tip can control the frequency content of the impact.



Hammer

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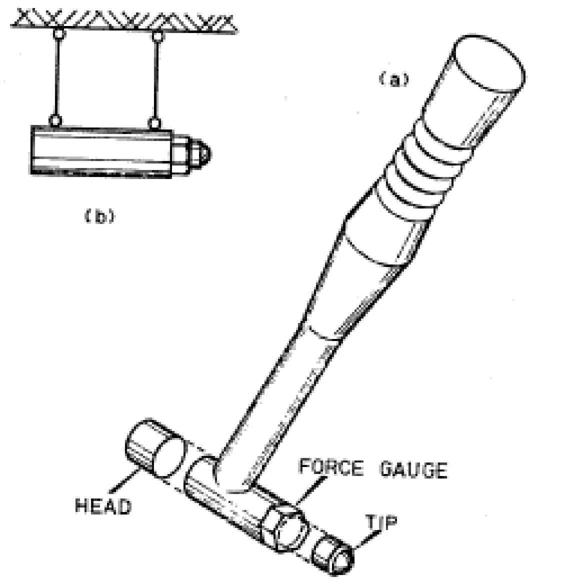


Fig 3.7 Impactor and Hammer Details

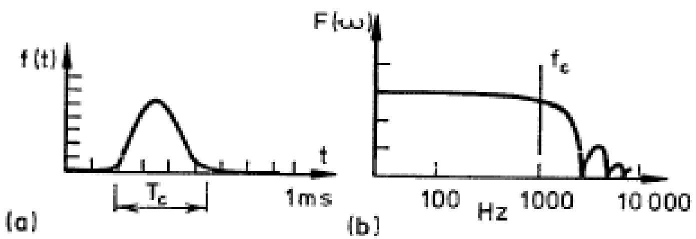
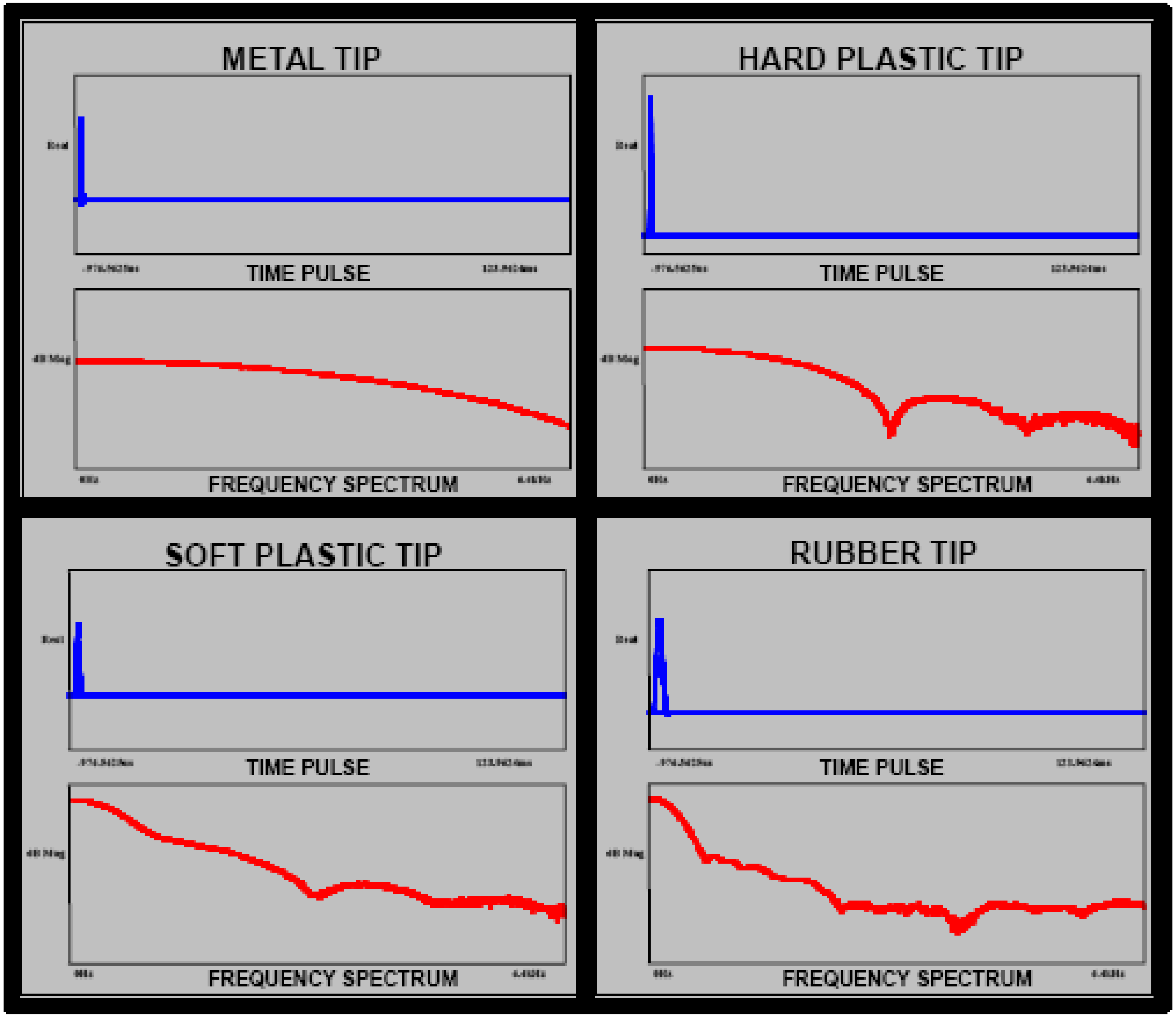


Fig 3.8 Typical Impact Force Pulse and Spectrum
(a) Time History
(b) Frequency Spectrum

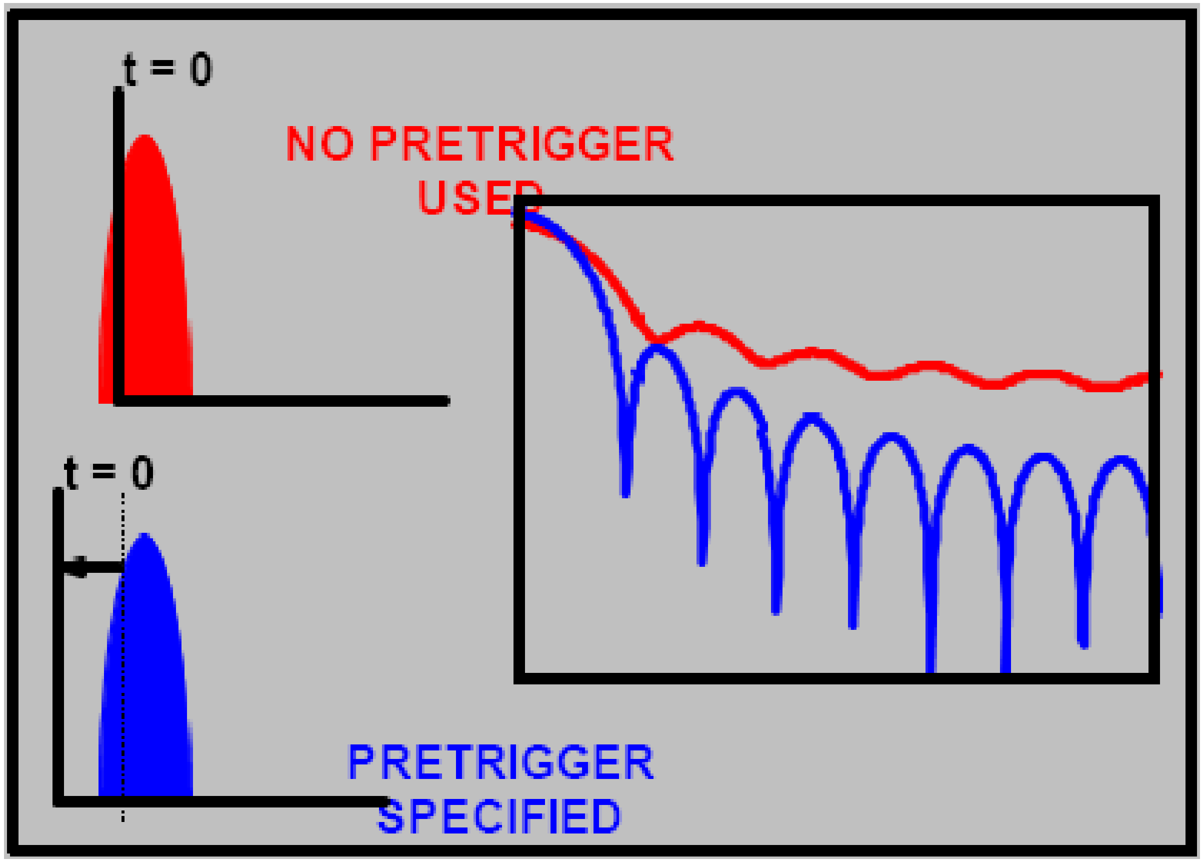
$$f_c = \left(\frac{K_{contatto}}{M_{impactor}} \right)^{0.5}$$

Hammer tip selection

The force spectrum can be customized to some extent through the use of hammer tips with various hardnesses. A hard tip has a very short pulse and will excite a wide frequency range (more similar to a Dirac impulse). A soft tip has a long pulse and will excite a narrow frequency range.
The duration of the pulse is also a function of the stiffness of the surface being hit.

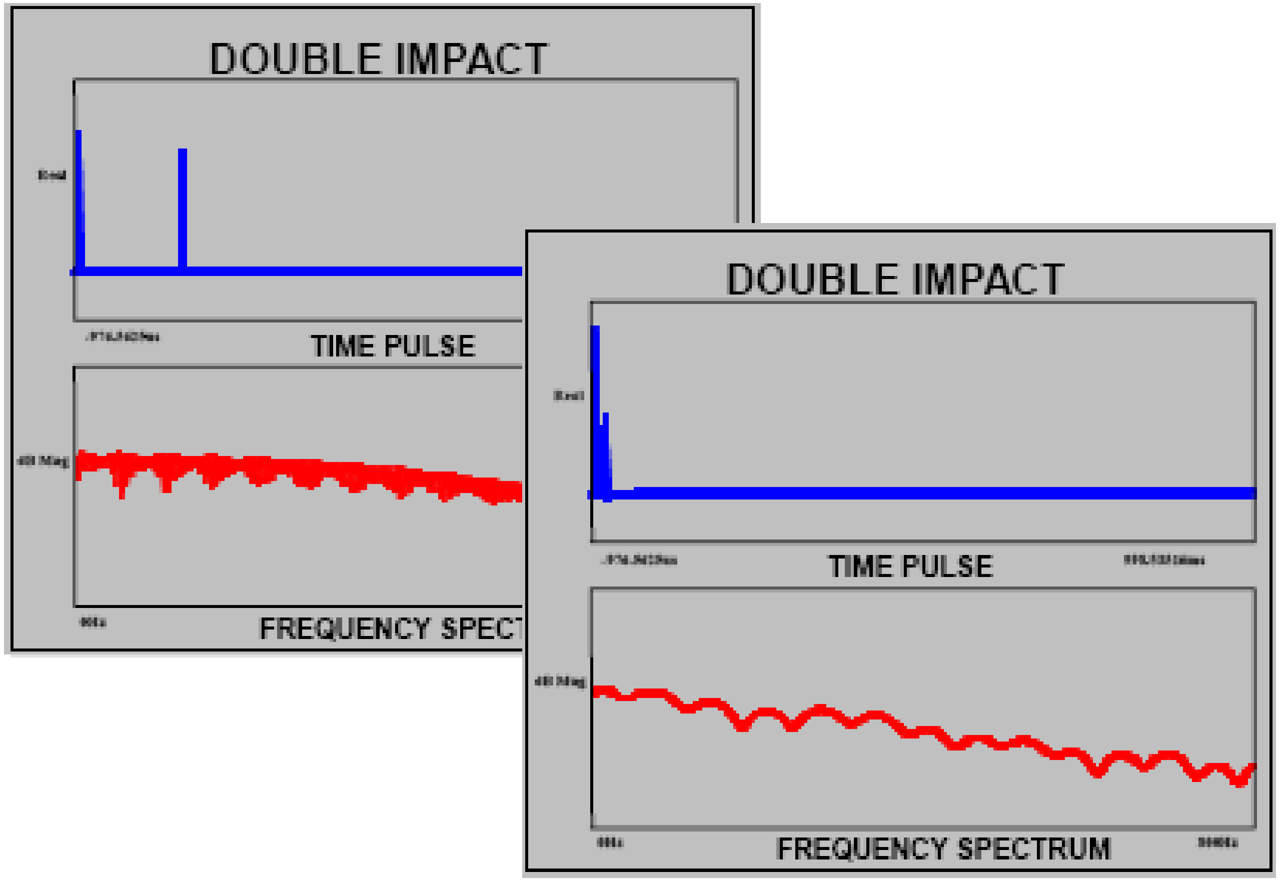


Pretrigger delay



Some signal analyzers have a feature referred to as pre-trigger delay (very important)
Data acquisition process is started from an input trigger from the impact hammer.
This type of trigger inherently clips the beginning portion of the impulse.
Pre-trigger delay allows for capture of the whole impulsive excitation. This is necessary so that the signal analyzer measures the actual input spectrum to the structure.

Double impacts

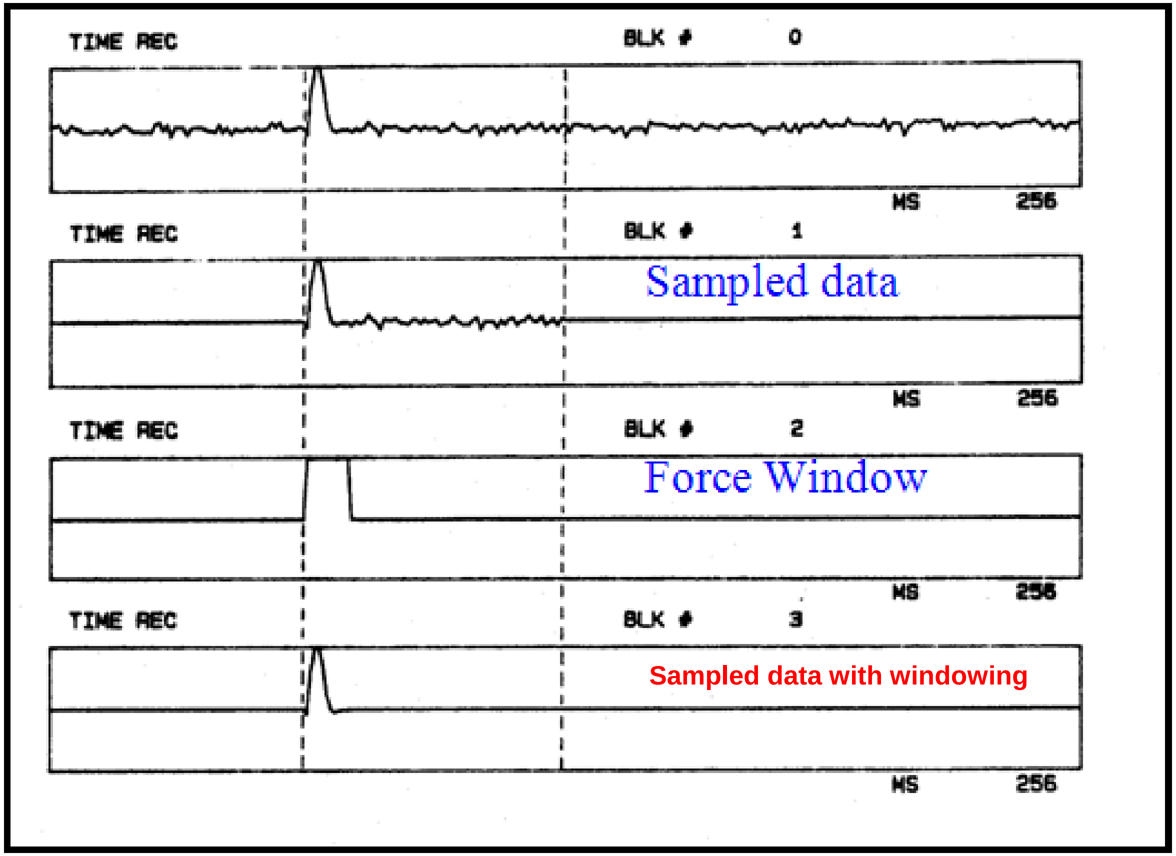


A double impact may occur during testing due to a sloppy hammer swing. Sometimes it is not possible to move the hammer away from a very responsive structure fast enough.

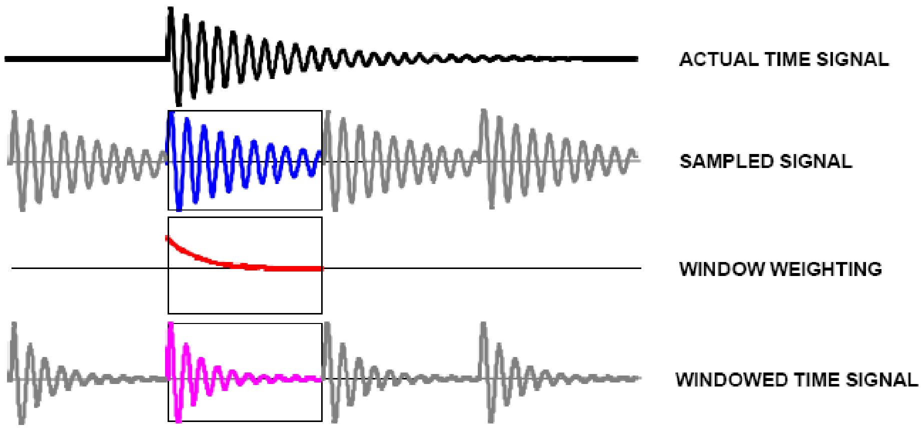
Avoid double impacts if at all possible. Never try to remove the effects of the double impact through the use of the force window. Just repeat the measurement.

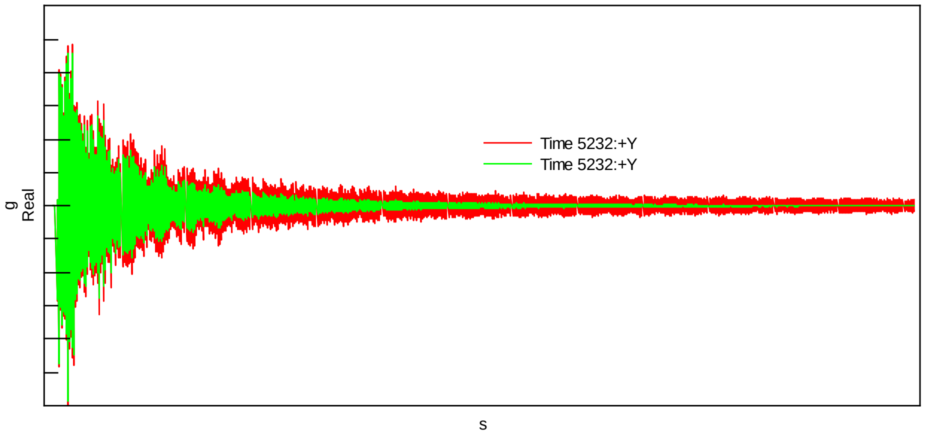
Impact testing
force window for input

Unwanted noise may exist on the impact channel
Force window may be needed to minimize this effect.
This window may be a rectangular window that exists over a portion of the block
Main use of the force window is to minimize the effect of spurious noise on the input channel.



Impact testing
Exponential window for response

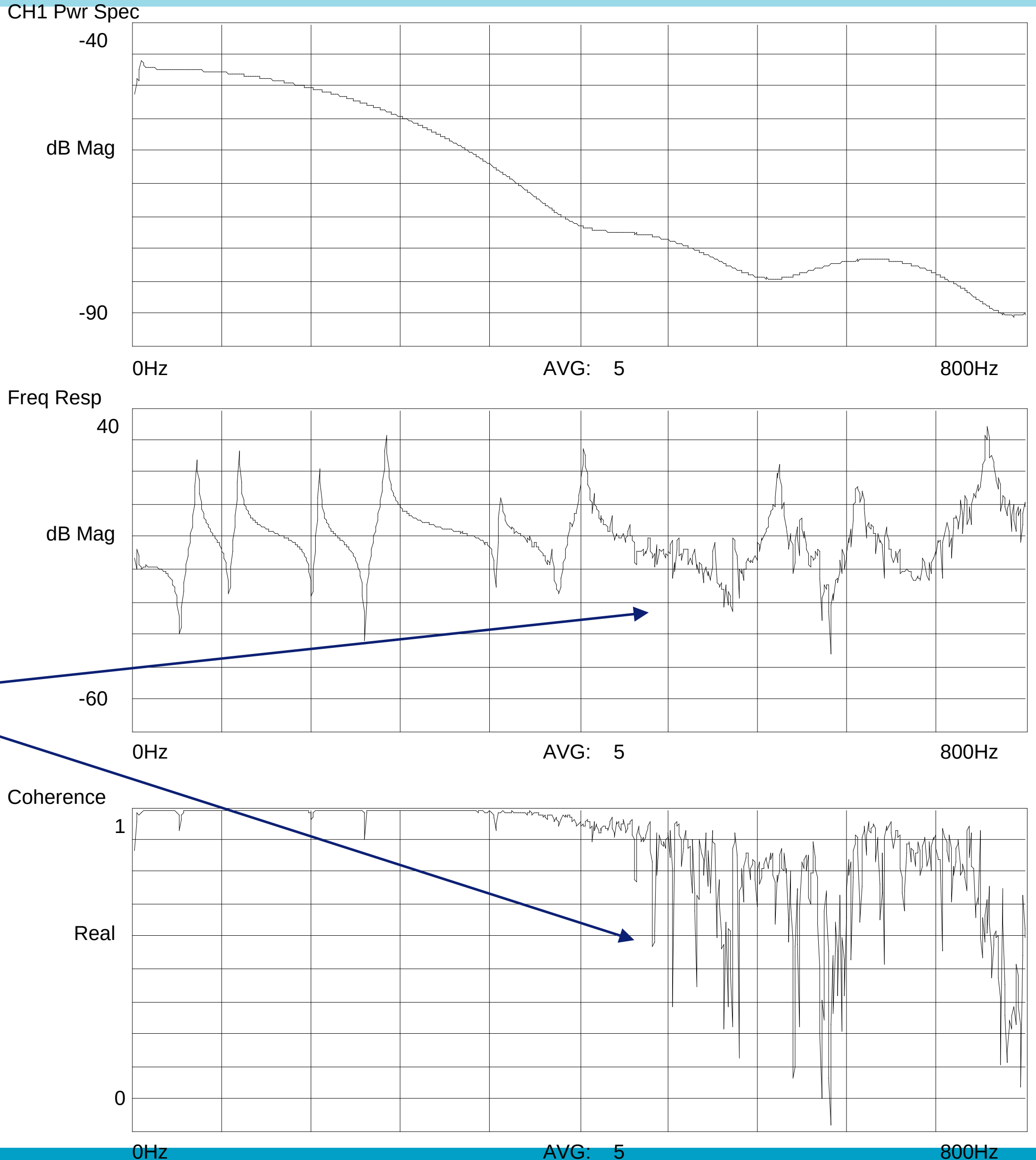




- Response signals should decay within observation period ("block")
- If not: leakage
- Remedy
  - Use exponential window
  - AND: wait for next impact until structure is in rest

IMPACT CONSIDERATIONS
Impact measurement with soft tip.

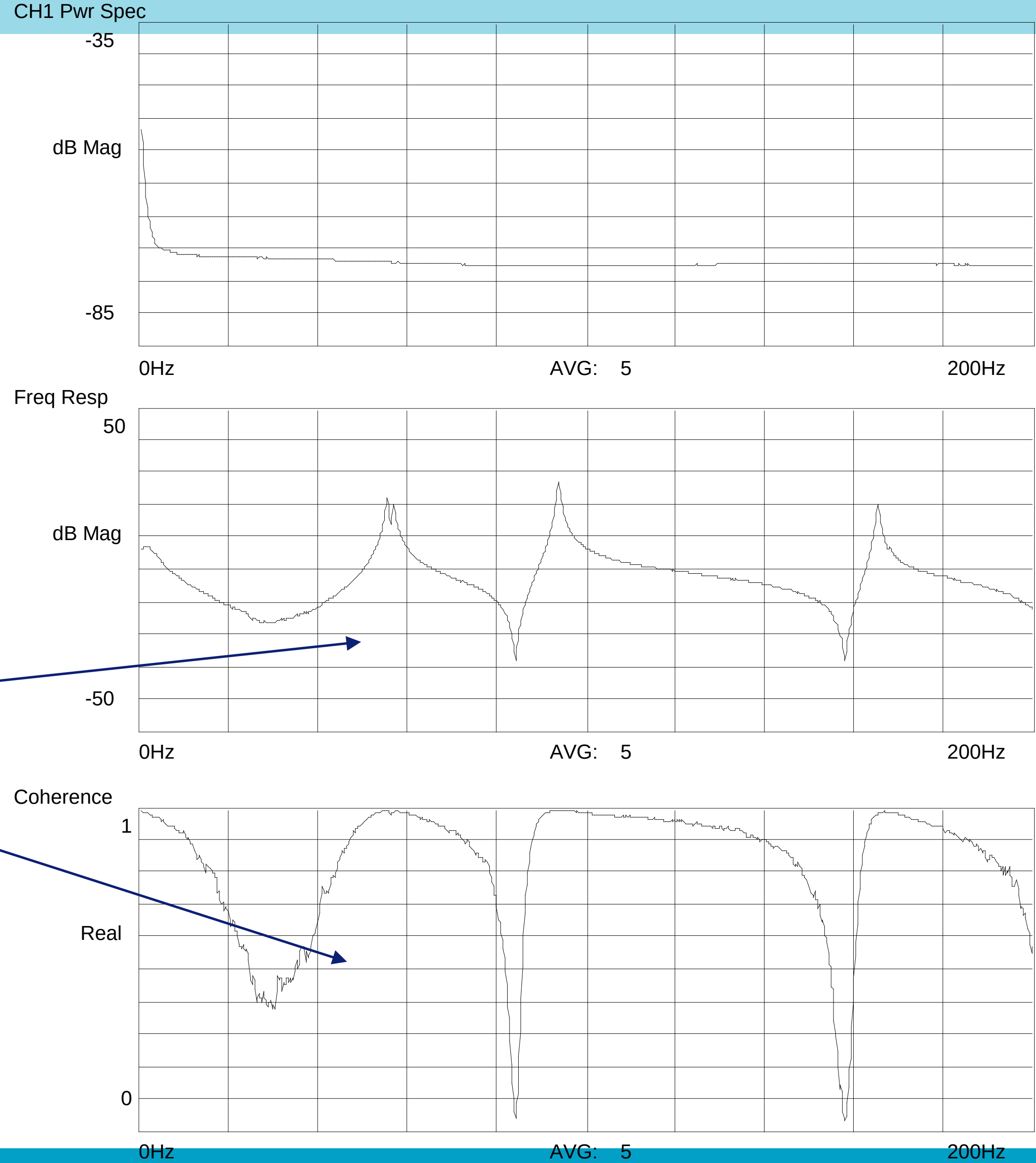
Example:
Input spectrum has insufficient energy for frequencies greater than 400 Hz.



Poor quality of the FRF and coherence

IMPACT CONSIDERATIONS
Impact measurement with very hard tip.

Example:
Input spectrum has
enough energy in
this frequency range



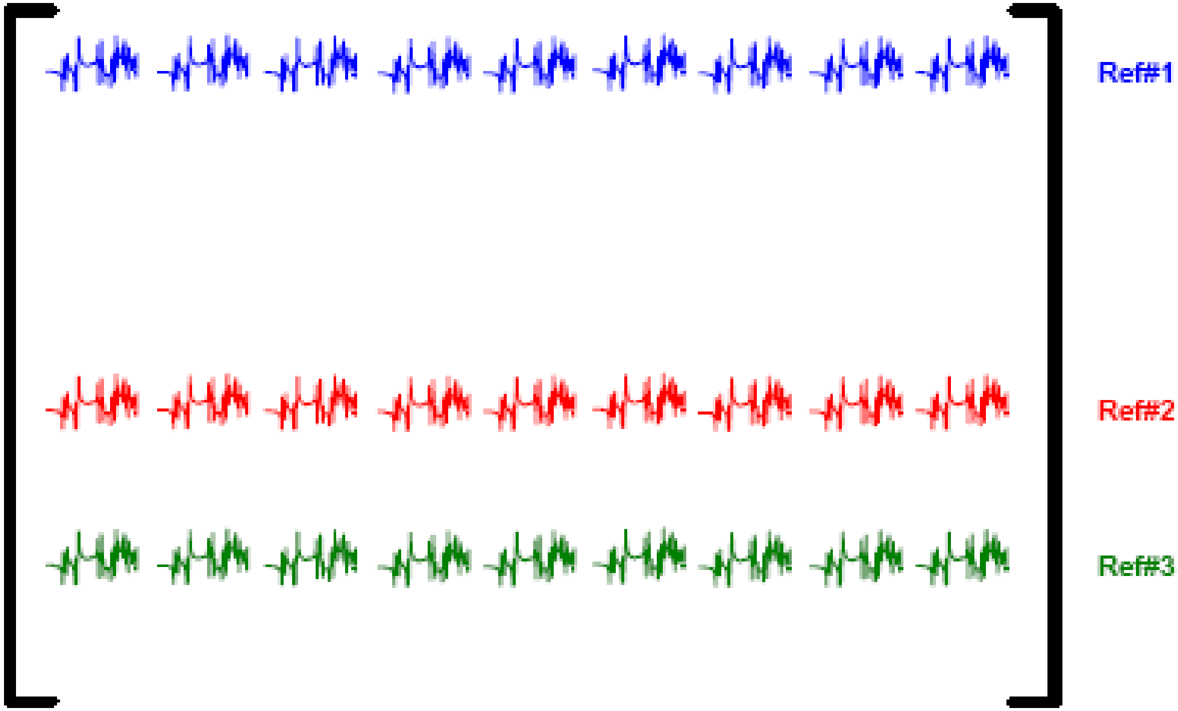
good quality of the FRF and coherence

Multiple reference impact testing

Mount a few accelerometers at key points on the structure where the majority of the modes can be observed.

Impact ALL points in ALL directions.

Multiple reference data is then obtained.

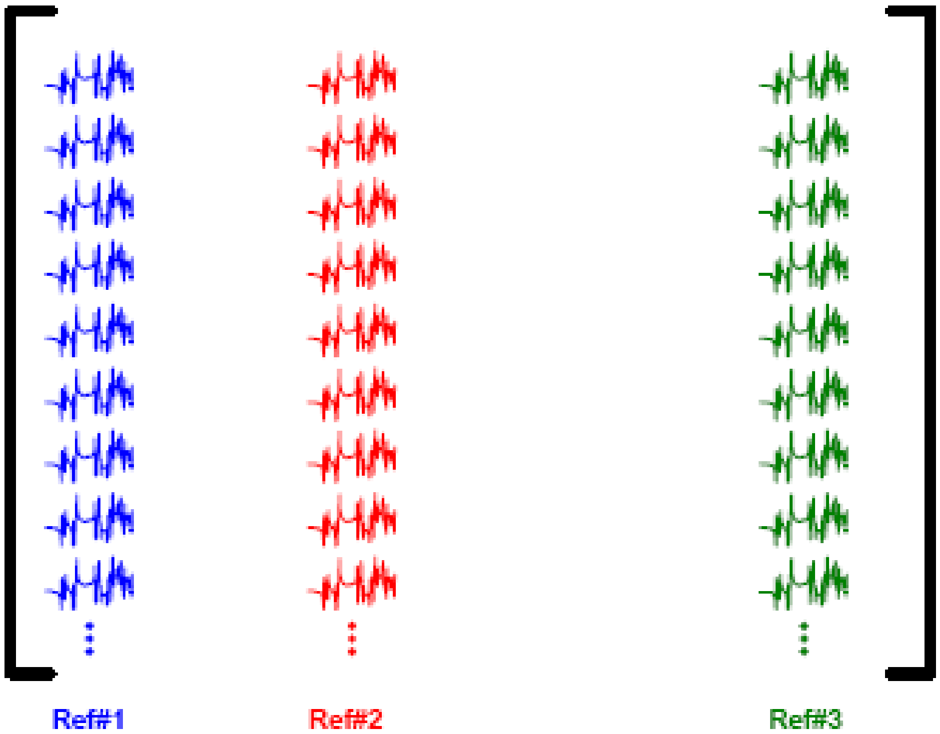


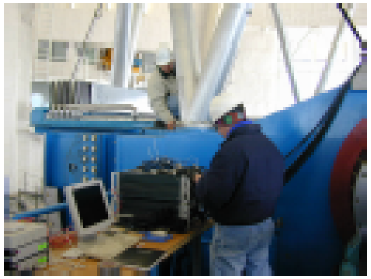
Multiple reference impact testing

Mount ALL the accelerometers at ALL points in ALL of the required directions.

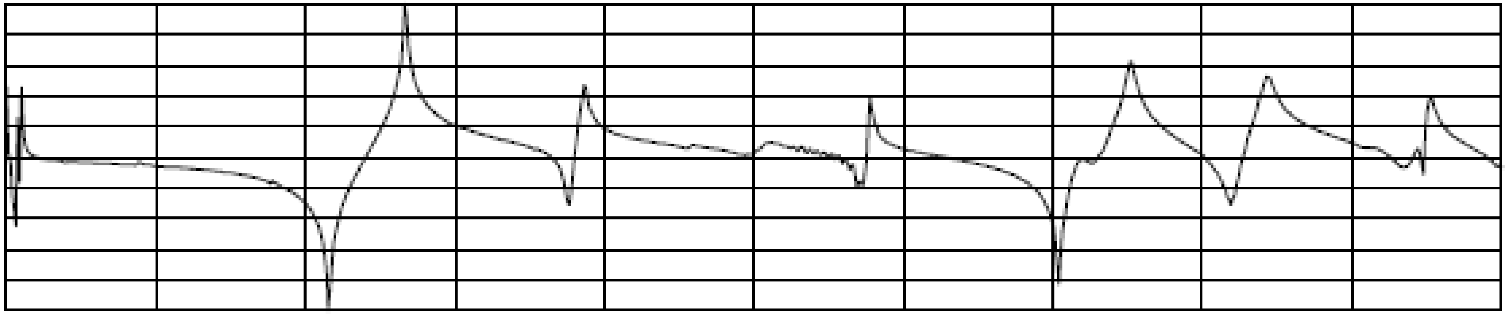
Impact a few key points where most of the desired modes can be observed.

Multiple reference data is then obtained.

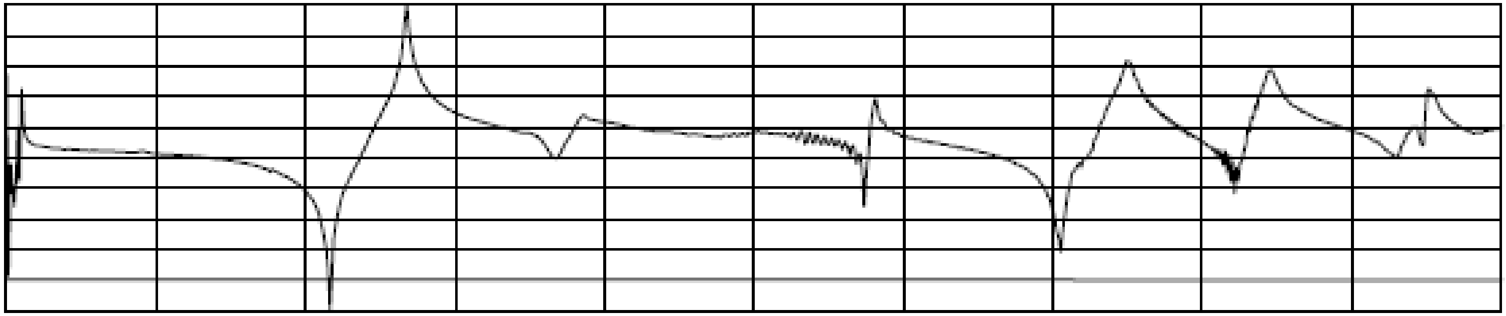




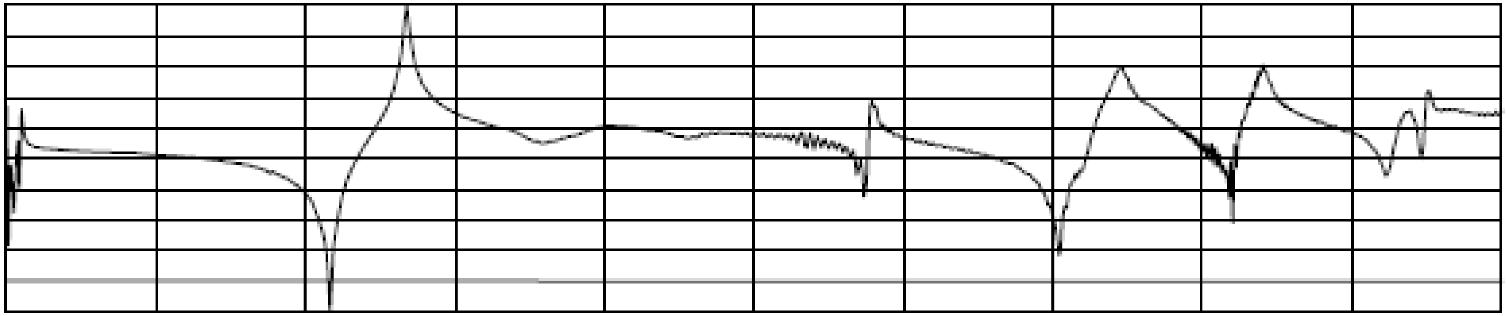
Linearity check (difficult for hammer testing)



ONE FORCE UNIT



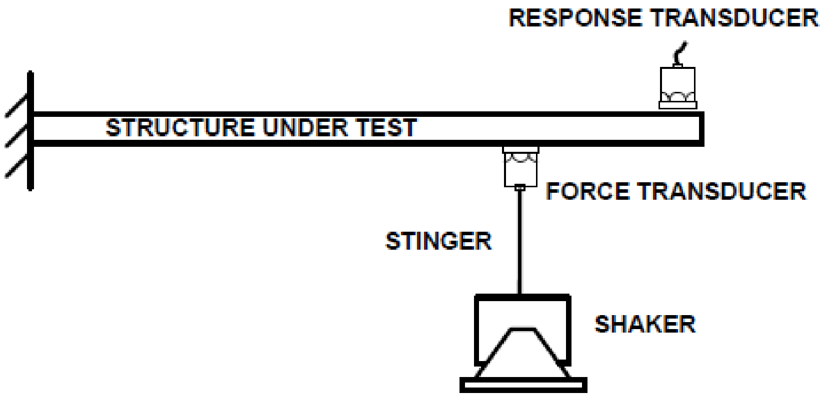
FIVE FORCE UNITS



TEN FORCE UNITS

Shaker

Excitation device is attached to the structure using a long rod called a stinger or quill



Excitation device is attached to the structure using a long rod called a stinger or quill

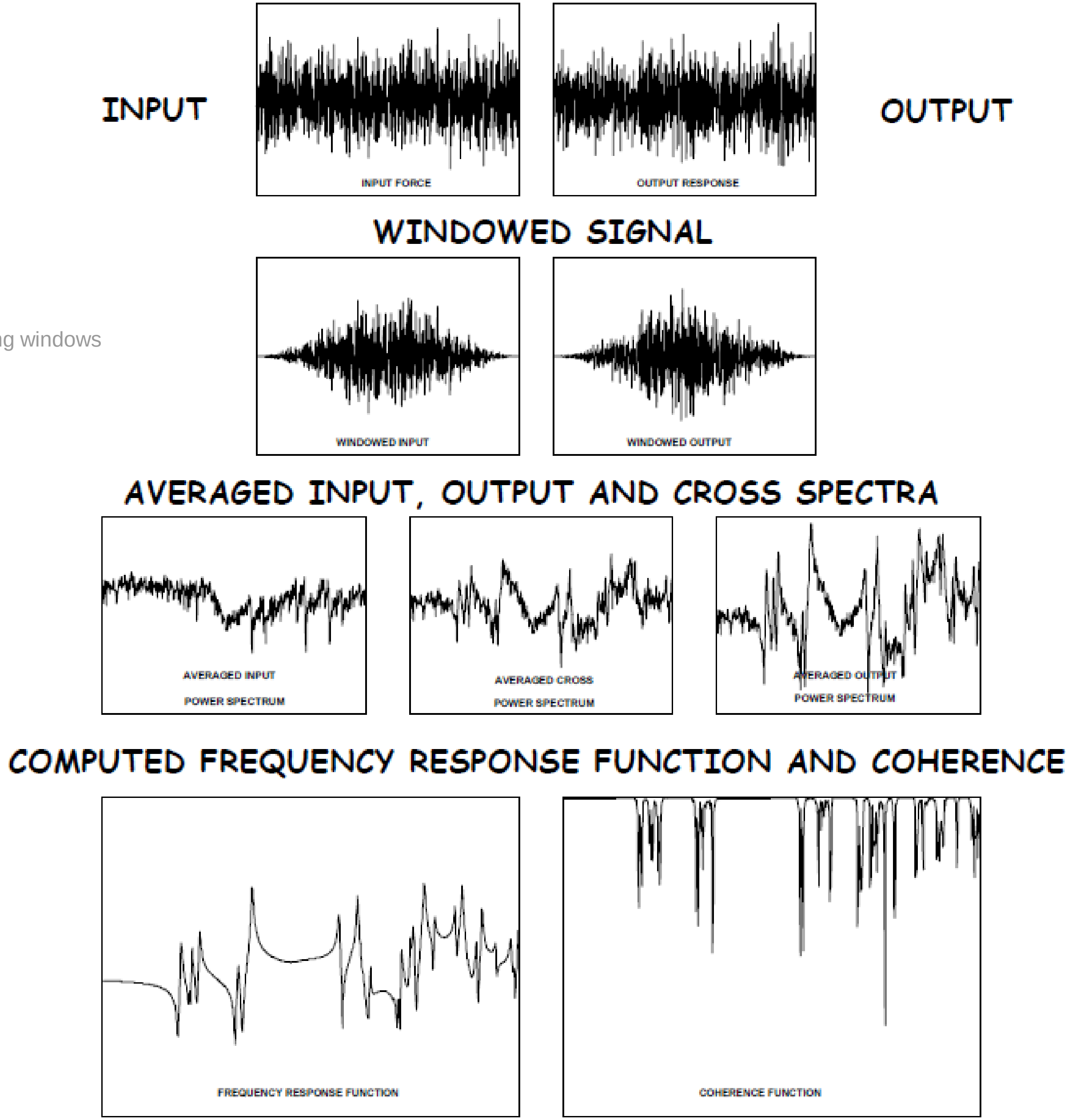
Its purpose is to provide input along the shaker excitation axis with essentially no excitation of the other directions

It is also intended to be flexible enough to not provide any stiffness to the other directions

The force gage is always mounted on the structure side of the quill

NOT ON THE SHAKER SIDE

Overall Measurement Process



Hanning windows

Shaker Excitation Methods

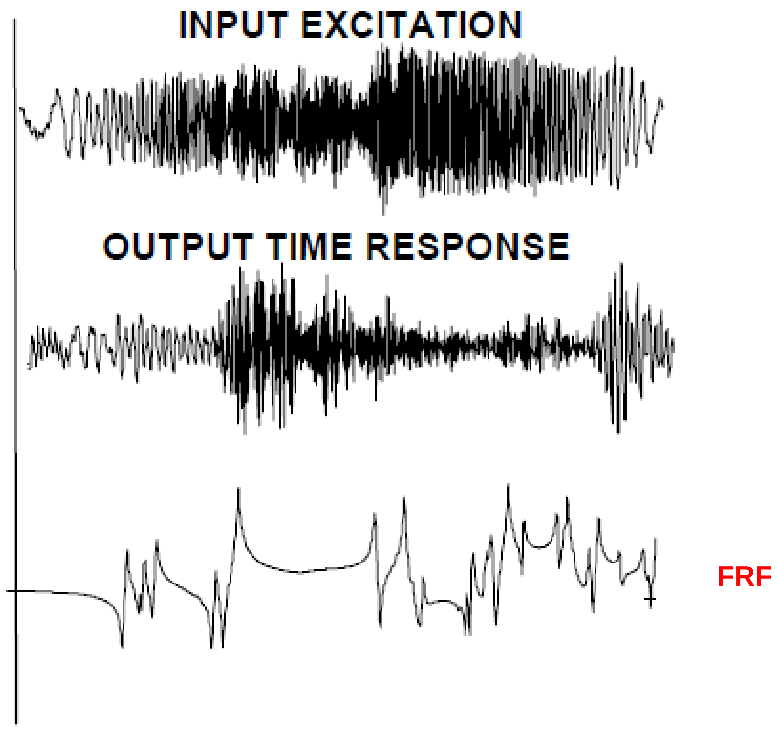
- Shaker Excitation techniques are divided into two general categories:

Deterministic Signals

Non-deterministic Signals

- **Deterministic Signals**
  - Conform to a particular mathematical relationship
  - Can be described exactly at any instant in time
  - System response can be exactly defined if the system character is known
  - Examples: **Swept Sine**, **Sine Chirp**
- **Non-deterministic Signals**
  - Do not conform to any mathematical relationship
  - Can only be estimated at any instant in time
  - Amplitude, phase, and frequency are varying at any instant in time
  - Examples: Pure Random, Periodic Random, Burst Random (Random Transient)

Swept Sine Excitation



Slowly changing sine signal sweeping from one frequency to another frequency

Swept Sine Excitation

A slowly changing sine output sweeping from one frequency to another frequency

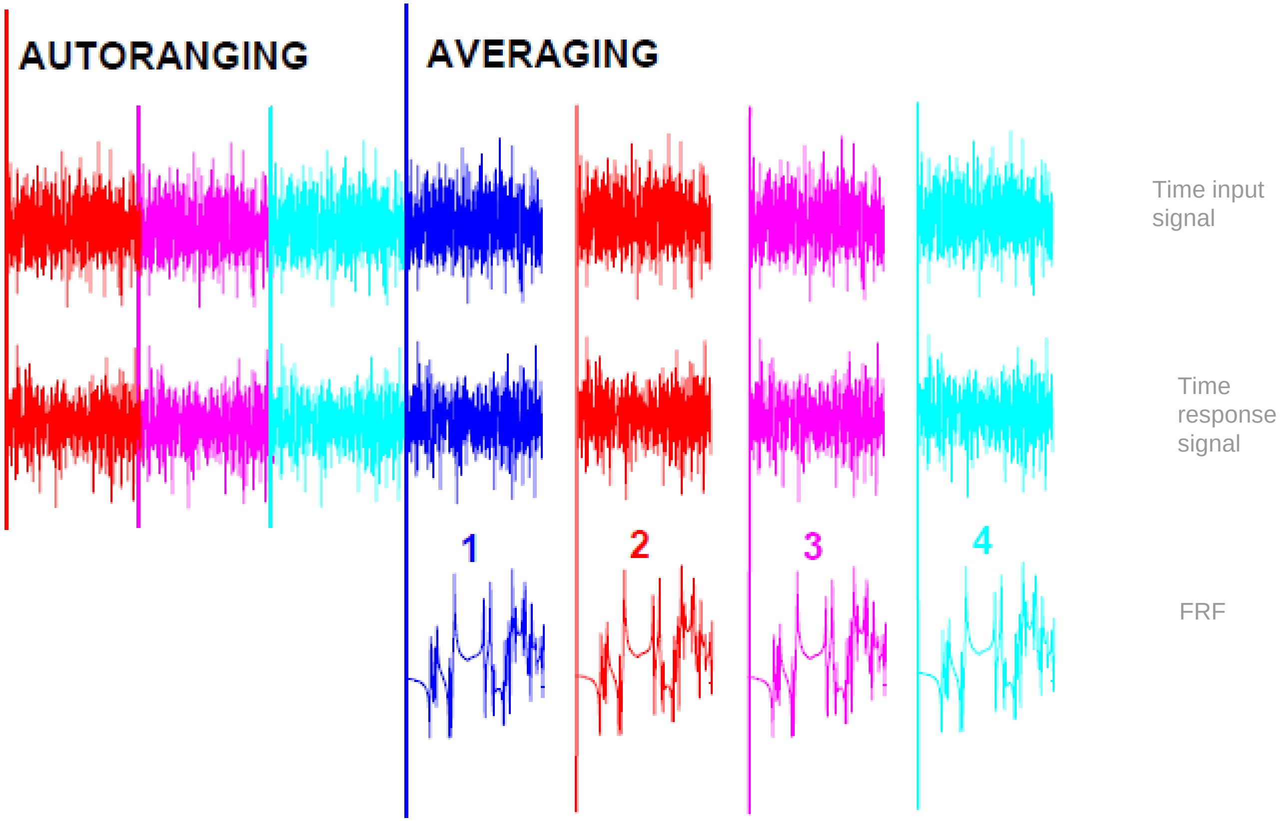
ADVANTAGES

- best peak to RMS level
- best signal to noise ratio
- good for nonlinear characterization
- widely accepted and understood

DISADVANTAGES

- slowest of all test methods
- leakage is a problem
- does not take advantage of speed of FFT process

Random Excitation



An ergodic, stationary signal with Gaussian probability distribution. Typically, has frequency content at all frequencies.

Random Excitation

An ergodic, stationary signal with Gaussian probability distribution. Typically, has frequency content at all frequencies.

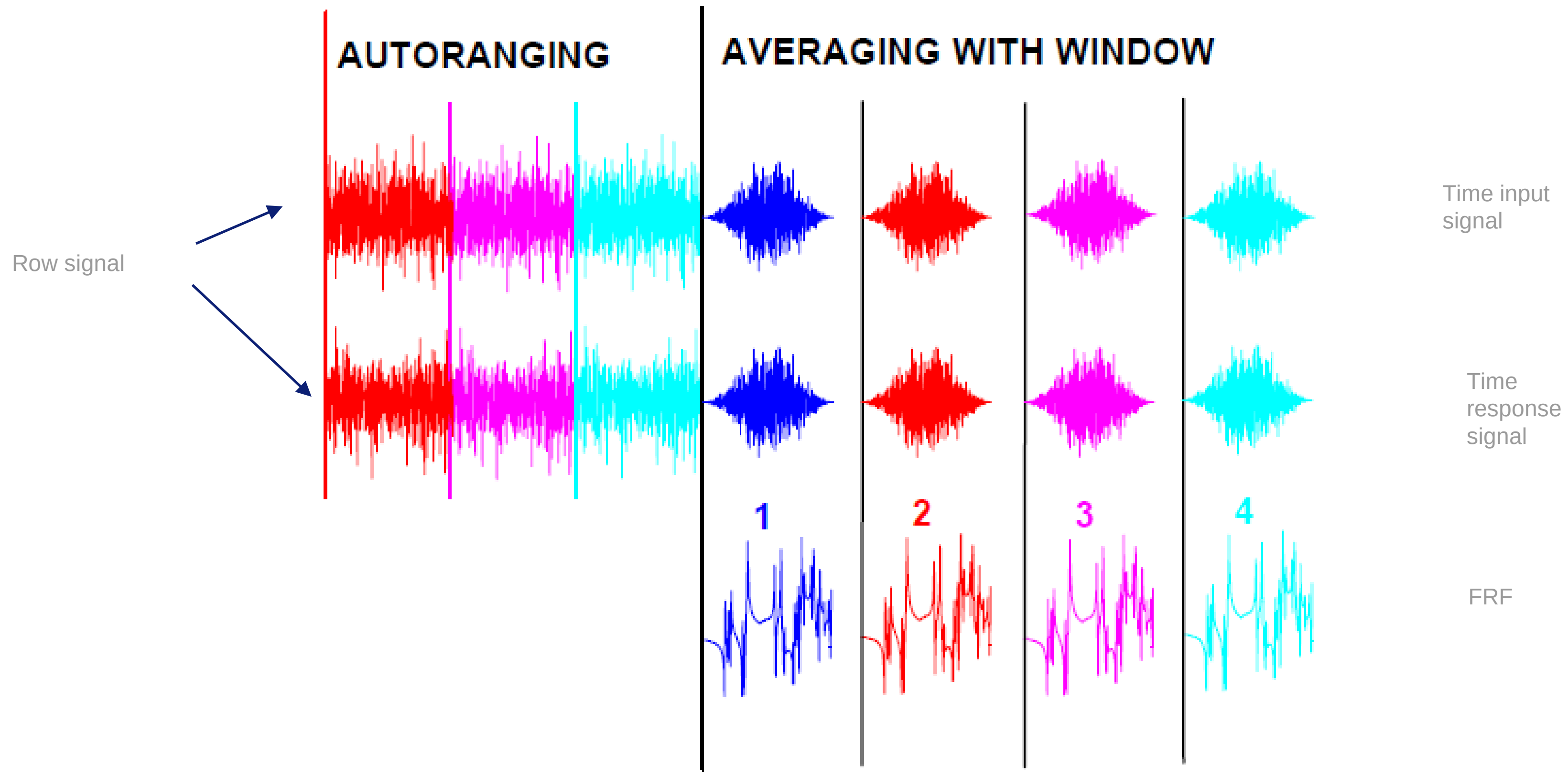
ADVANTAGES

- gives a good linear approximation for a system with slight non-linearities
- relatively fast
- relatively good general purpose excitation

DISADVANTAGES

- leakage is a very serious problem
- FRFs are generally distorted due to leakage

Random Excitation with Hanning Window



An ergodic, stationary signal with Gaussian probability distribution. Typically, has frequency content at all frequencies.

Random Excitation with Hanning Window

An ergodic, stationary signal with Gaussian probability distribution. Typically, has frequency content at all frequencies.

ADVANTAGES

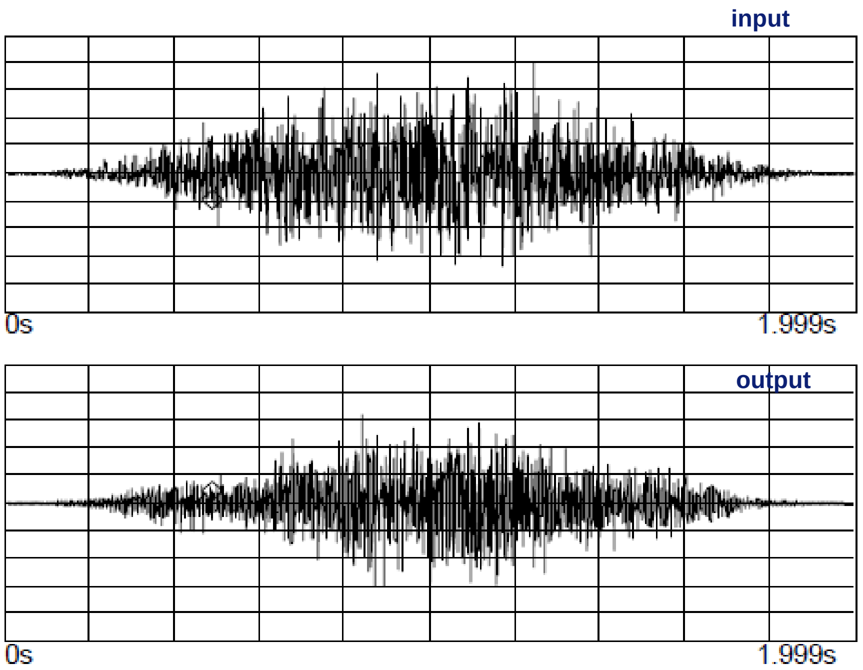
- gives a good linear approximation for a system with slight non-linearities
- relatively fast
- overlap processing can be used
- relatively good general purpose excitation

DISADVANTAGES

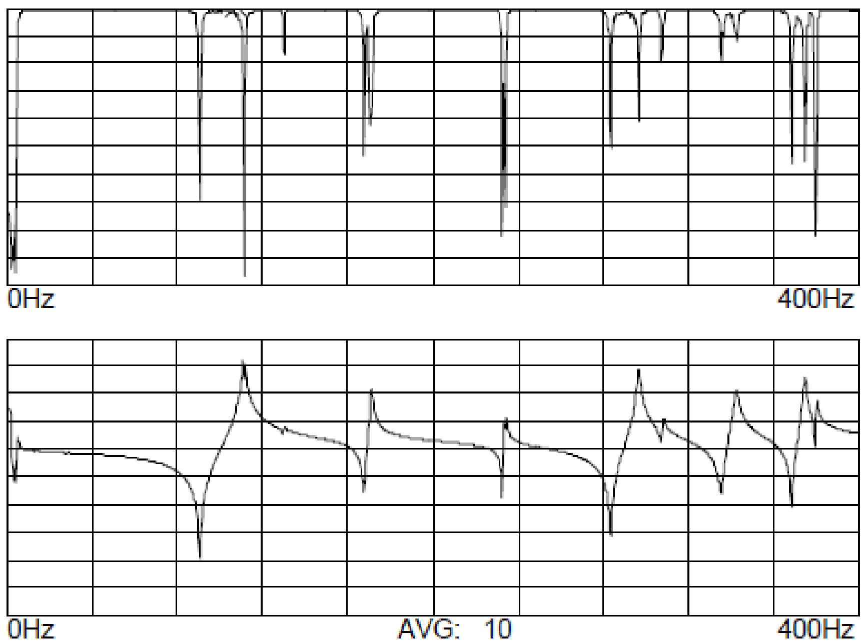
- even with windows applied to the measurement leakage is a very serious problem
- FRFs are generally distorted due to leakage with (significant distortion at the peaks)
- excessive averaging necessary to reduce variance on data

Random Excitation with Hanning Window

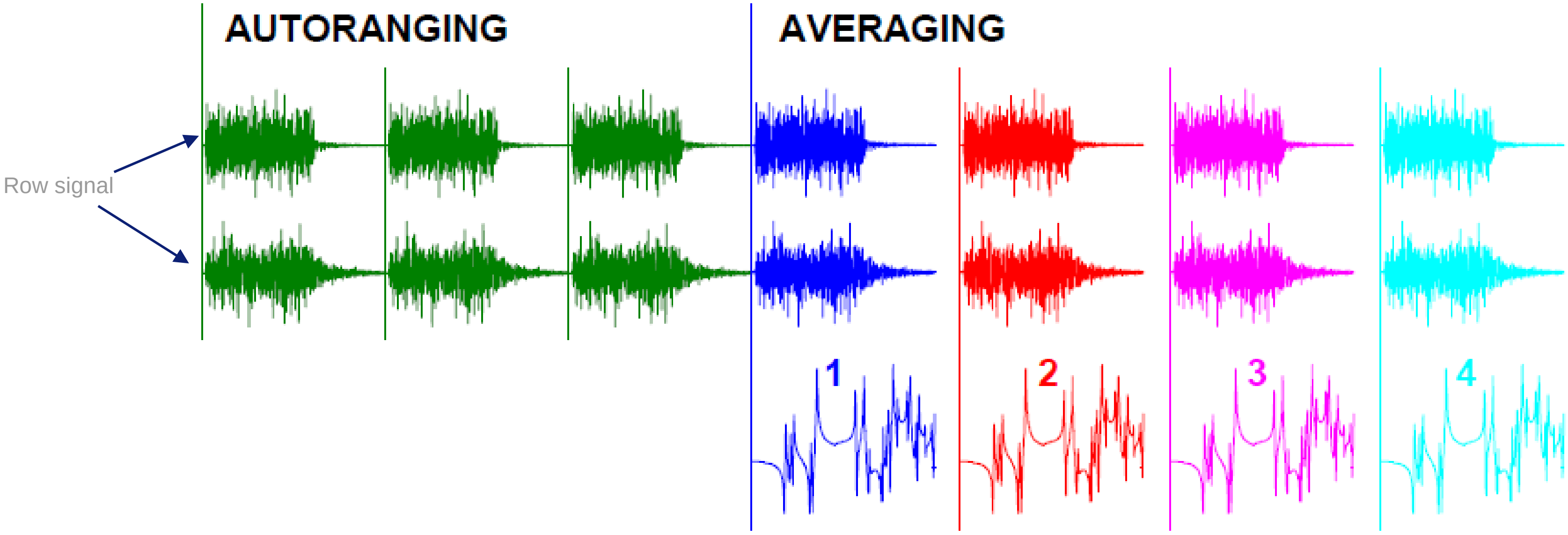
Time signal



Frequency Signal



Burst Random Excitation



A random excitation that exists over only a portion of the data block (typically 50% to 70%).

Burst Random Excitation

A random excitation that exists over only a portion of the data block (typically 50% to 70%)

ADVANTAGES

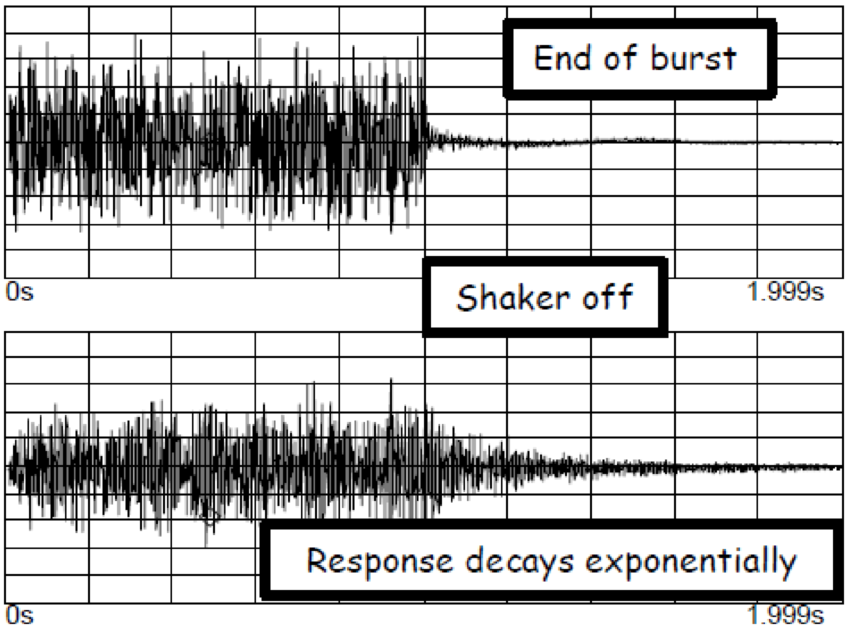
- has all the advantages of random excitation
- the function is self-windowing
- no leakage

DISADVANTAGES

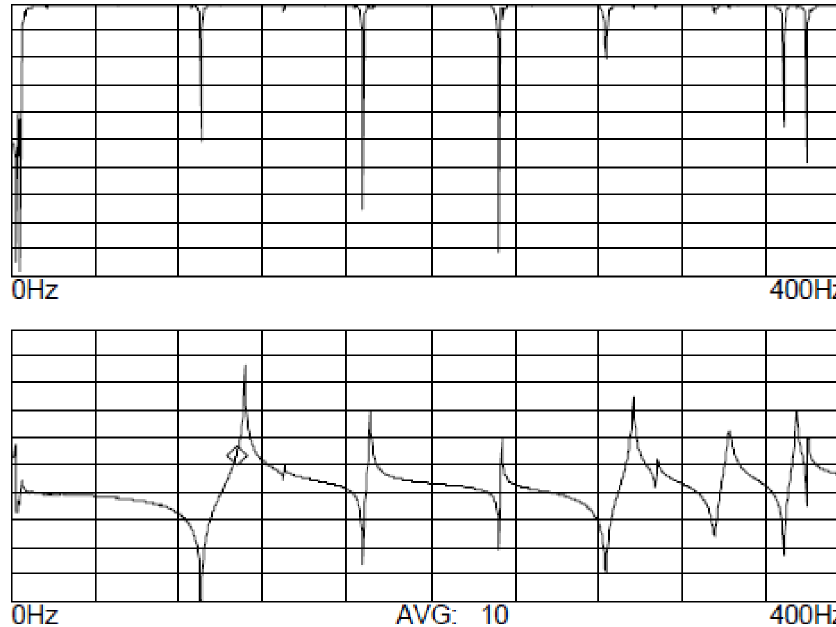
- if response does not die out within on sample interval, then leakage is a problem

Burst Random Excitation

Time signal

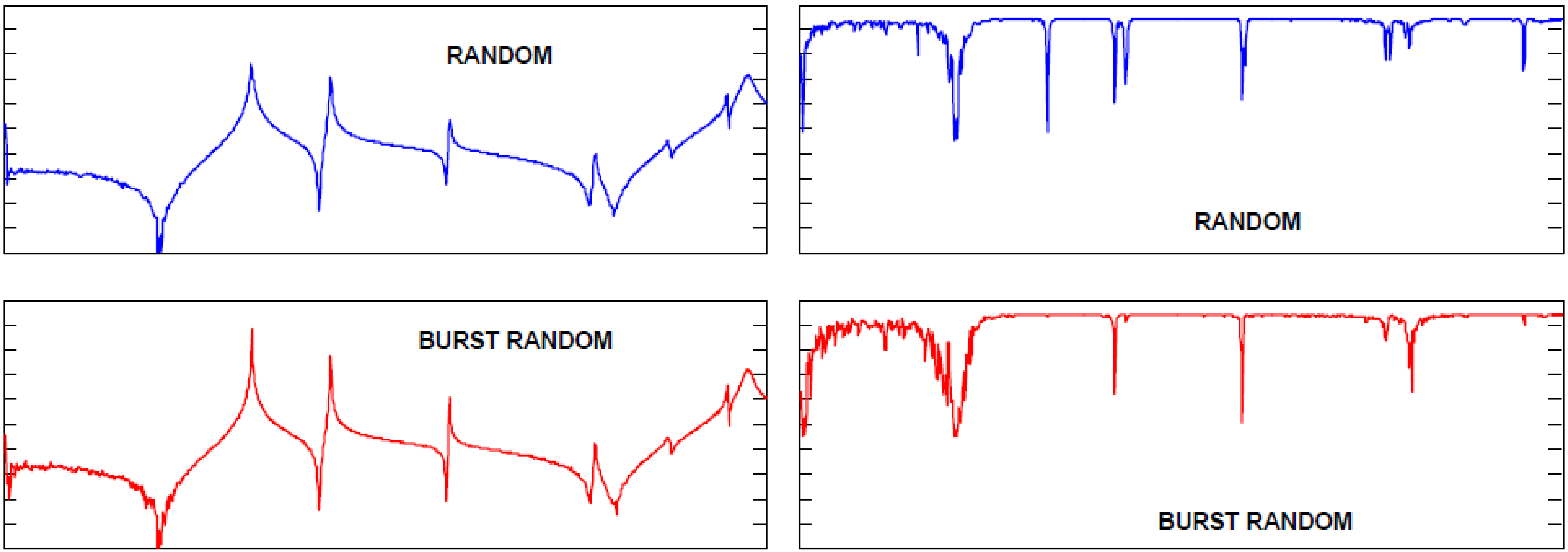


Frequency Signal



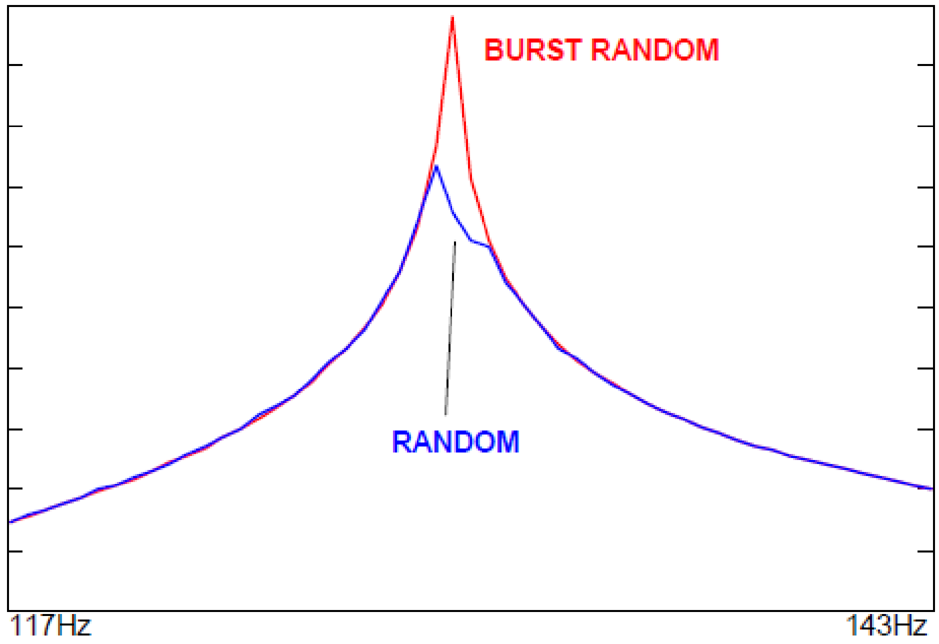
Random with Hanning Window vs Burst Random

Frequency Response Function | Coherence



When comparing the measurement with random and burst random, notice that the random excitation peaks are lower and appear to be more heavily damped when compared to the burst random. - also notice the coherence improvement at the resonant peaks.

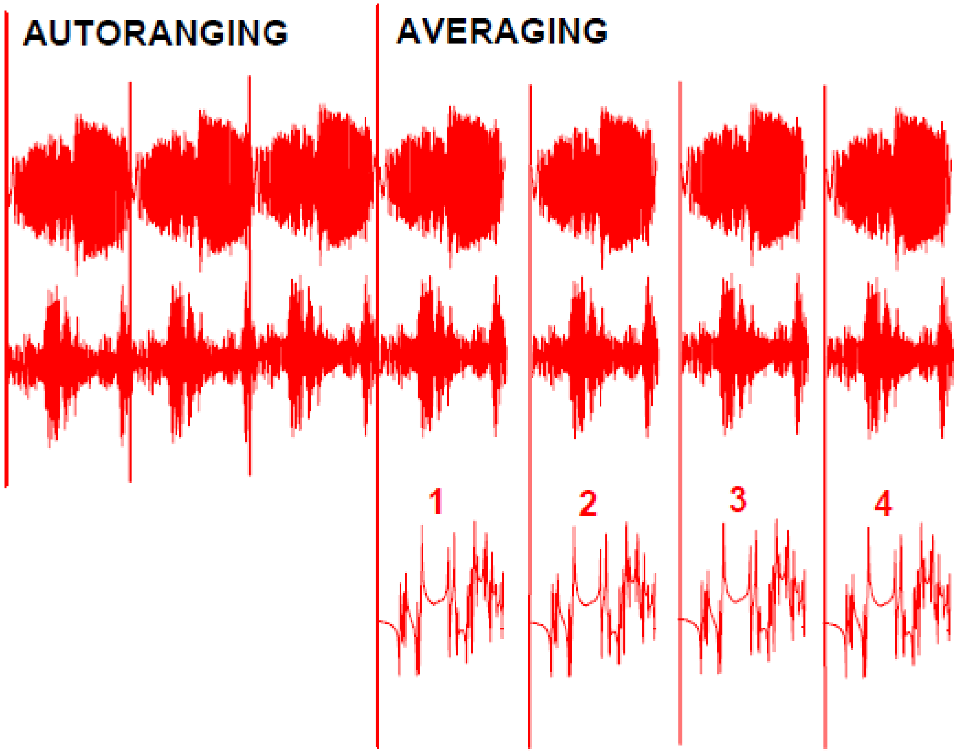
Random with Hanning Window vs Burst Random



In burst random no windows is apply and thus no distortion occurs.

- Windows will ***always*** have an effect on the measured FRF even when the same window is applied to both input and output signals
- There will ***always*** be a distortion at the peak and the appearance of higher damping
- ***Windows always, always, always, ... distort data!!!***

Sine Chirp Excitation



A very fast swept sine signal that starts and stops within one sample interval of the FFT analyzer

Sine Chirp Excitation

A very fast swept sine signal that starts and stops within one sample interval of the FFT analyzer

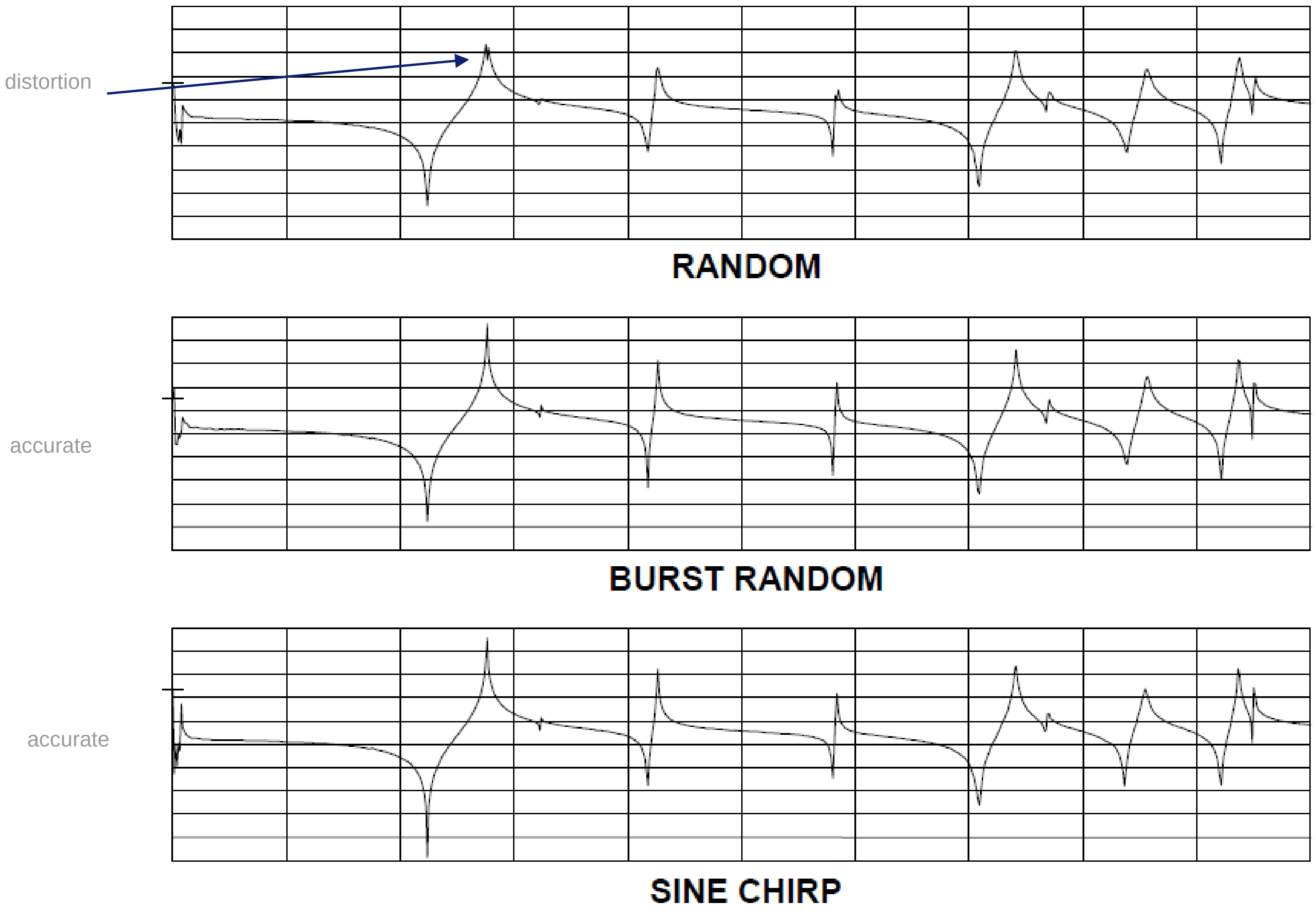
ADVANTAGES

- has all the same advantages as swept sine
- self windowing function
- good for nonlinear characterization

DISADVANTAGES

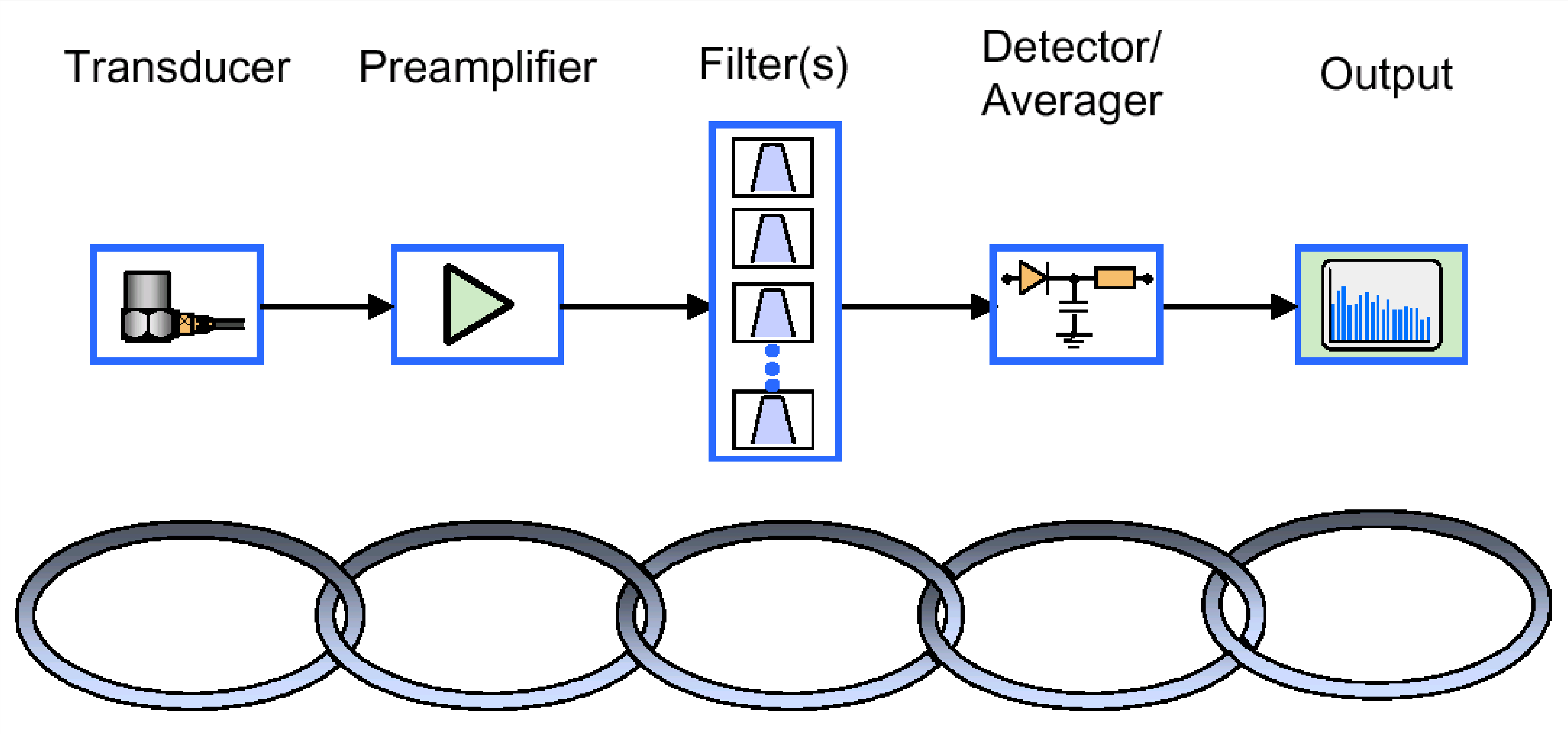
- nonlinearities will not be averaged out

Comparison - Random/Hann, Burst Random, Chirp

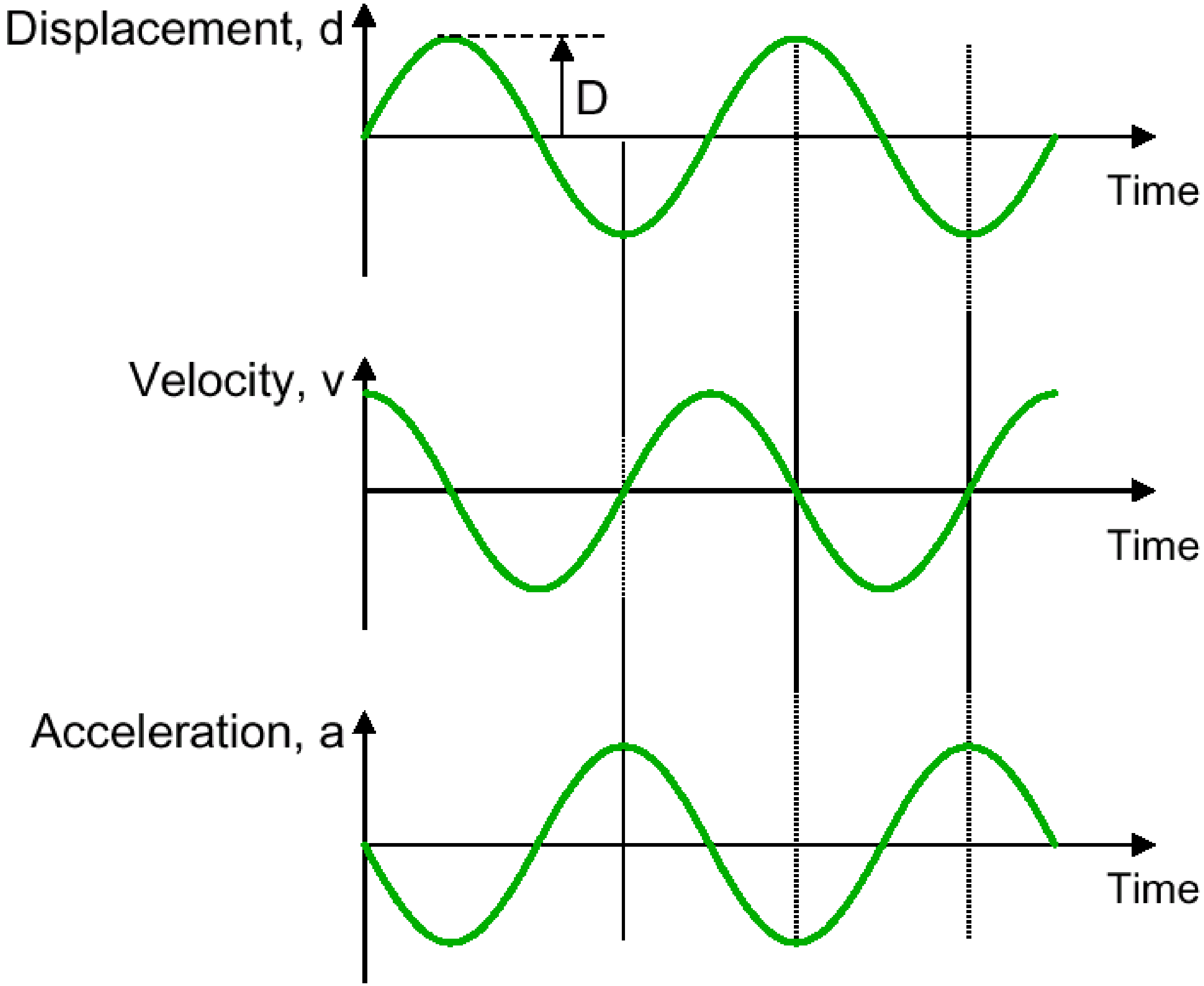


5. Measuring Vibrations

The Measurement Chain



Conversion from Displacement to Acceleration



$$d = D \sin \omega t$$

$$d = D$$

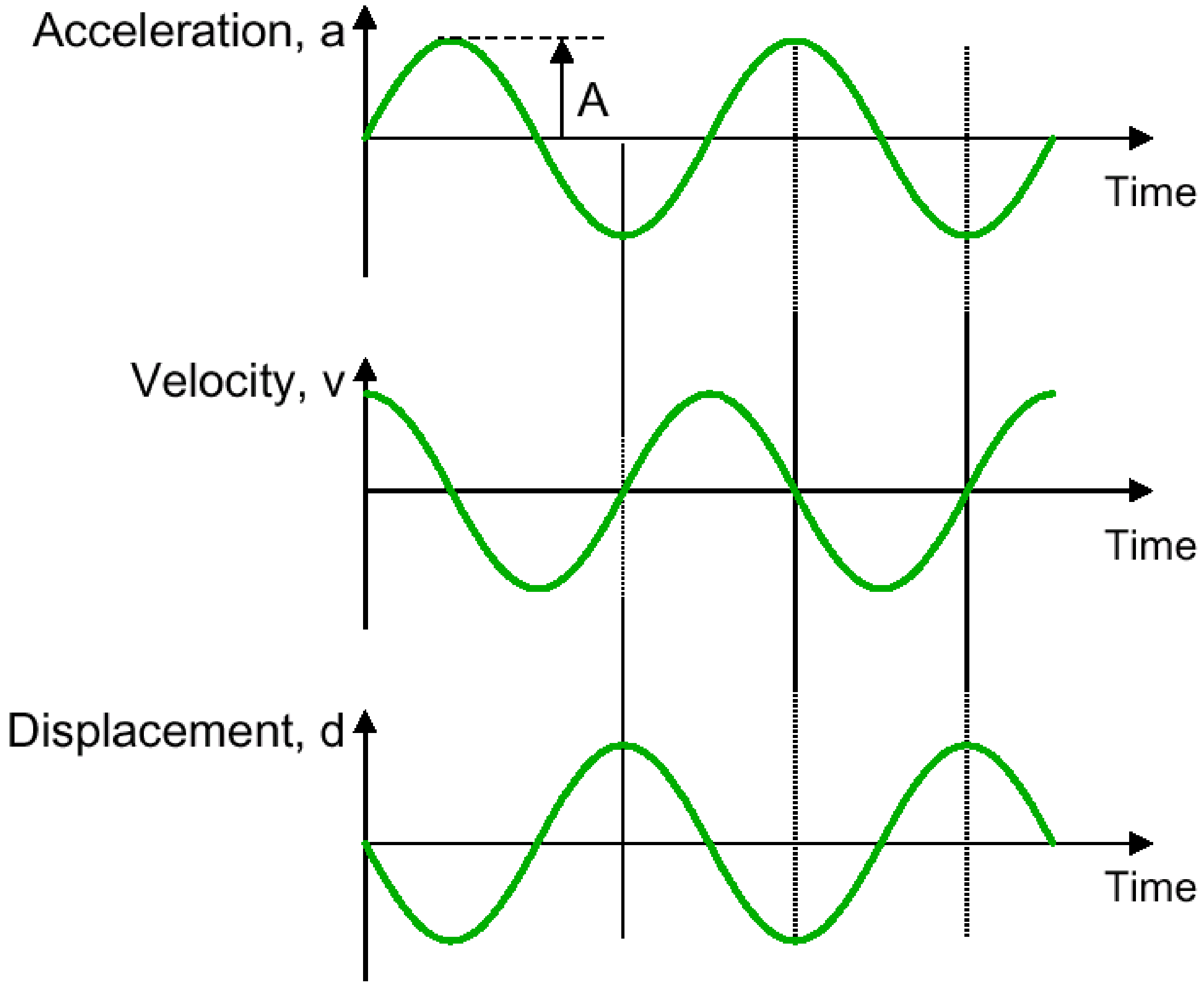
$$v = \frac{dd}{dt} = D\omega \cos \omega t$$

$$v = D\omega = D2\pi f$$

$$a = \frac{d^2d}{dt^2} = D\omega^2 \sin \omega t$$

$$a = D\omega^2 = D4\pi^2 f^2$$

Conversion from Acceleration to Displacement



$$a = A \sin \omega t$$

$$a = A$$

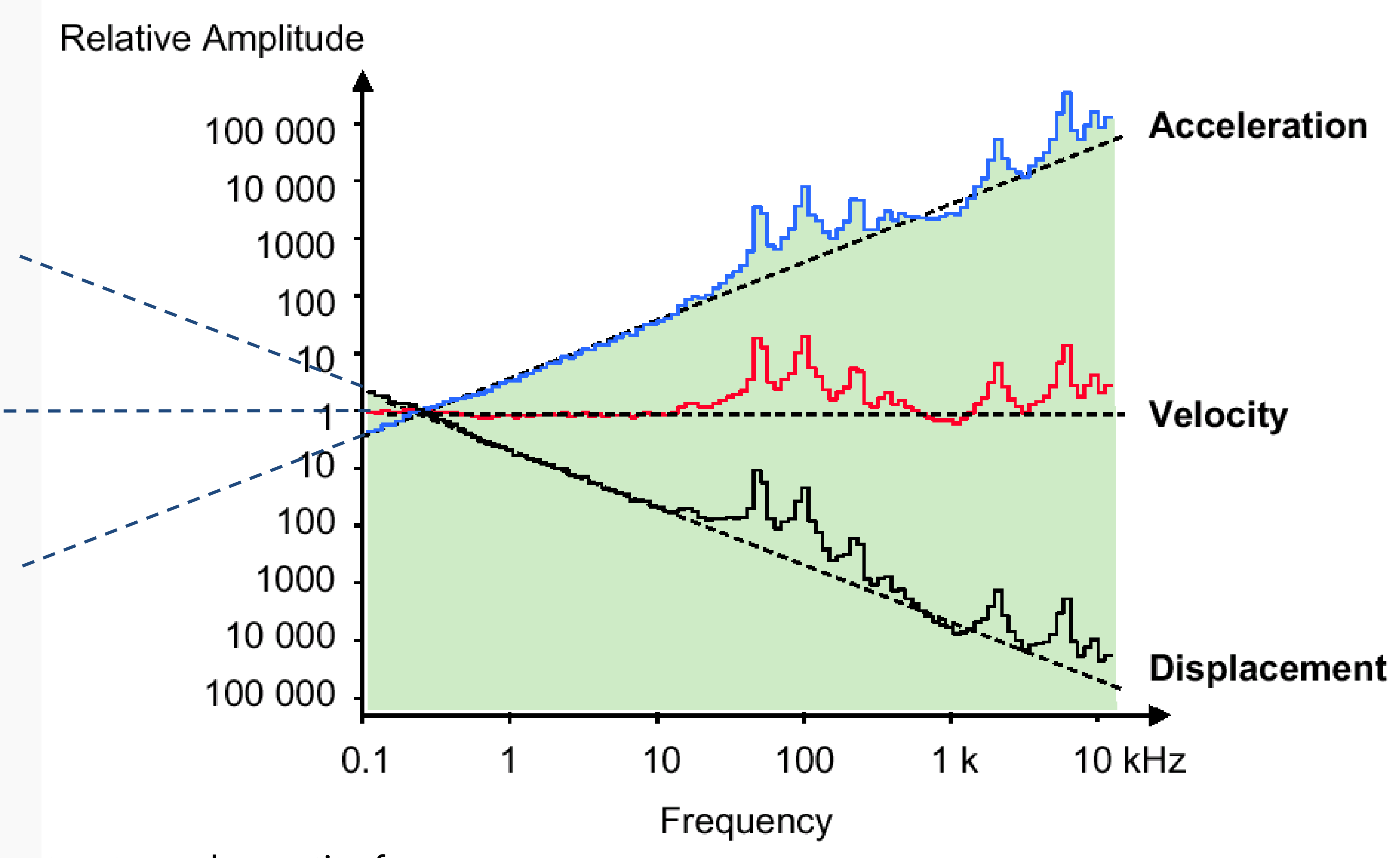
$$v = \int a \, dt = -\frac{A}{\omega} \cos \omega t$$

$$v = \frac{A}{\omega} = \frac{A}{2\pi f}$$

$$d = \iint a \, dt \, dt = -\frac{A}{\omega^2} \sin \omega t$$

$$d = \frac{A}{\omega^2} = \frac{A}{4\pi^2 f^2}$$

Vibration Parameters



Displacement not good quantity for high frequency measurement.
Acceleration not the good quantity for low frequency measurement

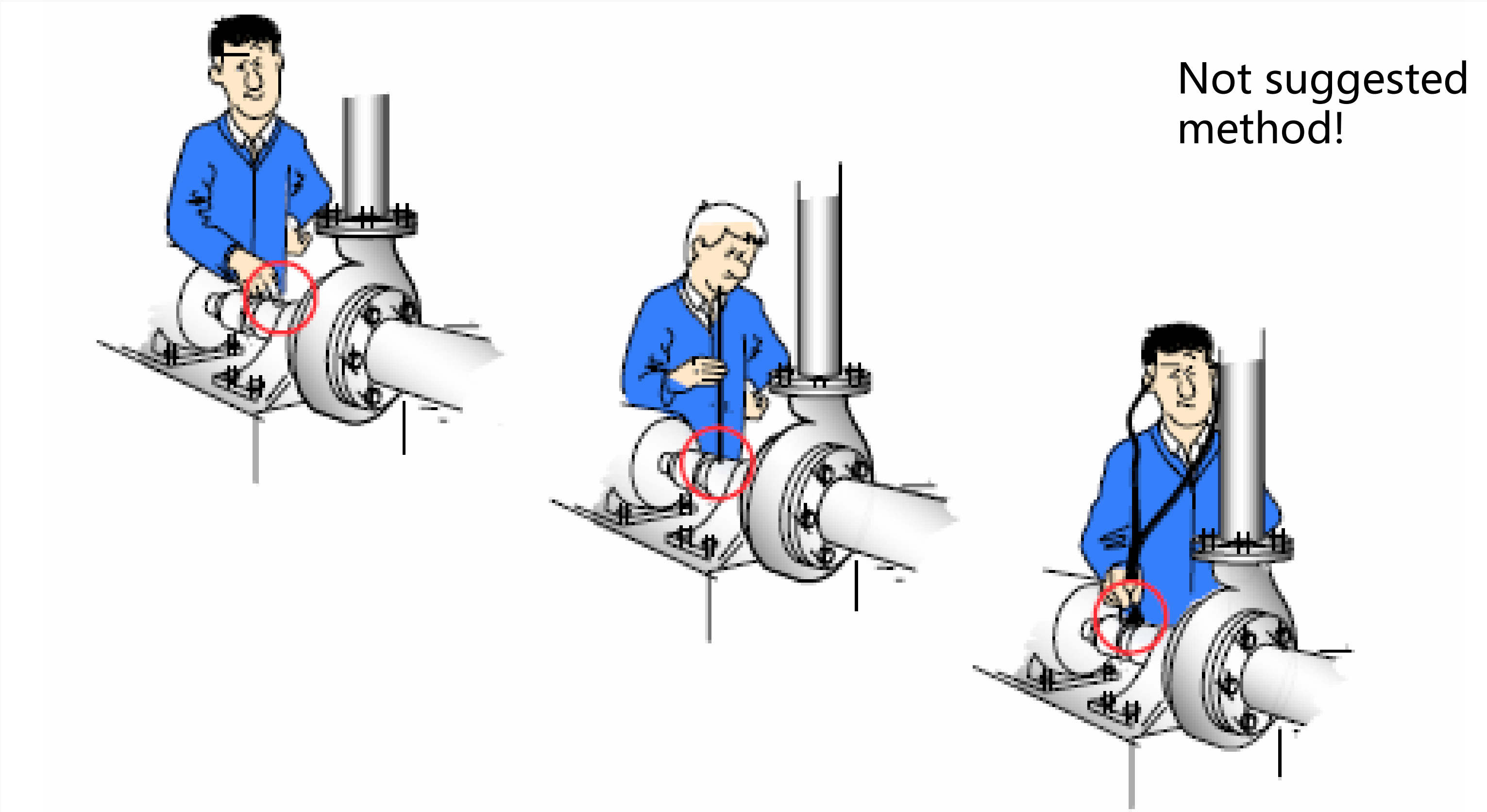
Units of Vibration Signals

Acceleration a	$1ms^{-2}$ (m/s^2)	= 0.102g = 39.4 in/s^2
Velocity v	$1ms^{-1}$ (m/s)	= 3.6 km/h = 39.4 in/s
Displacement d	1m	= 1000 mm = 39.4 in

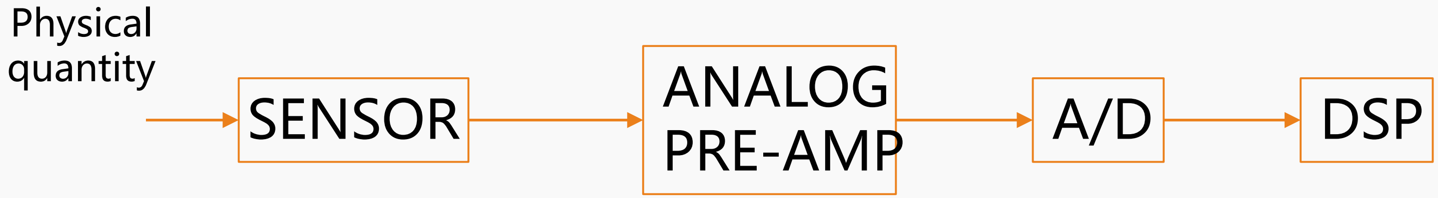
$$1g \equiv 9.80665\ ms^{-2}$$

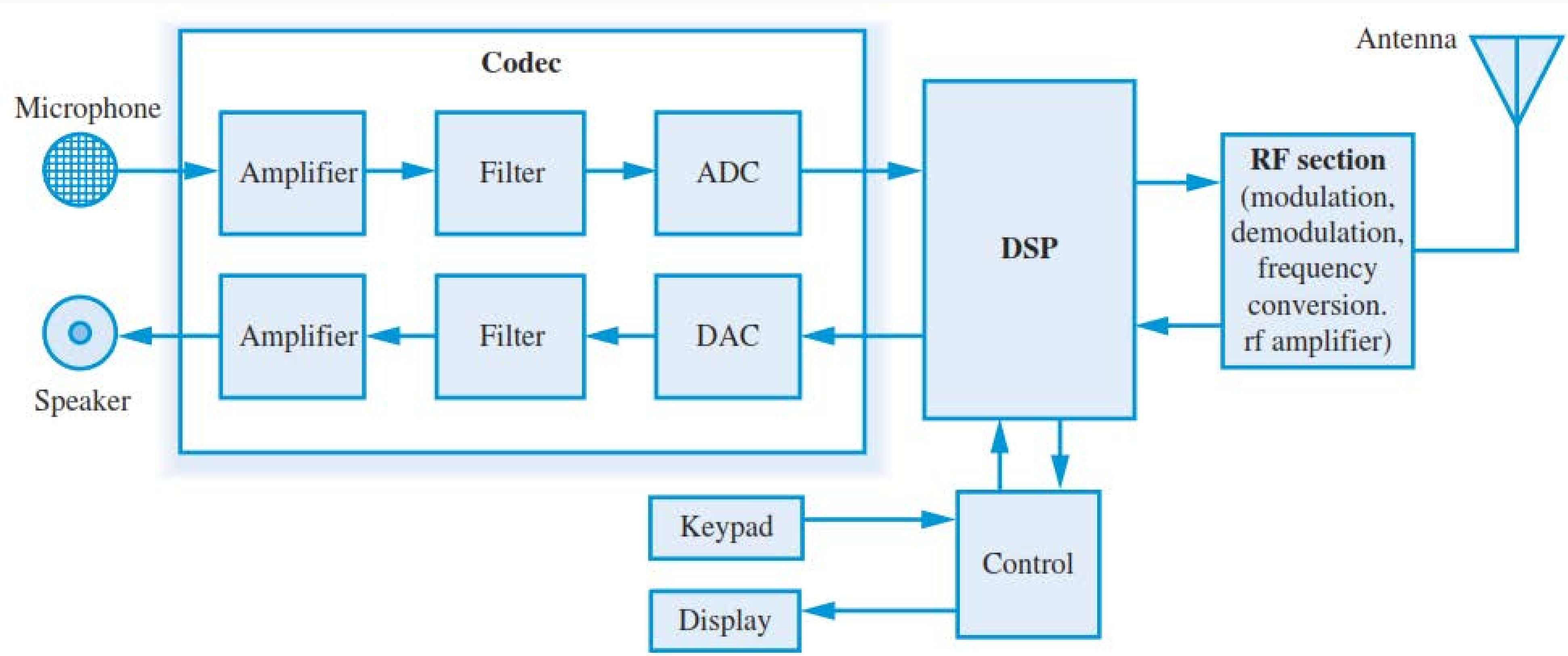
VIBRATION “MEASUREMENT”

In the absence of instruments, vibration has been “evaluated” by means of touching the machine; transfer of the vibration signal from the source to the head with the aid of a rod, or by using a doctor’s stethoscope. In each of these cases, the signal is evaluated by experience without the aid of numerical values to aid comparison.

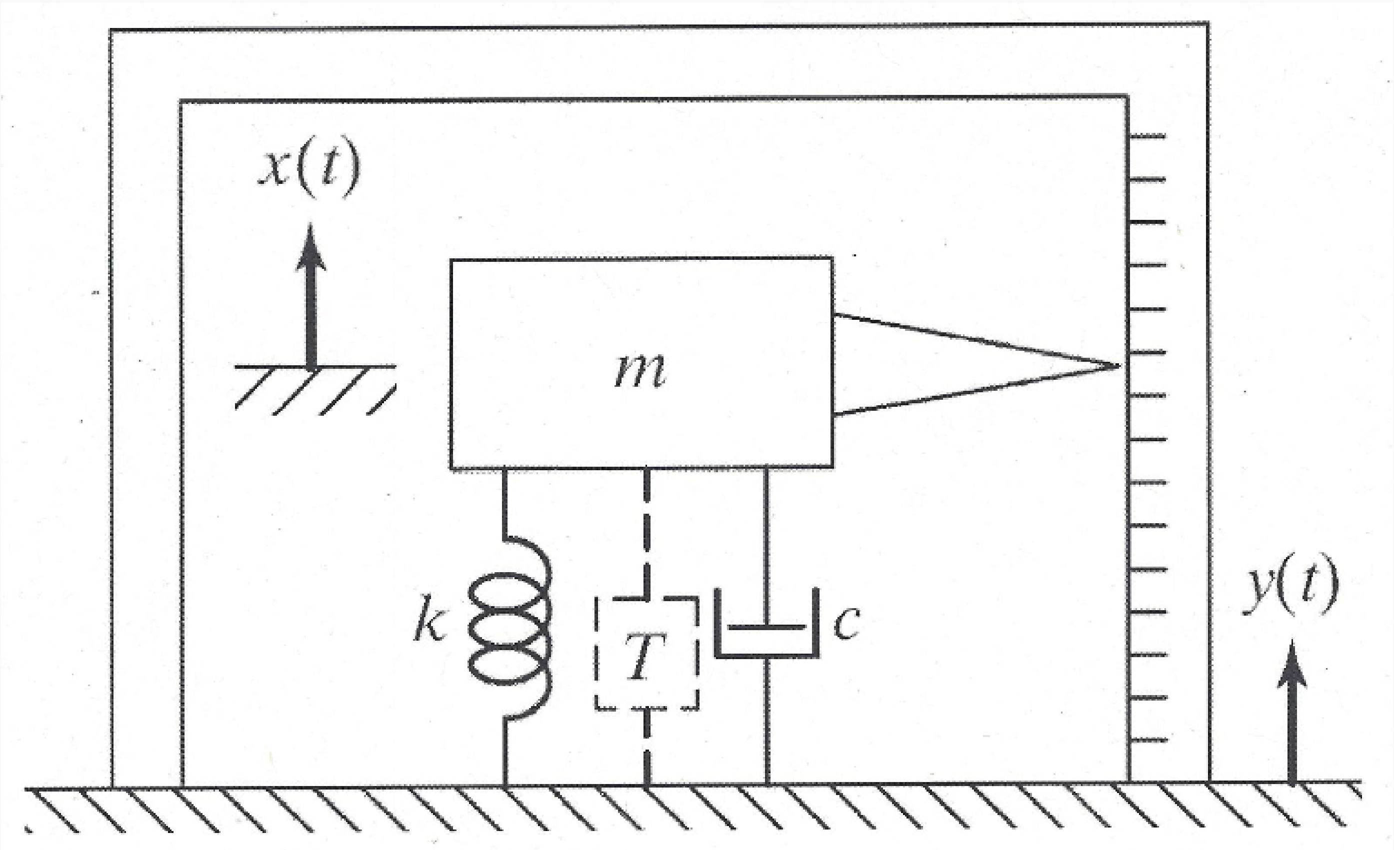


MEASUREMENT - DIGITAL SIGNAL ANALYSIS





ACCELEROMETER – HOW IT WORKS



x(t) displacement of the mass

y(t) displacement of the base (what we want to measure)

Relative displacement:

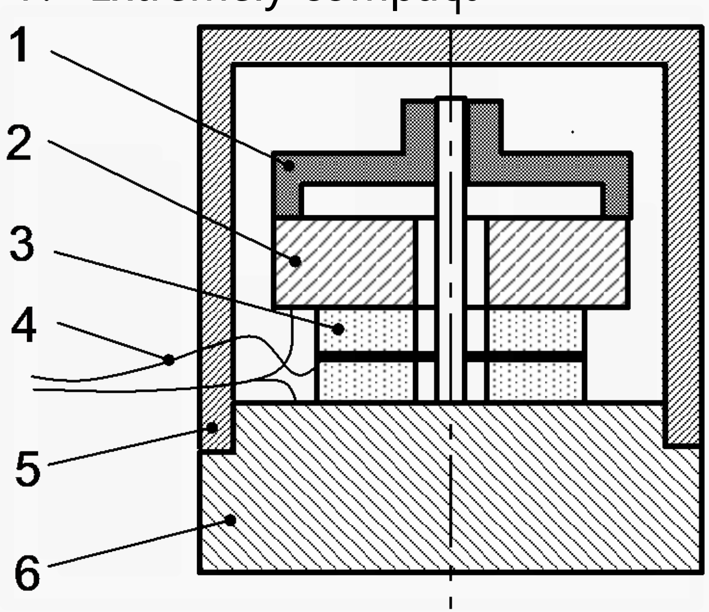
$$z(t) = x(t) - y(t)$$

If $x(t) \approx 0$ $\rightarrow$ $z(t) \approx -y(t)$

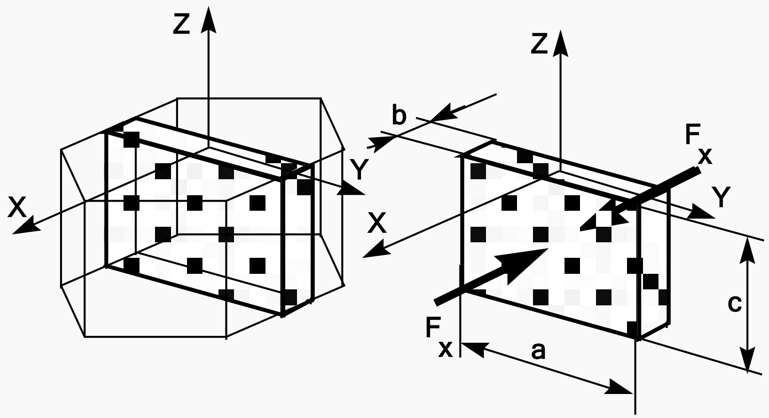
PIEZOELECTRIC ACCELEROMETER

The piezoelectric accelerometer is widely accepted as the best available transducer for the absolute measurement of vibration. This is a direct result of these properties

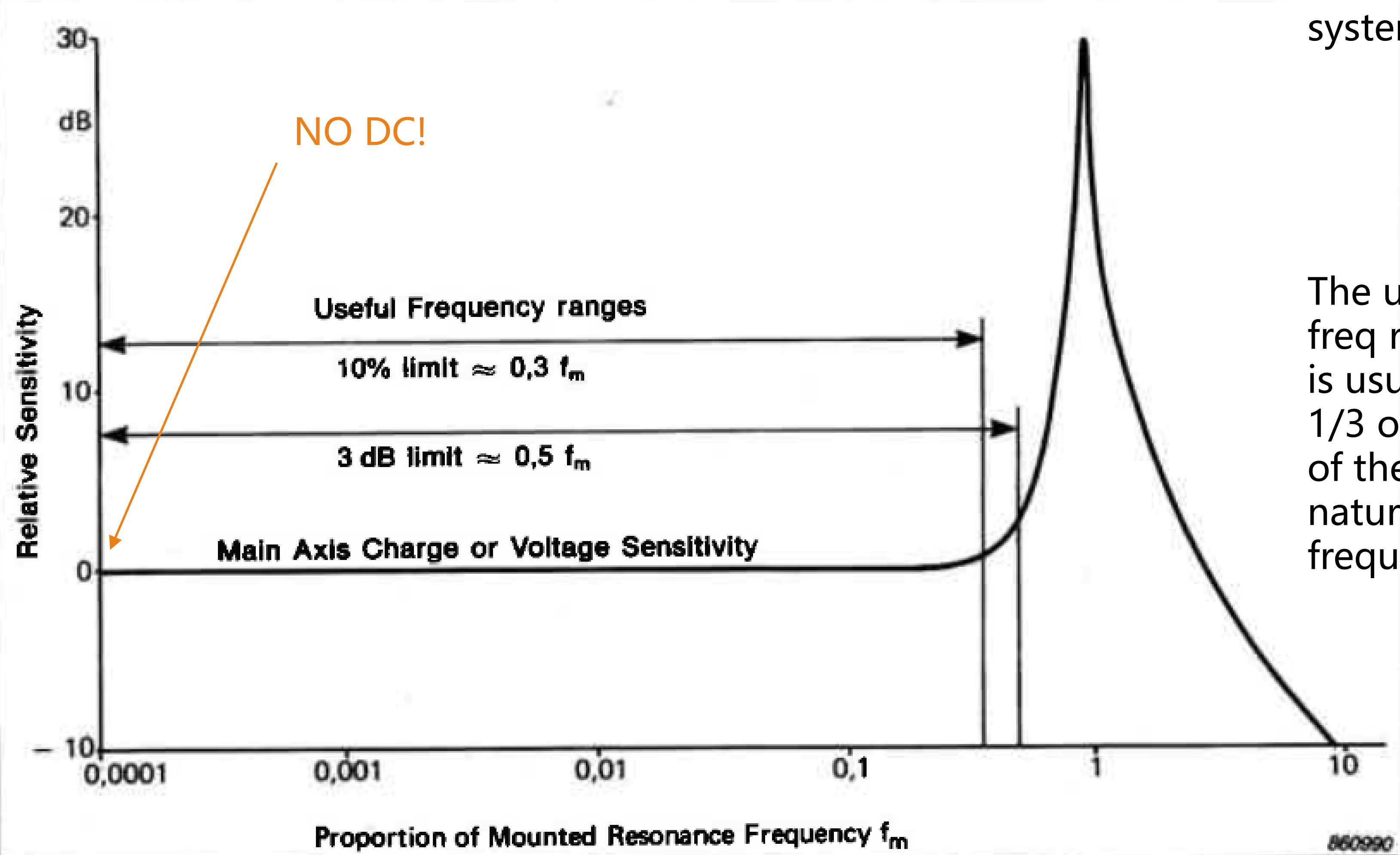
1. Usable over very wide frequency ranges.
2. Excellent linearity over a very wide dyamic range.
3. Acceleration signal can be electronically integrated to provide velocity and displacement data
4. Vibration measuments are possible in a wide range of environmental conditions while still maintaining excellent accuracy
5. Self-generating so no external power supply is required
6. No moving parts hence extremely durable.
7. Extremely compact



1. Spring
2. Seismic mass
3. Piezoelectric material
4. Cables
5. Case
6. Base



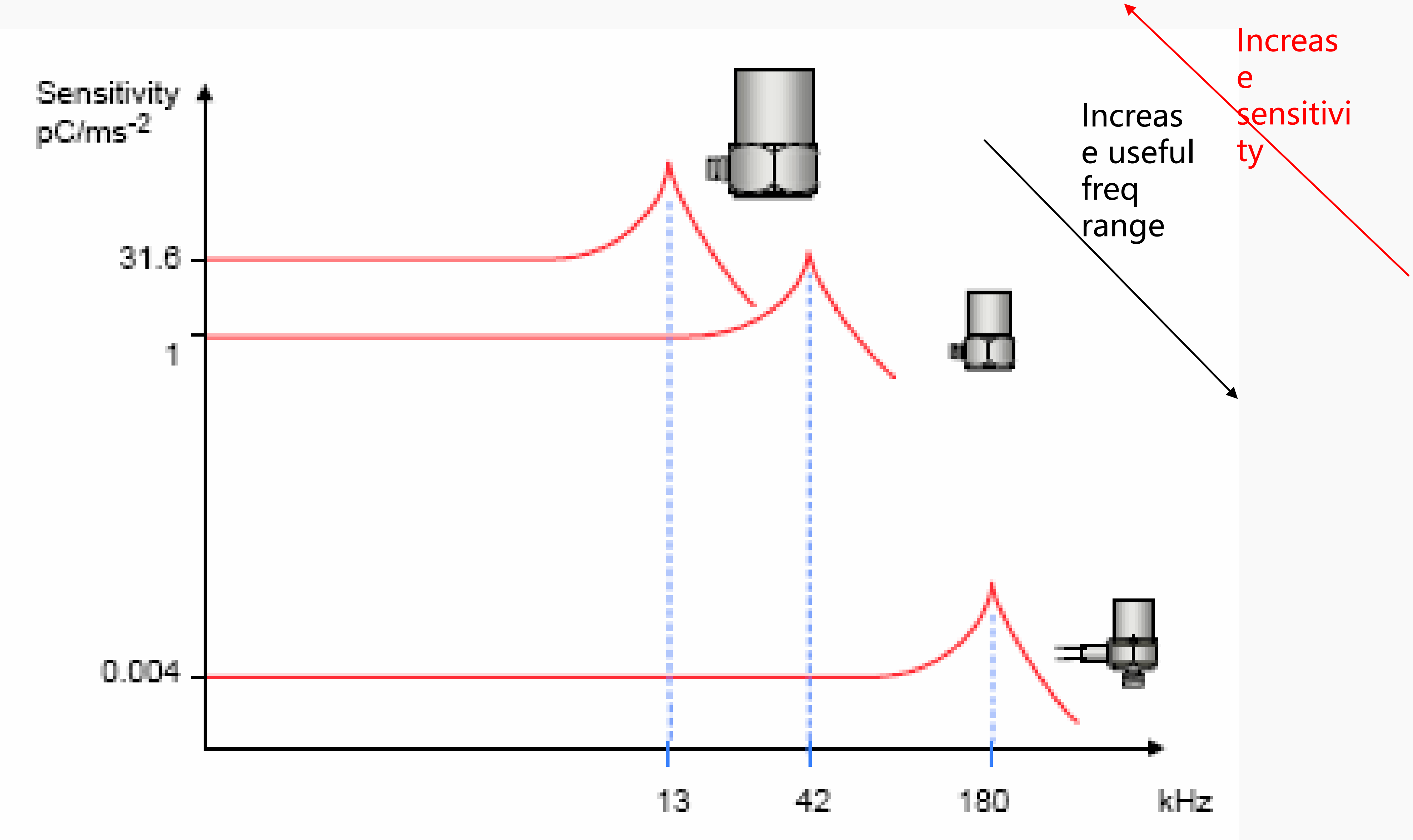
ACCELEROMETER PERFORMANCE IN PRACTICE



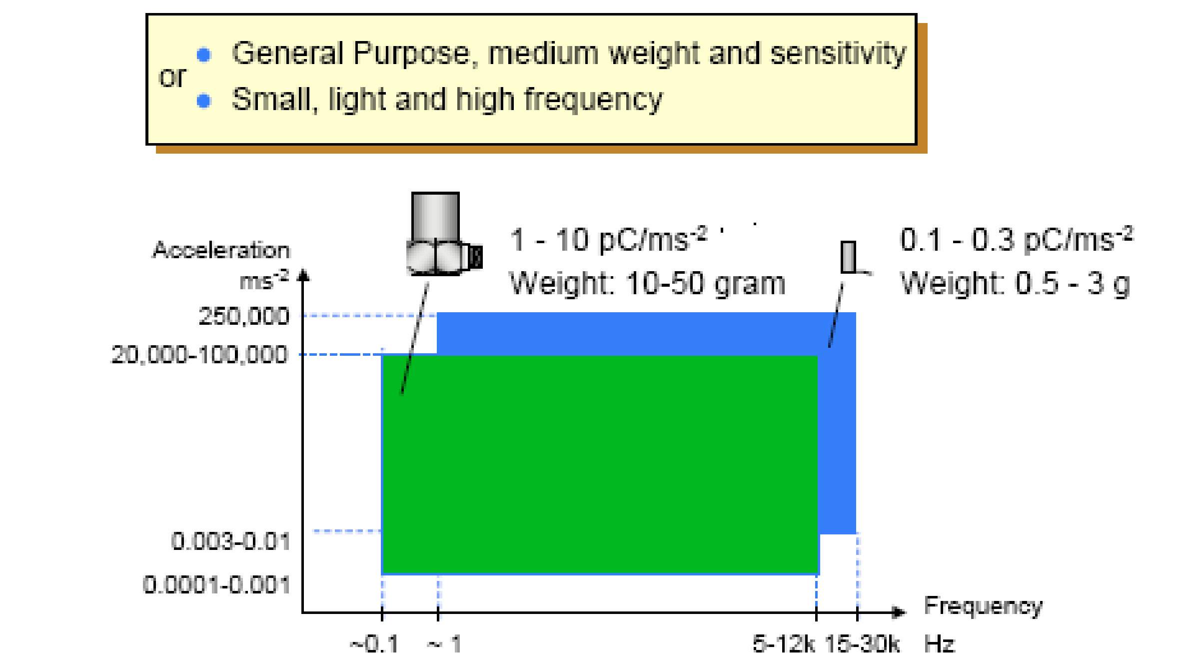
Similar to 1 DOF system

The useful freq range is usually 1/3 or 1/5 of the natural frequency

ACCELEROMETER PERFORMANCE IN PRACTICE



ACCELEROMETER PERFORMANCE IN PRACTICE



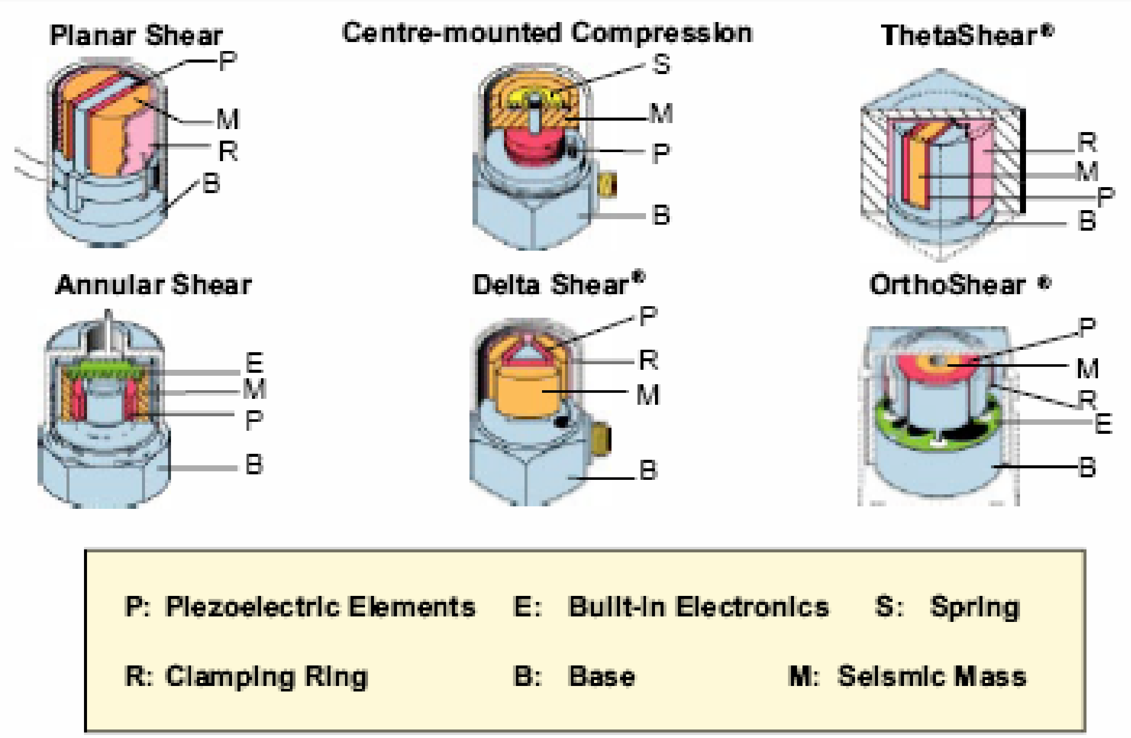
Selection of an Accelerometer

The range of operation is the first to be considered when selecting an accelerometer.

The graph shows two typical groups of accelerometers with typical specifications:

- General Purpose Type Accelerometers
- Small (miniature) Accelerometers

ACCELEROMETER PERFORMANCE IN PRACTICE



The main difference regards the orientation of the piezoelectric material.

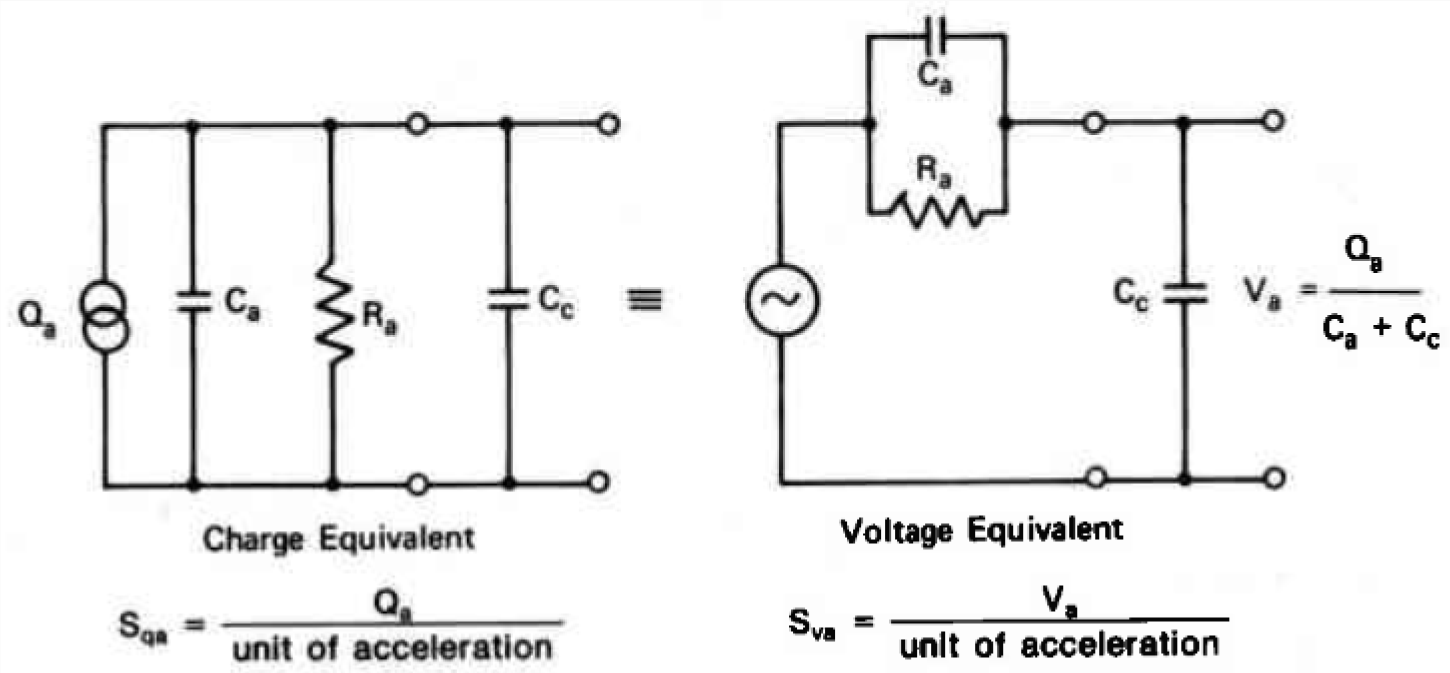
Delta Shear Design. This design gives a high sensitivity-to-mass ratio compared to other designs and has a relatively high resonance frequncy and high isolation from base strains and temperature transients. The excellent overall characteristics of this design make it ideal for both general purpose accelerometers and more specialized types

Centre-mounted Compression design. This design gives a moderately high sensitivity-to-mass ratio. However, any dynamic changes in the base such as bending or thermal expansions can cause stresses in the piezoelectric elements and hence erroneous outputs. For these reasonos this type of accelerometer is used for high level measurements (i.e. shock measurements) where the erroneous output is small compared with the vibration signal. This accelerometer is also used in the controlled environment of accelerometer calibration.

ACCELEROMETER PERFORMANCE IN PRACTICE

The accelerometer can be regarded as either a charge source or a voltage source.

The voltage produced by the accelerometer is divided between the accelerometer capacitance and the cable capacitance. Hence a change in the cable capacitance, caused either by a different type of cable and/or a change in the cable length, will cause a change in the voltage sensitivity. A sensitivity recalibration will therefore be required.



Charge sensitivity:

$$S_{qa} = \frac{pC}{ms^{-2}}$$

Voltage sensitivity:

$$S_{va} = \frac{mV}{ms^{-2}}$$

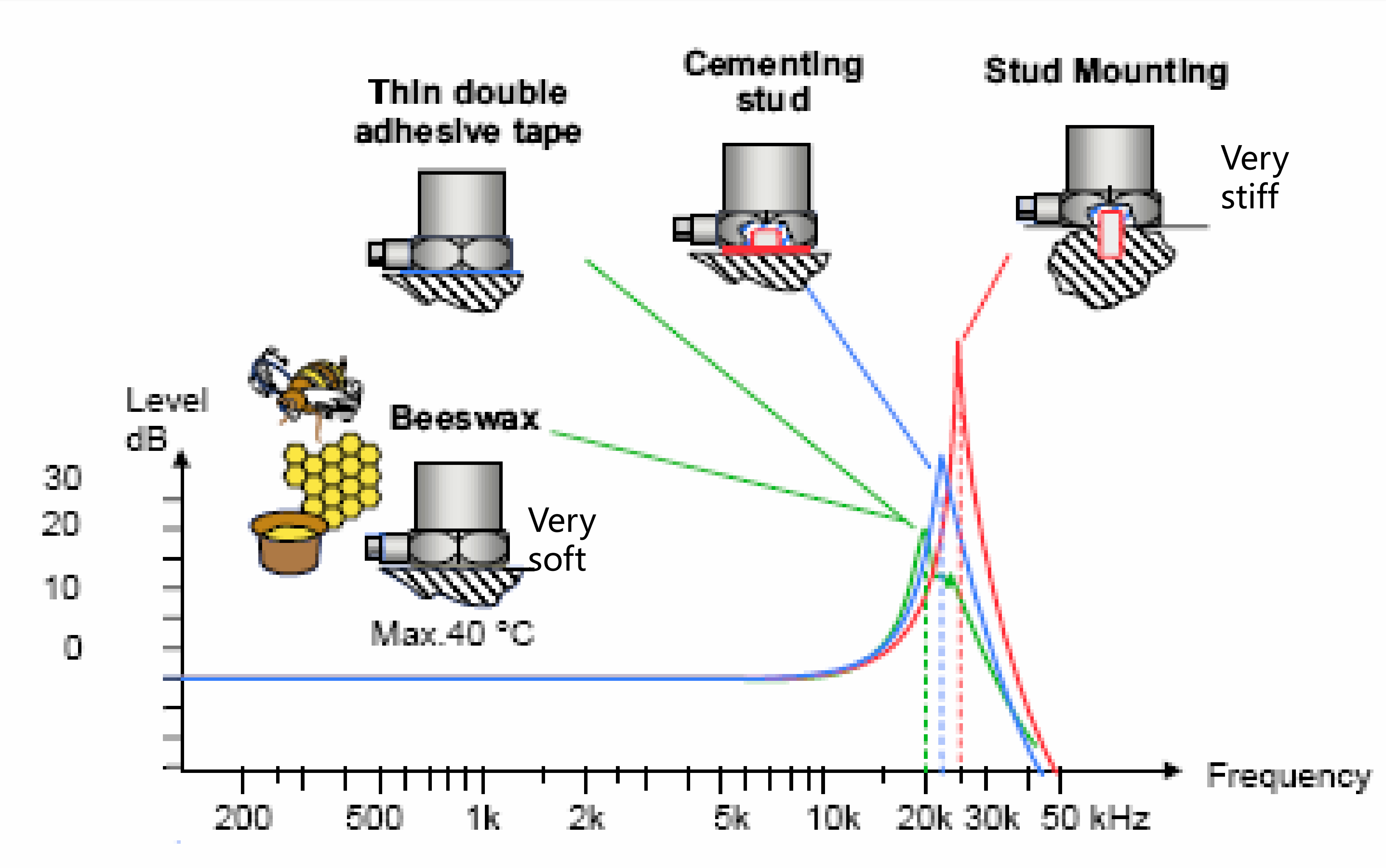
ACCELEROMETER PERFORMANCE IN PRACTICE

Accelerometer manufacturers use different names for the pre-amplified version:

Manufacturer	Brand name
Brüel & Kjær	Deltatron
Dytran Instruments	LIVM
Endevco	Isotron
Kistler Instrument Corp.	Piezotron
PCB Piezotronics Inc.	ICP

ACCELEROMETER PERFORMANCE IN PRACTICE

The natural freq of the accelerometer depends on the mounting methods!



ACCELEROMETER PERFORMANCE IN PRACTICE

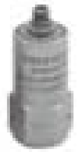
Suggested mounting:

1. Stud
2. Cyanoacraylate (epoxy adhesives)
3. Beeswax (low frequency)
4. Magnet(low frequency)

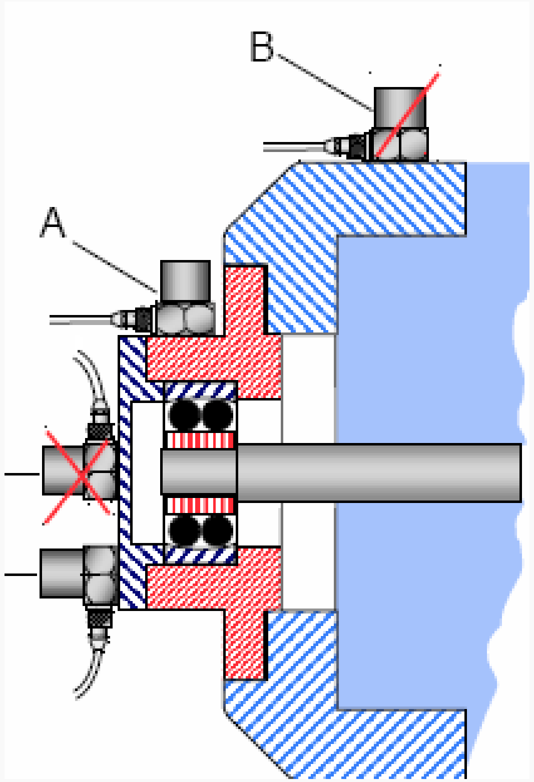
Not Suggested mounting:

1. Self-Adhesive
2. Probes

ACCELEROMETER PERFORMANCE IN PRACTICE

		 Enlarge	 Enlarge	 Enlarge	 Enlarge
		Product Data	Product Data	Product Data	Product Data
Product Type		2273A	2273AM1	2273AM20	2276
Order No		EE 0019	EE 0019-002	EE 0019-003	EE 0020
Description		Side Connector	Side Connector	Top Connector	Side Connector
Sensitivity		3 pC/g	10 pC/g	10 pC/g	10 pC/g
Frequency Range	Hz	0.5 to 8000	0.5 to 6000	10 to 6000	0.5 to 7000
Resonance Frequency	kHz	30	27	27	27
Residual Noise Level in Spec Freq Range (rms)	mg	0.21	0.11	0.12	0.12
Temperature Range (C)	C	-184 to 399	-55 to 399	-55 to 399	-55 to 482
Temperature Range (F)	F	-300 to 750	-67 to 750	-67 to 750	-67 to 900
Maximum Operational Level (peak)	'g'	1000	500	500	500
Maximum Shock Level (± peak)	'g'	10000	3000	3000	3000
Weight	gram	25	32	32	30
Connector, Electrical		10-32 UNF	10-32 UNF	10-32 UNF	10-32 UNF
Mounting		Stud	Stud	Stud	Stud
Accessory Included (Selected)		3075M6-120	3075M6-120	3075M6-120	3075M6-120
Clip/Stud/Screw included		2981-12	2981-12	2981-12	2981-12
Output		Charge/PE	Charge/PE	Charge/PE	Charge/PE

ACCELEROMETER PERFORMANCE IN PRACTICE



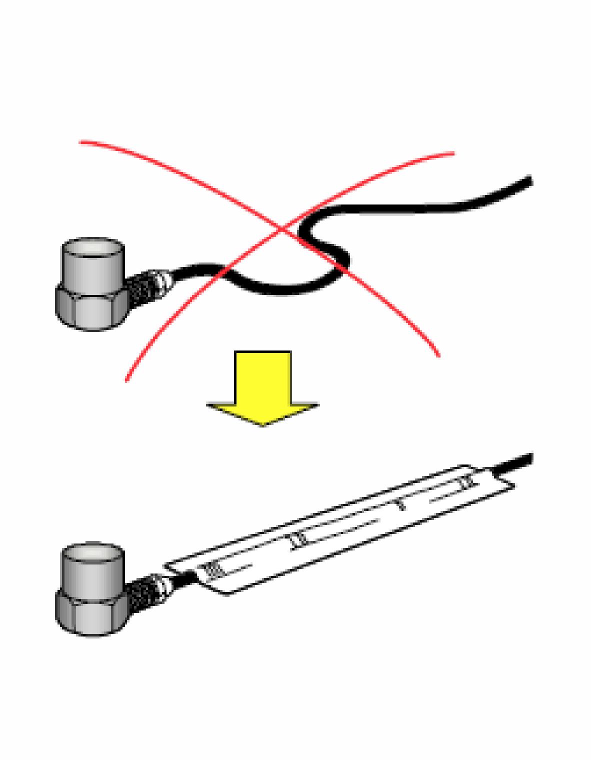
The accelerometer should be mounted so that the desired measuring direction coincides with the main sensitivity axis. Accelerometers are slightly sensitive to vibrations in the transverse direction, but this can normally be ignored as the maximum transverse sensitivity is typically only a few percent of the main axis sensitivity.

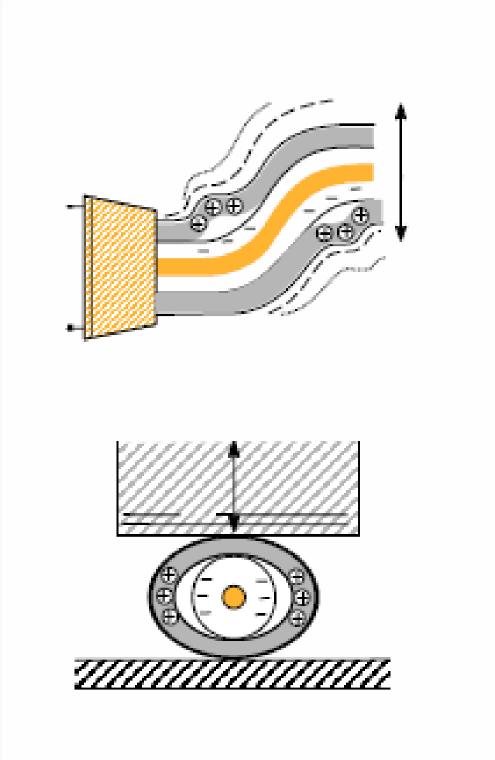
The reason for measuring vibration will normally dictate the position of the accelerometer. In the figure the reason is to monitor the condition of the shaft and bearing. The accelerometer should be positioned to maintain a direct path for the vibration from the bearing.

Accelerometer “A” thus detects the vibration signal from the bearing predominant over vibrations from other parts of the machine, but accelerometer “B” receives the bearing vibration modified by transmission through a joint, mixed with signals from other parts of the machine. Likewise, accelerometer “C” is positioned in a more direct path than accelerometer “D”.

It is very difficult to give general rules about placement of accelerometers, as the response of mechanical objects to forced vibrations is a complex phenomenon, so that one can expect, especially at high frequencies, to measure significantly different vibration levels and frequency spectra, even on adjacent measuring points on the same machine element.

ACCELEROMETER PERFORMANCE IN PRACTICE

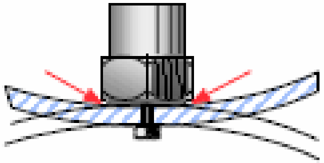




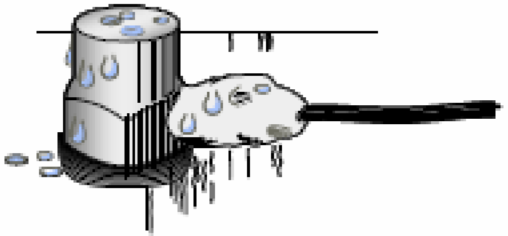
Movement (vibration) of the accelerometer cable during use can cause the screen of the cable to be separated from the insulation around the inner core of the cable. A varying electrical field is thereby created between the conducting screen and the non-conducting insulation, causing a minute current to flow in the screen which will be superimposed on the accelerometer signal as a noise signal. This phenomenon can be prevented by using low noise (or super low noise, which has similar precautions around the center conductor) accelerometer cables and fixing them to the test object e.g. with the aid of adhesive tape near the accelerometer, and let them leave the structure at a point with minimum motion.

ACCELEROMETER PERFORMANCE IN PRACTICE

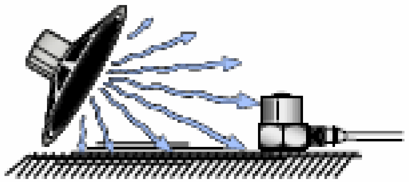
- Base Strain



- Humidity

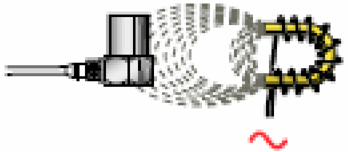


- Acoustic noise

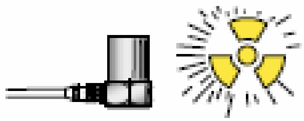


- Corrosive substances

- Magnetic fields



- Nuclear radiation



<u>*Base Strain:*</u> Base strain sensitivity has been reduced by the use of a very thick base in the accelerometers. Delta Shear accelerometers are best in this respect as the elements are not in direct connection with the base.

<u>*Humidity:*</u> The accelerometer itself is sealed, so moisture can only enter the connector. In wet conditions this effect can be prevented by the use of a silicon rubber sealant.

<u>*Acoustic Noise:*</u> Has normally negligible influence on the vibration signal from the accelerometer.

Corrosive Substances: Special materials which are resistant to most corrosive substances are used in the construction of the accelerometer.

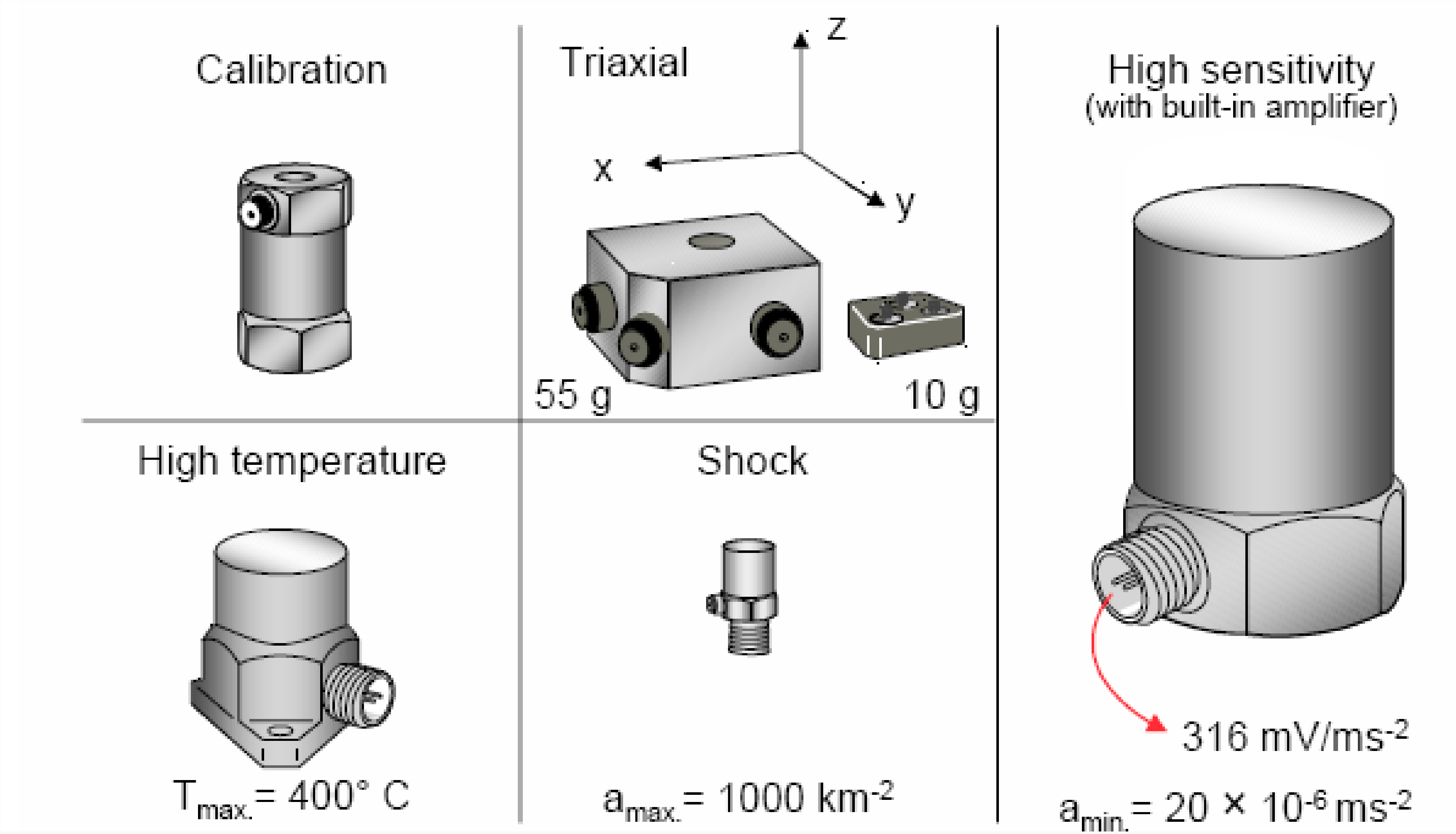
<u>*Magnetic Fields:*</u> The magnetic sensitivity is typically in the range

0.5 to 30 ms^{-2}/Tesla and thus normally not causing any problems.

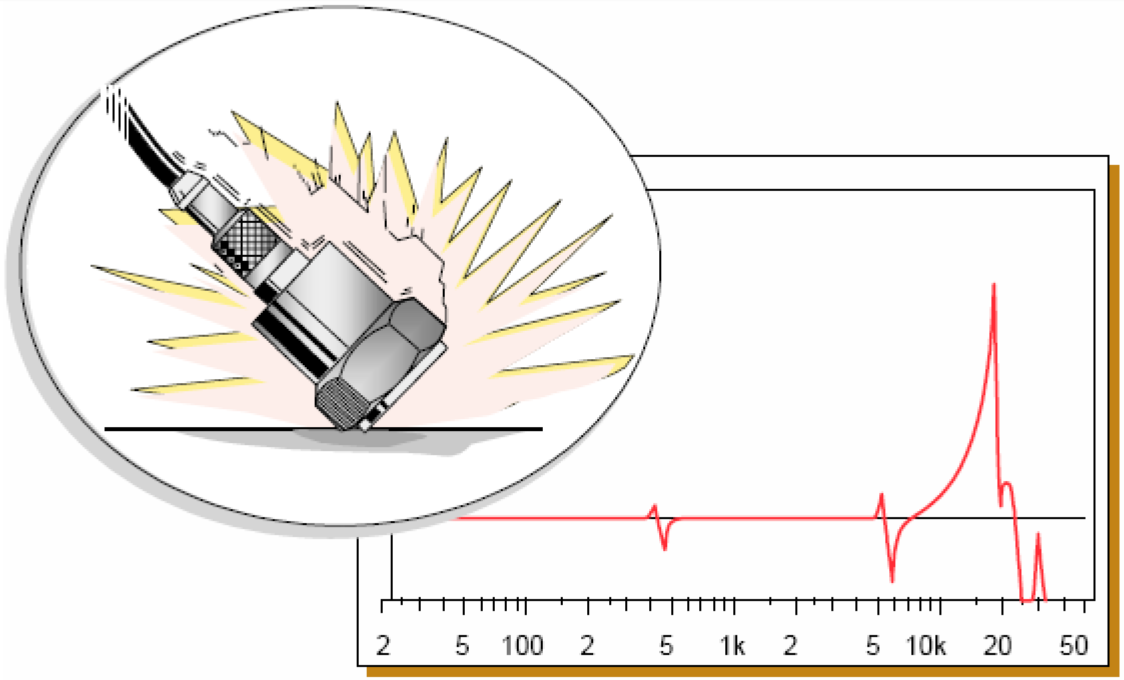
<u>*Nuclear Radiation:*</u> Most accelerometers can be used under gamma radiation of 100 kRad/h up to accumulated doses of 100 MRad without significant change in characteristics. High temperature (400°C) accelerometers can be used up to 1000 MRad.

<u>*Influence of Temperature Transients:*</u> Temperature transients (rapid fluctuations) can cause an electrical output from the accelerometer, but this effect has been considerably reduced in the Delta Shear accelerometer. The charges developed on the piezoelectric material due to temperature transients are mainly developed on surfaces normal to the polarisation of the piezoelectric material and are thus not measured.

ACCELEROMETER PERFORMANCE IN PRACTICE



ACCELEROMETER PERFORMANCE IN PRACTICE



Although most accelerometers are specified to withstand several thousand g's it is quite possible to attain such levels if the accelerometer is handled carelessly. A drop on a hard floor or a hit against a machine part might create shocks of several thousands of g. This could mean change in sensitivity or even severe damage to the accelerometer.

If it is known that the accelerometer has been subjected to such treatment it is advisable to recalibrate the accelerometer, preferably with a check of the frequency response curve.

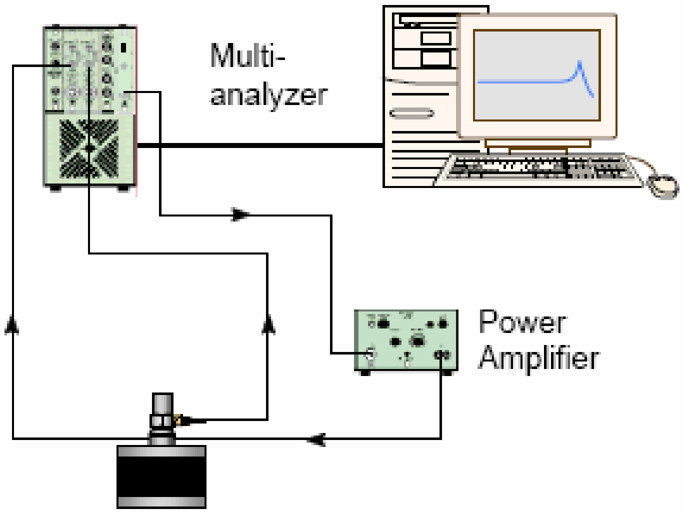
ACCELEROMETER PERFORMANCE IN PRACTICE

- In the field
  - Sensitivity check
  - Total system check
- In the lab
  - Frequency Response
  - Sensitivity Calibration

Frequency = 159.2 Hz
ω = 1000 rad/sec

Acceleration = 10 ms^{-2}

calibrator



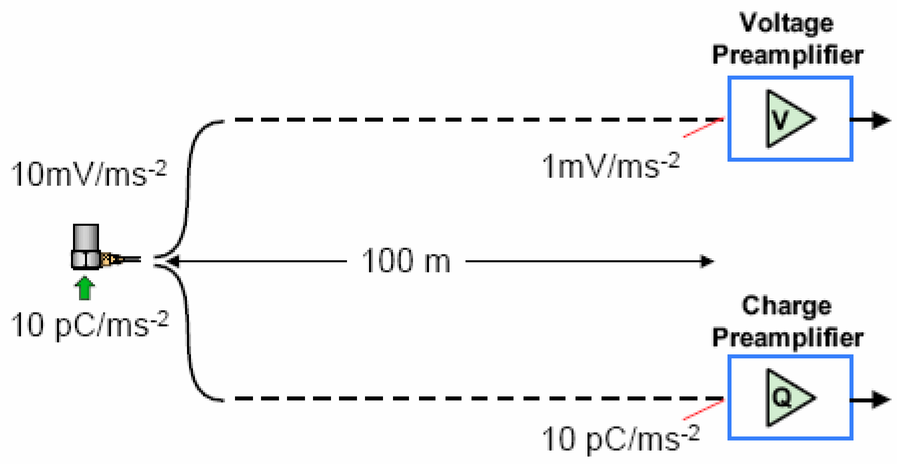
Calibration Exciter with built-in
or external reference accelerometer

Usually, accelerometer sensitivity remains the same for several years (also decades)

Check of accelerometer sensitivity and system setup

A small portable calibrator providing e.g. 10 ms^{-2} at ω = 1000 rad/sec is ideal for checking accelerometer sensitivity and the whole setup of a measuring chain.

ACCELEROMETER PERFORMANCE IN PRACTICE



With new low impedance cables, this is not true anymore!

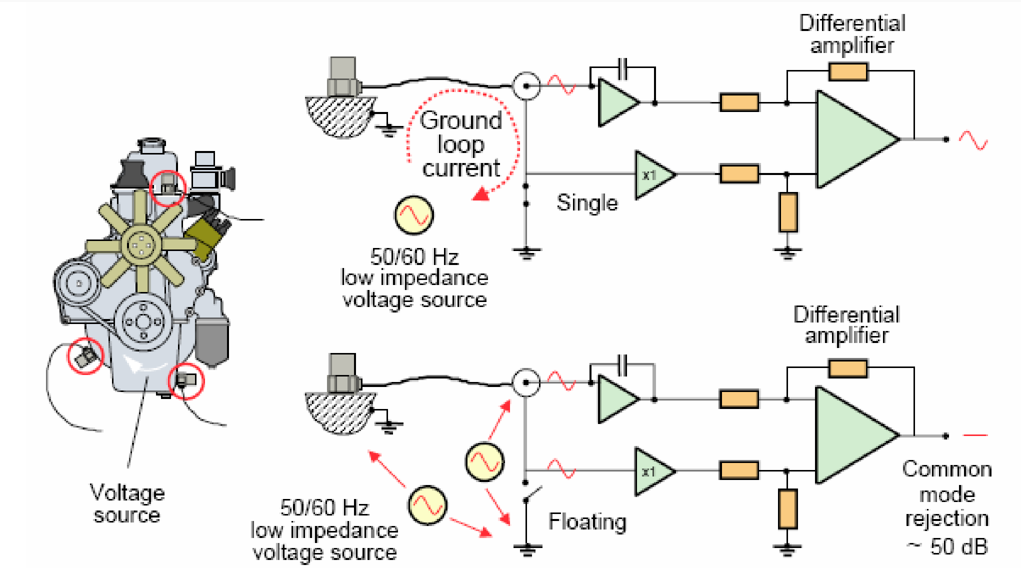
In principle both voltage and charge preamplifiers can be used to make the necessary impedance conversion etc.

However, as indicated on the figure, the sensitivity seen by the amplifier varies dramatically with cable length when voltage amplifiers are used. This means that a new calibration (or calculation) has to be made if the cable used is changed. Furthermore the lower limiting frequency can be affected by cable length and resistance.

Therefore the majority of preamplifiers used today are charge amplifiers as they are not affected by cable length or resistance changes within reasonable limits.

For input stages in built-in preamplifiers this is not quite as clear a choice, but for the best performance charge amplifiers are still to prefer.

ACCELEROMETER PERFORMANCE IN PRACTICE



Ground Loops

If the accelerometer is fixed to a test object which is connected to the instrument ground, e.g. through another accelerometer channel, then a ground loop is formed. This can cause noise to be superimposed upon the vibration signal from the accelerometer if any ground potential differences or electromagnetic fields are present.

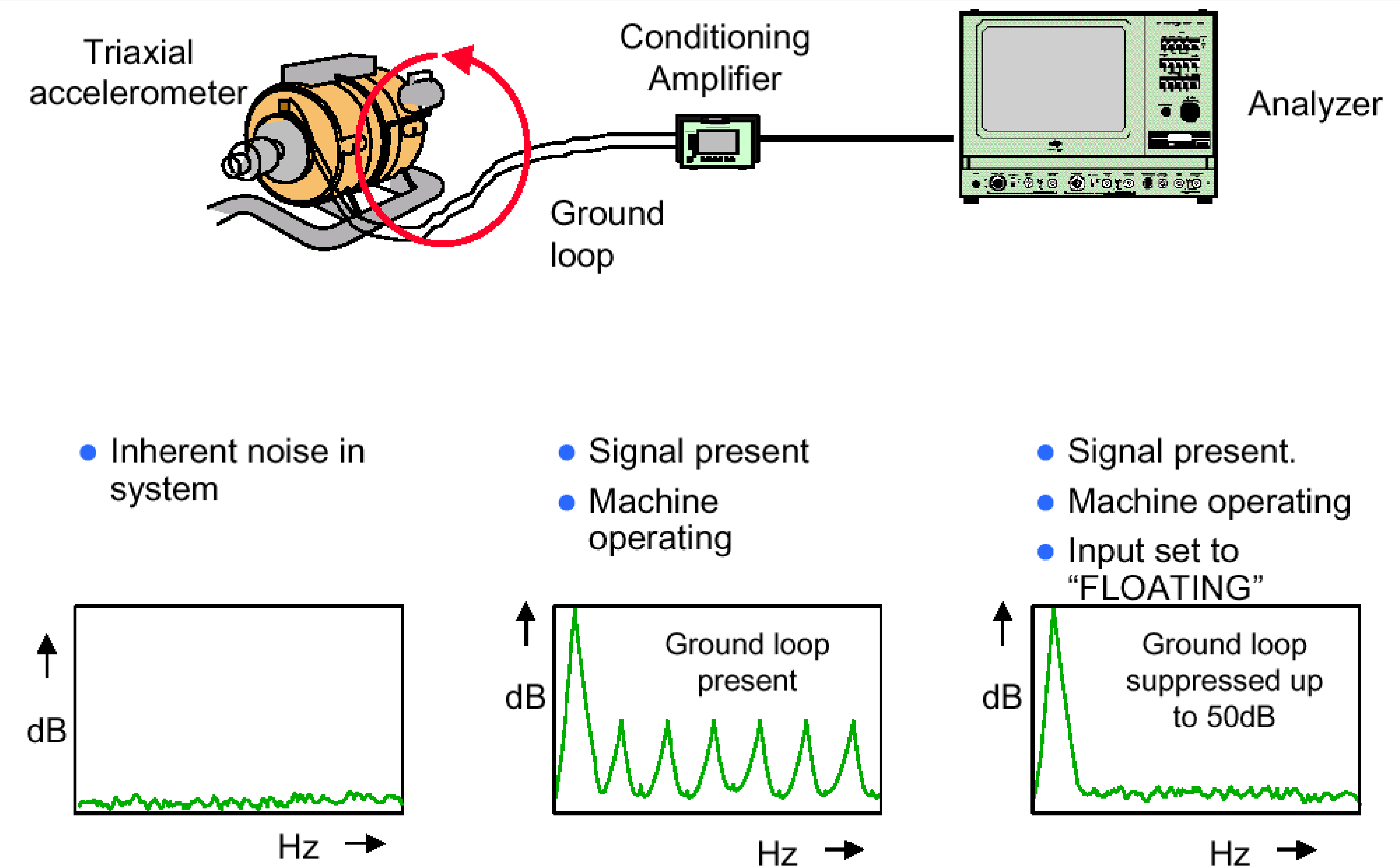
This situation can be avoided by:

Mounting the accelerometer by the aid of an isolating mounting method as discussed earlier. This is normally the most efficient method.

Use of an accelerometer which has its piezoelectric material isolated from the housing e.g. in the form of a differential output (requires differential preamplifier) or a double housing.

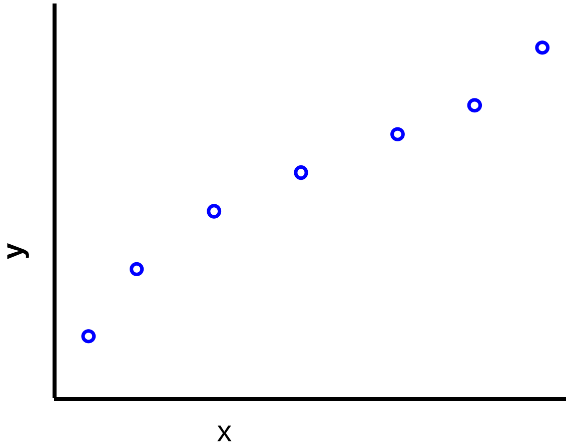
Using a preamplifier having a floating input as illutrated in the slide.

Ground loop solution

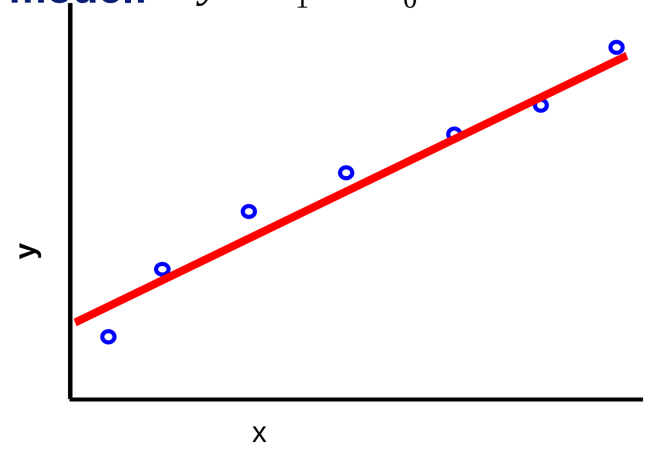


6. Parameter estimation methods: SDOF and MDOF methods.

Curve-Fitting



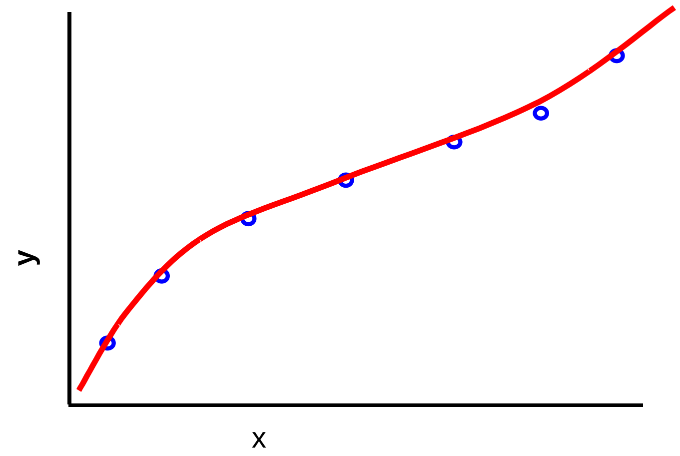
Assumed model: $y = a_1 x + a_0$



measurements

x_1	y_1
x_2	y_2
x_3	y_3
…	…
x_N	y_N

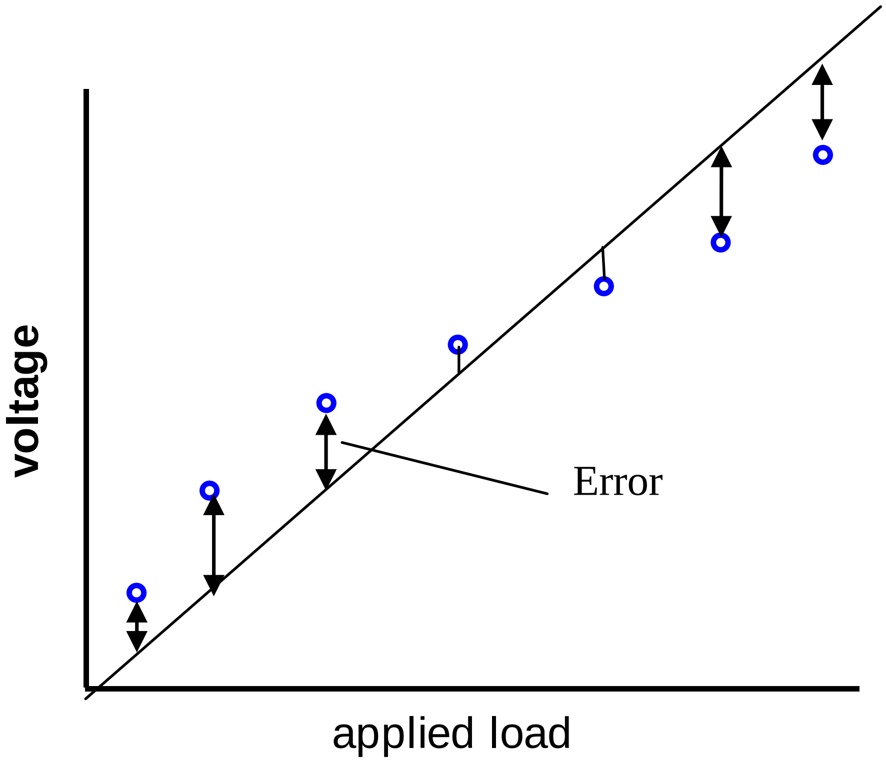
Assumed model: $y = a_3 x^3 + a_2 x^2 + a_1 x + a_0$



Curve-fitting = finding *model parameters* based on experimental data

Least Squared Error Concepts

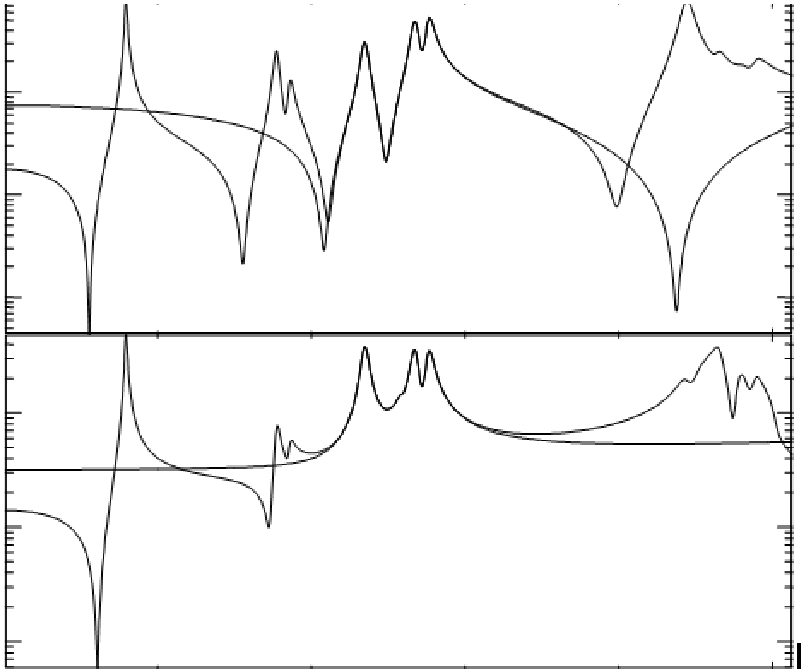
A least squared error curve fitting method is formulated in such a way that the sum of the error squared between the actual (measured) data and the fit function is a minimum.



FRF model

From Modal Theory a mathematical model for the Frequency Response Function was developed:

$$H_{ij}(j\omega) = \sum_{k=1}^{n} \left(\frac{A_{ij,k}}{(j\omega - \lambda_k)} + \frac{A^*_{ij,k}}{(j\omega - \lambda^*_k)} \right) + UR + LR$$



Data supplied by the analyst:

$H_{ij}(j\omega)$ measured FRF's

n the number of modes (order of the model)

Estimated Parameters:

λ_k the kth pole

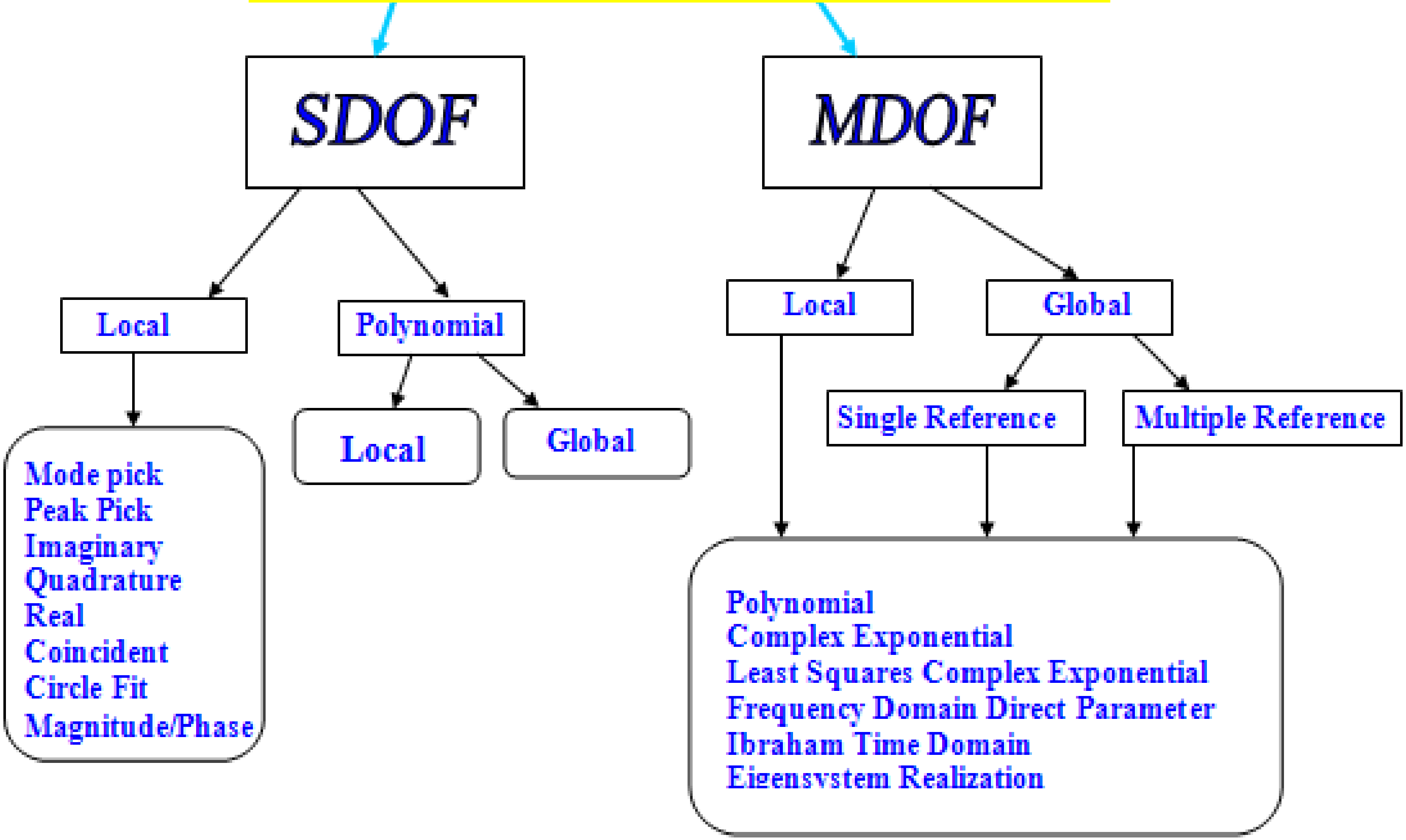
$A_{ij,k}$ the residue for kth mode, response j, input i

LR the lower residual term

UR the upper residual term

Classification of Modal Parameter Estimation Techniques

$$H_{ij}(j\omega) = \sum_{k=1}^{n} \left(\frac{A_{ij,k}}{(j\omega - \lambda_k)} + \frac{A^*_{ij,k}}{(j\omega - \lambda^*_k)} \right) + UR + LR$$



SDOF vs. MDOF

SDOF curvefitting model

$$H_{ij}(j\omega) = \frac{A_{ij}}{\left(j\omega - \lambda\right)} + \frac{A^*_{ij}}{\left(j\omega - \lambda^*\right)} + UR + LR$$

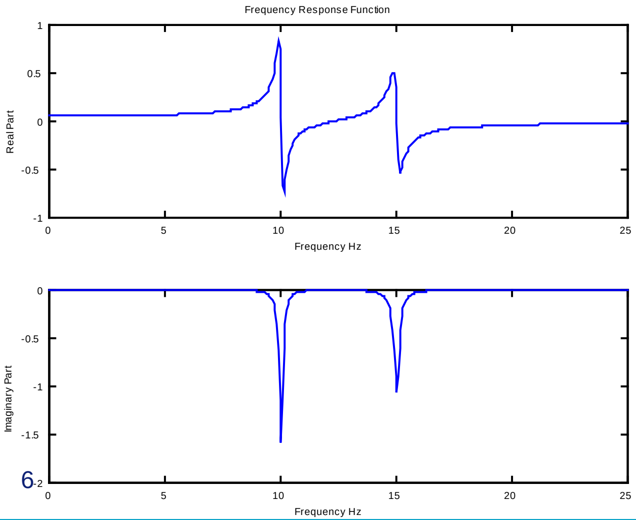
MDOF curvefitting model

$$H_{ij}(j\omega) = \sum_{k=1}^{n} \left(\frac{A_{ij,k}}{\left(j\omega - \lambda_k\right)} + \frac{A^*_{ij,k}}{\left(j\omega - \lambda^*_k\right)} \right) + UR + LR$$

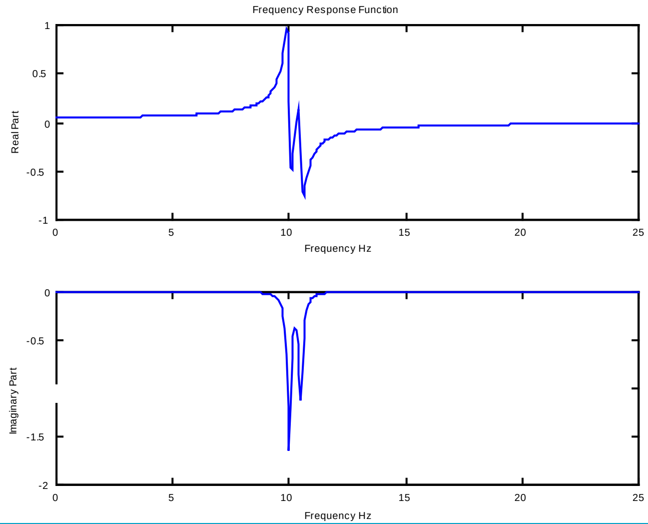
SDOF methods should be used in frequency ranges where the FRF can be approximated as a single degree of freedom system. Otherwise MDOF assumptions are used and the number of modes "n" must be selected.

These characteristics are governed by modal density and modal damping.

Well separated, Lightly damped



Closely Spaced, Lightly Damped

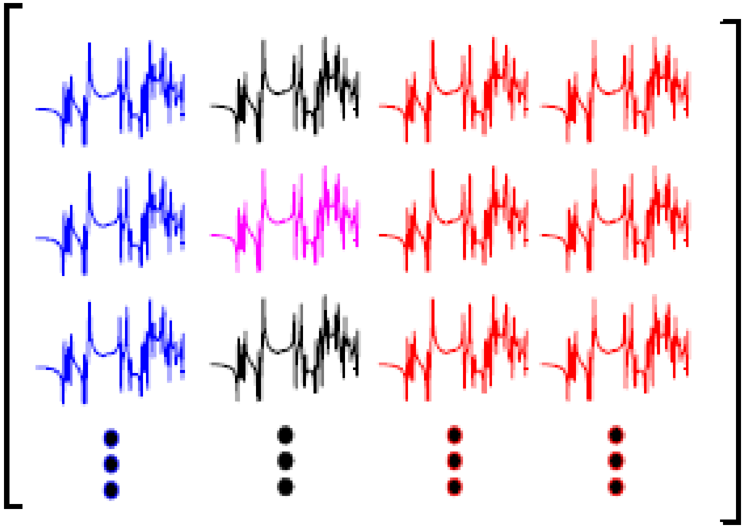


Parameter extraction considerations

The FRF matrix contains redundant information regarding the system frequency, damping and mode shapes

Multiple referenced data can be used to obtain better estimates of modal parameters

MULTIPLE REFERENCE FRF MATRIX DEVELOPMENT



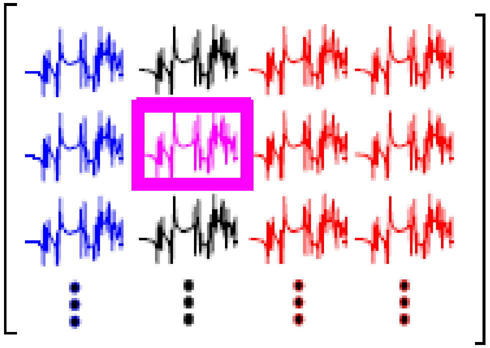
LOCAL CURVEFITTING

GLOBAL CURVEFITTING

POLYREFERENCE CUVREFITTING

Local curvefitting

Local Curvefitting



- Each measurement is curvefit to estimate the frequency, damping and residue for each FRF
- The frequency and damping is allowed to vary for each measurement and may not be the same for every measurement

ADVANTAGES

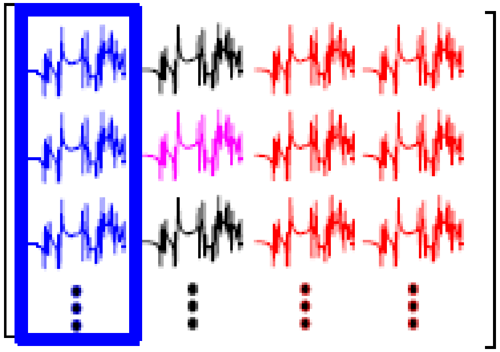
- Good for systems where the poles are not global

DISADVANTAGES

- Frequency and damping is different for the system
- Local modes/node points are not characterized well

Global curvefitting

Global Curvefitting



- A set of measurements are curvefit to estimate the frequency and damping
- The residue is estimated in a second pass

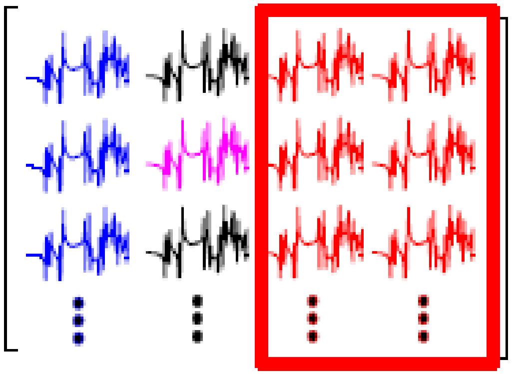
ADVANTAGES

- Good for systems where the poles are global
- Better estimate of the frequency and damping
- Local modes are better characterized

DISADVANTAGES

- Frequency and damping must be global in FRFs

Polyreference curvefitting



Polyreference Curvefitting

- A set of measurements are curvefit to estimate the frequency and damping
- The residue is estimated in a second pass and is based on redundant FRF matrix information

ADVANTAGES

- Good for systems where the poles are global
- Better estimate of the frequency and damping
- Repeated roots can be identified

DISADVANTAGES

- Frequency and damping must be global in FRFs

Time or frequency domain

- FRF matrix

$$[H(\omega)] = \sum_{k=1}^{n} \frac{[A_k]}{j\omega - \lambda_k} + \frac{[A_k^*]}{j\omega - \lambda_k^*}$$

  - 1 element

Inverse Fourier transform

- Impulse response function matrix

$$[h(t)] = \sum_{k=1}^{n} [A_k] e^{\lambda_k t} + [A_k^*] e^{\lambda_k^* t}$$

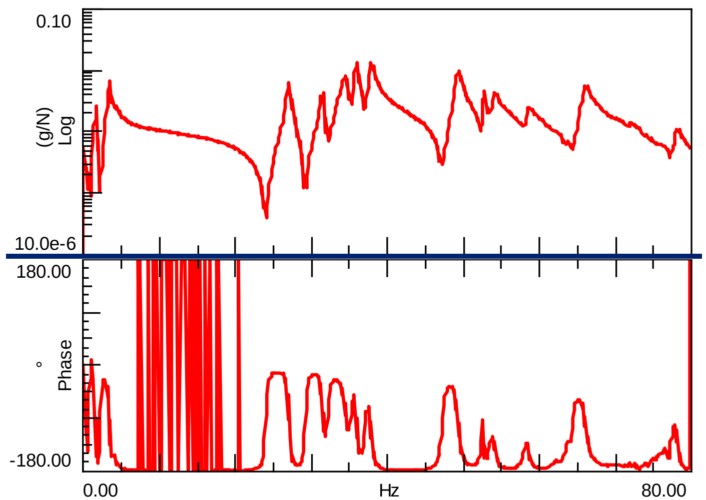
  - 1 element

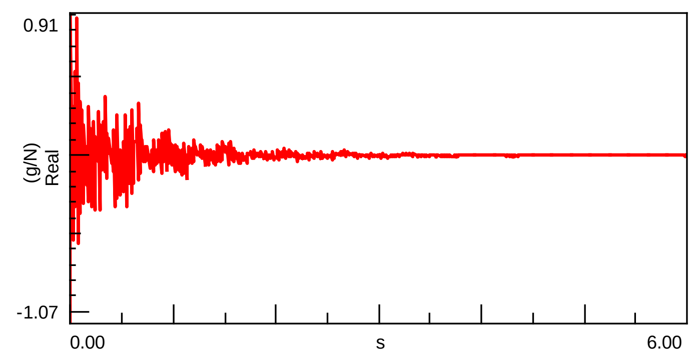
$$H_{ij}(\omega) = \sum_{k=1}^{n} \frac{A_{ij,k}}{j\omega - \lambda_k} + \frac{A_{ij,k}^*}{j\omega - \lambda_k^*}$$

$$h_{ij}(t) = \sum_{k=1}^{n} A_{ij,k}\, e^{\lambda_k t} + A_{ij,k}^*\, e^{\lambda_k^* t}$$

$$|H_{ij}(j\omega)|$$

$$\angle H_{ij}(j\omega)$$





Frequency domain

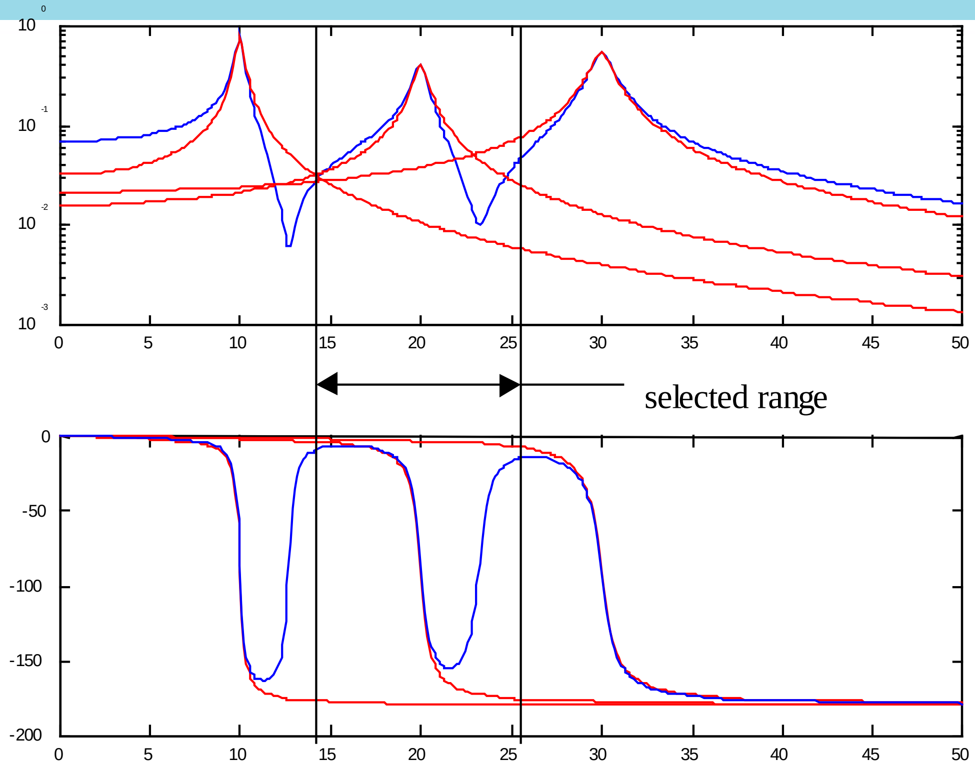
Time domain

Time or frequency domain

The modal information is the same in either domain (pole, residue).
The equations can be cast into a form that offers some numerical advantage that can result in increased efficiency and/or accuracy depending on the characteristics of the data.

Frequency Domain Methods	**Time Domain Methods**
Modal Peak / Peak Pick	Complex Exponential
Coincident / Quadrature	**Least Squares Complex Exponential**
Magnitude / Phase	Ibraham Time Domain
Circle Fit	Eigensystem Realization
Polynomial	
Frequency Domain Direct Parameter	

Residual Terms



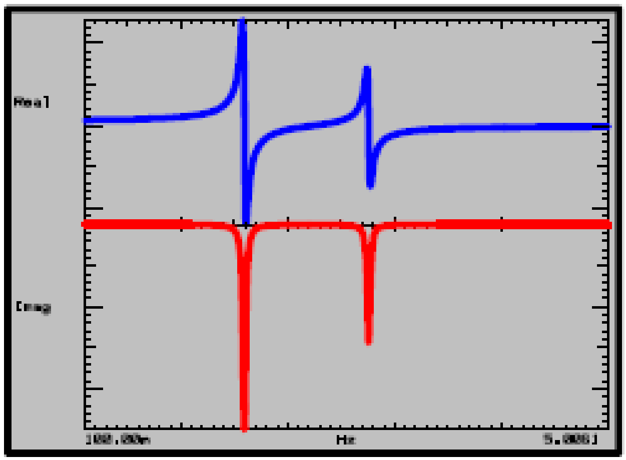
Curvefitting methods are applied to a selected frequency range, but a mode of vibration participates over the entire frequency range $(0 \leq f \leq \infty)$

.

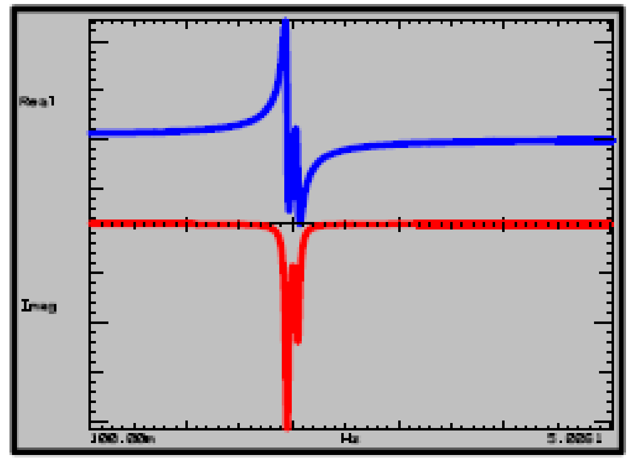
Residual terms account for the effect of the participation of "out of band" modes in the selected frequency range. The lower residual term accounts for mass effects and the upper residual accounts for flexibility effects.

Classification of modes

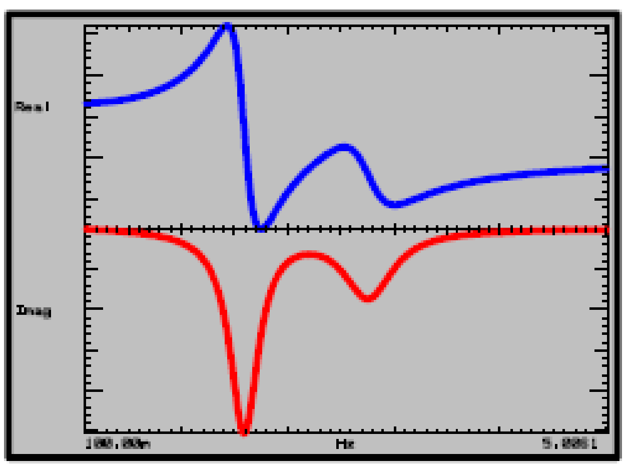
Well separated - lightly damped



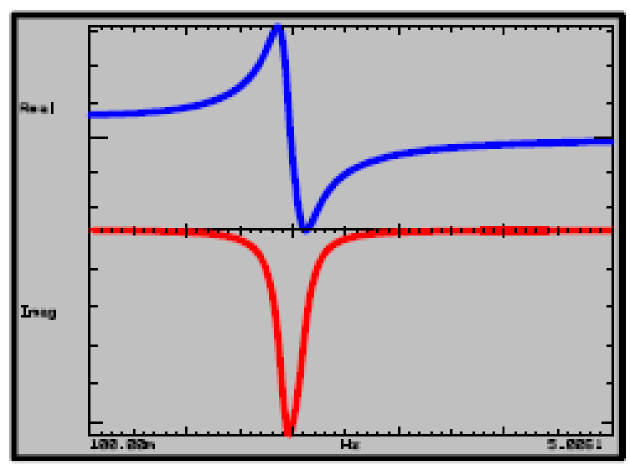
Closely spaced - lightly damped



Well separated - heavily damped



Closely spaced - heavily damped

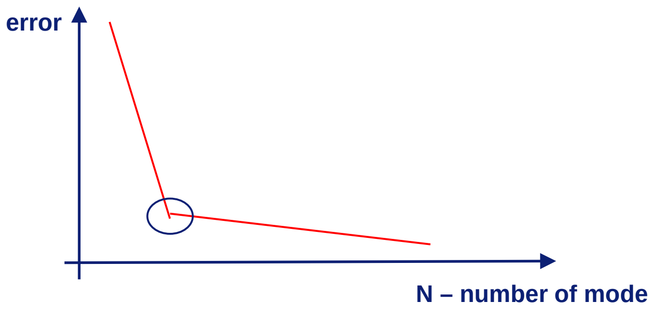


Very difficult to estimate!

Least Square Complex Exponenetial (LSCE)

The LSCE method tries to reduce the error between the measured FRFs and the synthetized FRFs in the band of interest. The methodology is as follows:

- estimation of the number of DOFs
- Estimation of the Impulse Response Function (leakage problem can occur)
- An iterative procedure is established. The number of mode is increased and the frequency and damping is estimated in a least square sense. The stabilization diagram is built. When the number of considered modes in the summation is equal to the number of physical mode in the band of interest, the error between the measured FRFs and the synthetized FRFs strongly decreases. Thus, damping and frequency are stable. Therefore, the correct number of mode is obtained in order to reconstruct the measured FRFs.



$$\left[h(t)\right]=\sum_{k=1}^{n}\left[A_k\right]e^{\lambda_k t}+\left[A_k^*\right]e^{\lambda_k^* t}$$

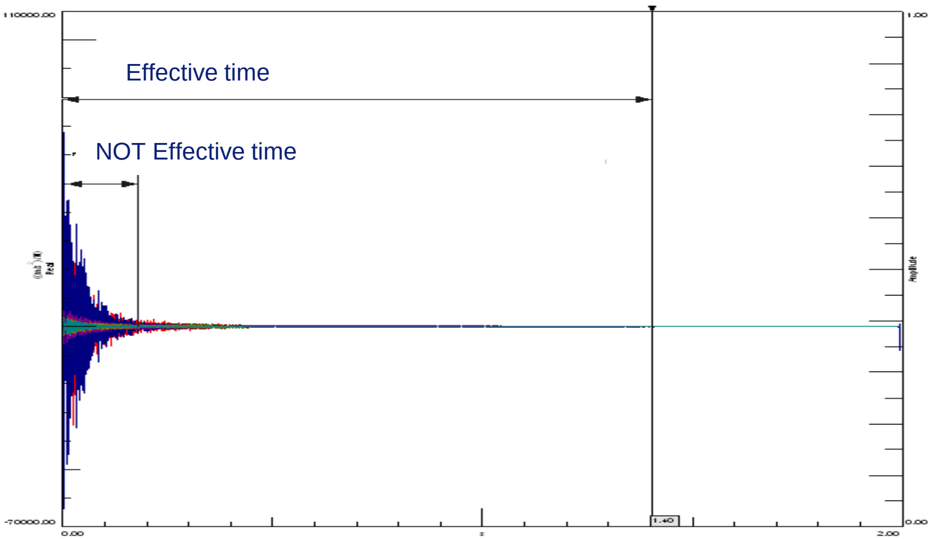
Least Square Complex Exponential (LSCE)

- Time windows definition to be done for LSCE method

where
•N_S = number of samples to select in the T*ime Window*;
•*t* = effective time;
•*Δf* = frequency band of interval.

$$N_s = 2t\Delta f$$

Impulse Response Function



Polymax: Frequency-Domain Curve-Fitting

- *Non-linear optimisation problem*

$$[H(\omega)] = \sum_{i=1}^{n} \frac{\{v_i\} < l_i^T >}{j\omega - \lambda_i} + \frac{\{v_i^*\} < l_i^H >}{j\omega - \lambda_i^*}$$

measured **unknowns**

- *Can be linearised*

$$[H(\omega)] = [B(\omega)][A(\omega)]^{-1} = \text{"} \frac{[\beta_p](j\omega)^p + [\beta_{p-1}](j\omega)^{p-1} + \ldots + [\beta_0](j\omega)^0}{[\alpha_p](j\omega)^p + [\alpha_{p-1}](j\omega)^{p-1} + \ldots + [\alpha_0](j\omega)^0} \text{"}$$

measured **unknowns**

$[\alpha_r]$ $\rightarrow$ Poles & participation factors

$[\beta_r]$ $\rightarrow$ Mode shapes

PolyMAX - Linear Least Squares

$$[H(\omega)] = [B(\omega)][A(\omega)]^{-1} = \text{"}\frac{[\beta_p]z^p + [\beta_{p-1}]z^{p-1} + \ldots + [\beta_0]z^0}{[\alpha_p]z^p + [\alpha_{p-1}]z^{p-1} + \ldots + [\alpha_0]z^0}\text{"}$$

- Linearisation

$$\text{error} = [B(\omega)] - [H(\omega)][A(\omega)]$$

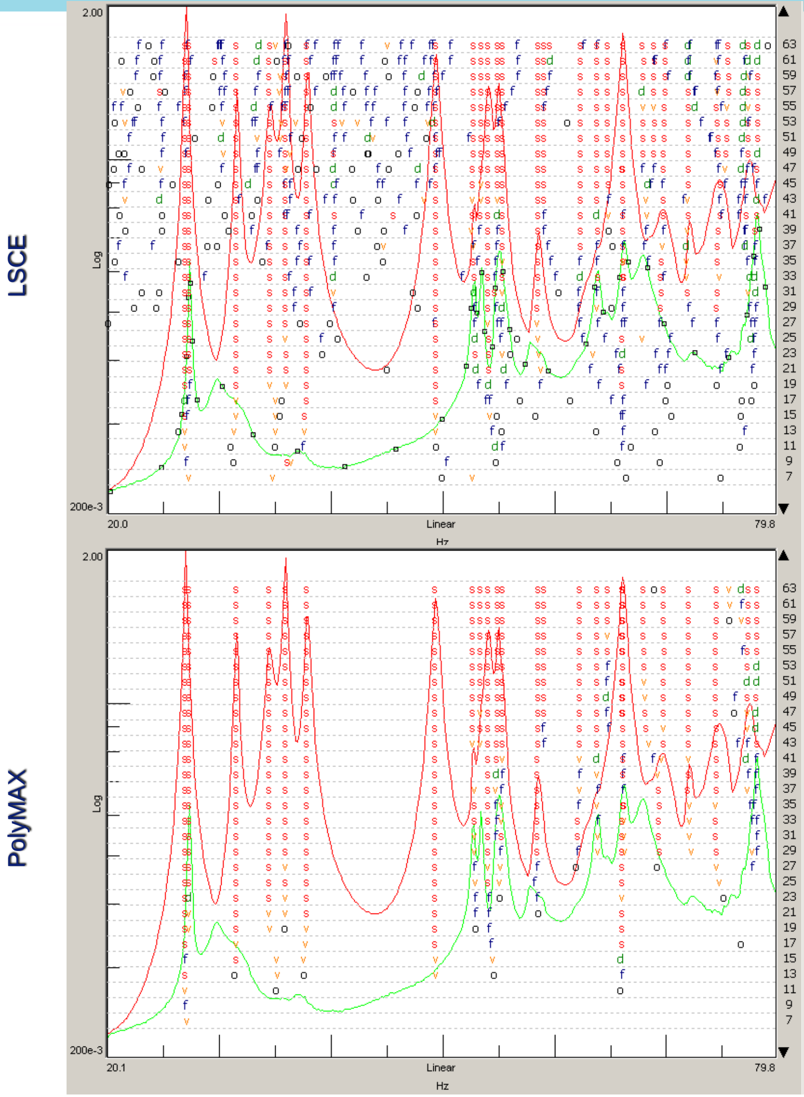
  - By minimising the error (in a *linear least squares* sense), the model $[A(\omega)], [B(\omega)]$ can be found from the data $[H(\omega)]$

- Algorithm optimisation
  - Problem size reduction (Final dimension not related to the number of DOFs, but on the order of the linear problem: very large-size problems can be tackled)
  - Memory and speed optimisation

LMS PolyMAX vs. LSCE

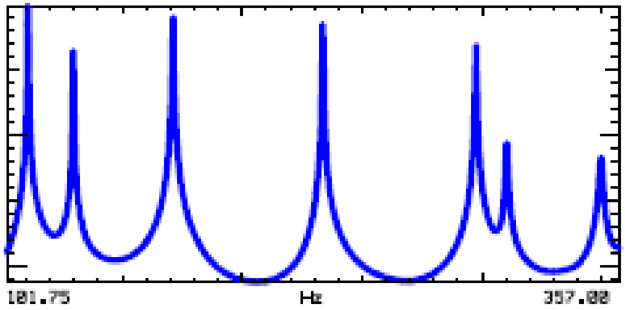
- Time domain LSCE vs. PolyMAX
  - Both use a 2-step procedure
  - Step 1 differs
    - LSCE uses impulse responses
    - PolyMAX uses FRF
  - Step 2 is a linear frequency domain fit for the modeshapes
  - Large difference in stabilisation diagram

Usually time-domain methods are preferred in case of low damping, and frequency-domain methods in case of high damping.

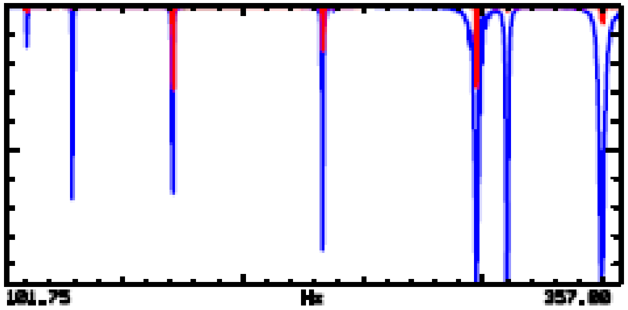


Mode determination tools

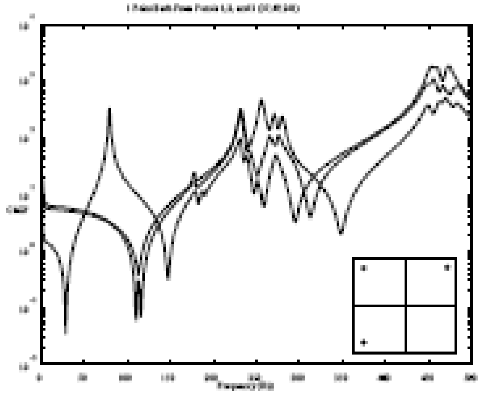
Summation



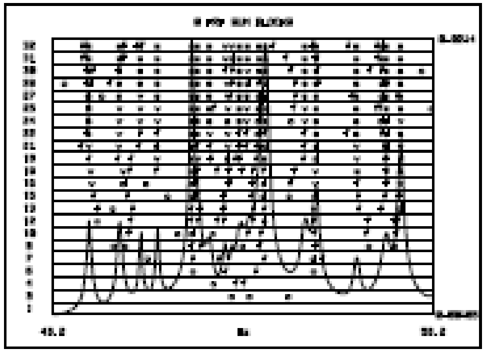
MIF



A variety of tools assist in the determination and selection of modes in the structure

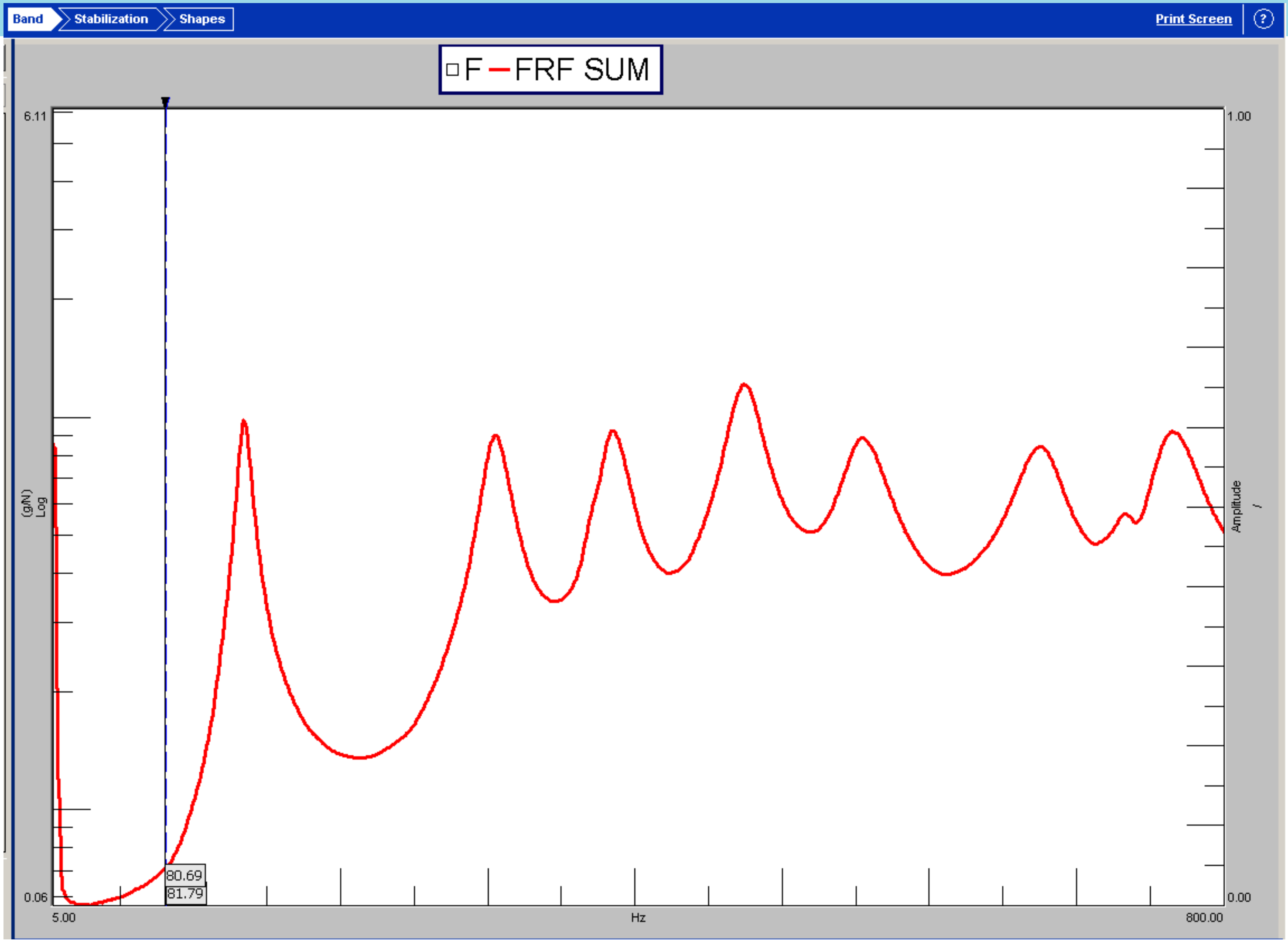


CMIF



Stability Diagram

Summation of FRFs

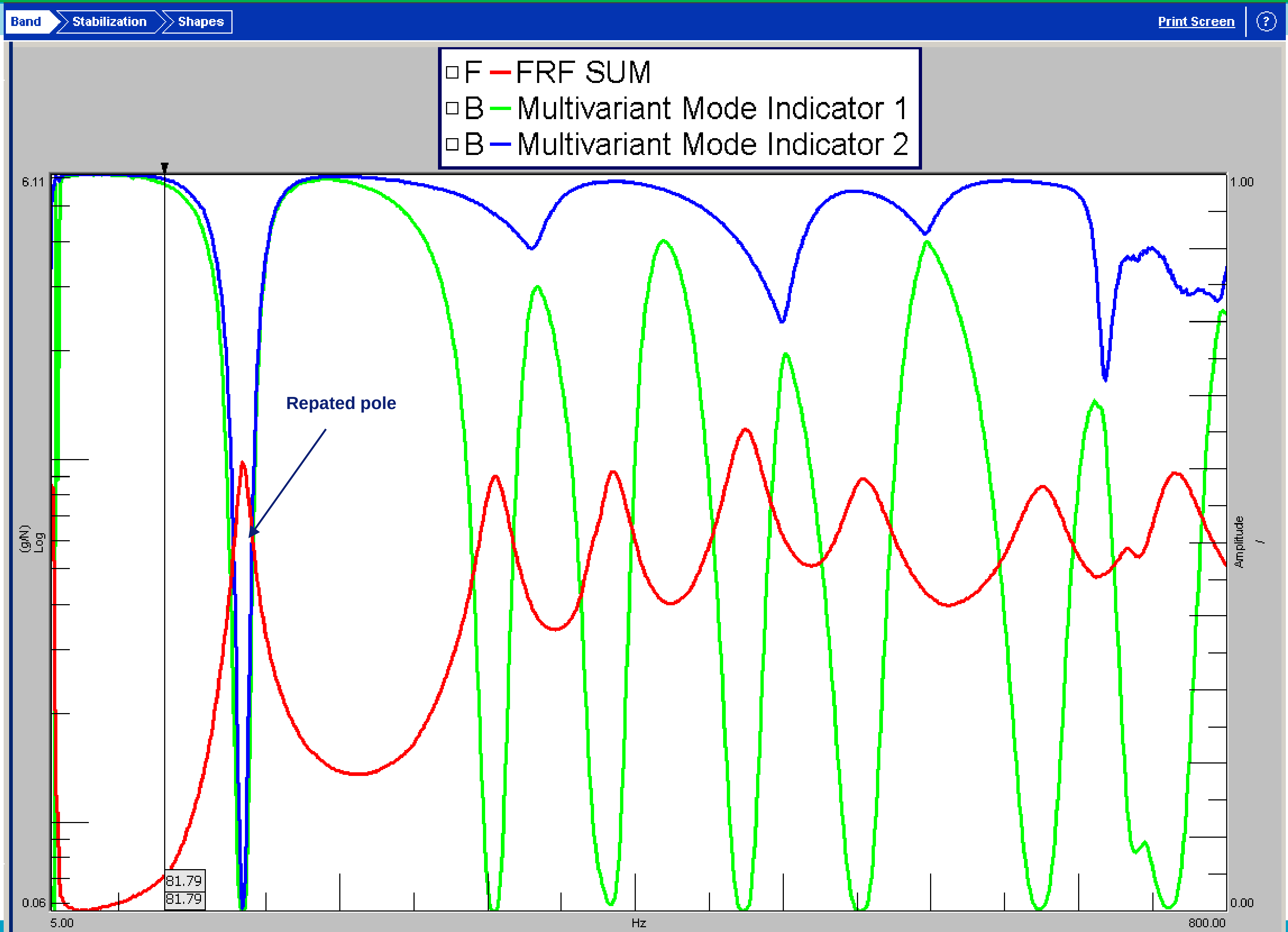


- Complex sum of all the FRFs

MIF – Mode indicator function

- MIF are frequency domain curves that have maxima or minima in correspondence of natural frequencies (they are obtained by SVD methods)
- The number of MIF available depends on the number of input (number of columns available).
- The 1st MIF has a minima or maxima for each natural frequency. The 2^{nd} MIF has a minima or maxima in correspondence of repeated roots (i.e. if more than one mode has same frequency but different mode shape, or modes with very similar natural frequencies)
- In Test.Lab a number of MIF are available:
  - Multivariant MIF
  - Complex MIF
  - Real MIF
  - Imaginary MIF
  - Coincident MIF
  - Modified Real MIF

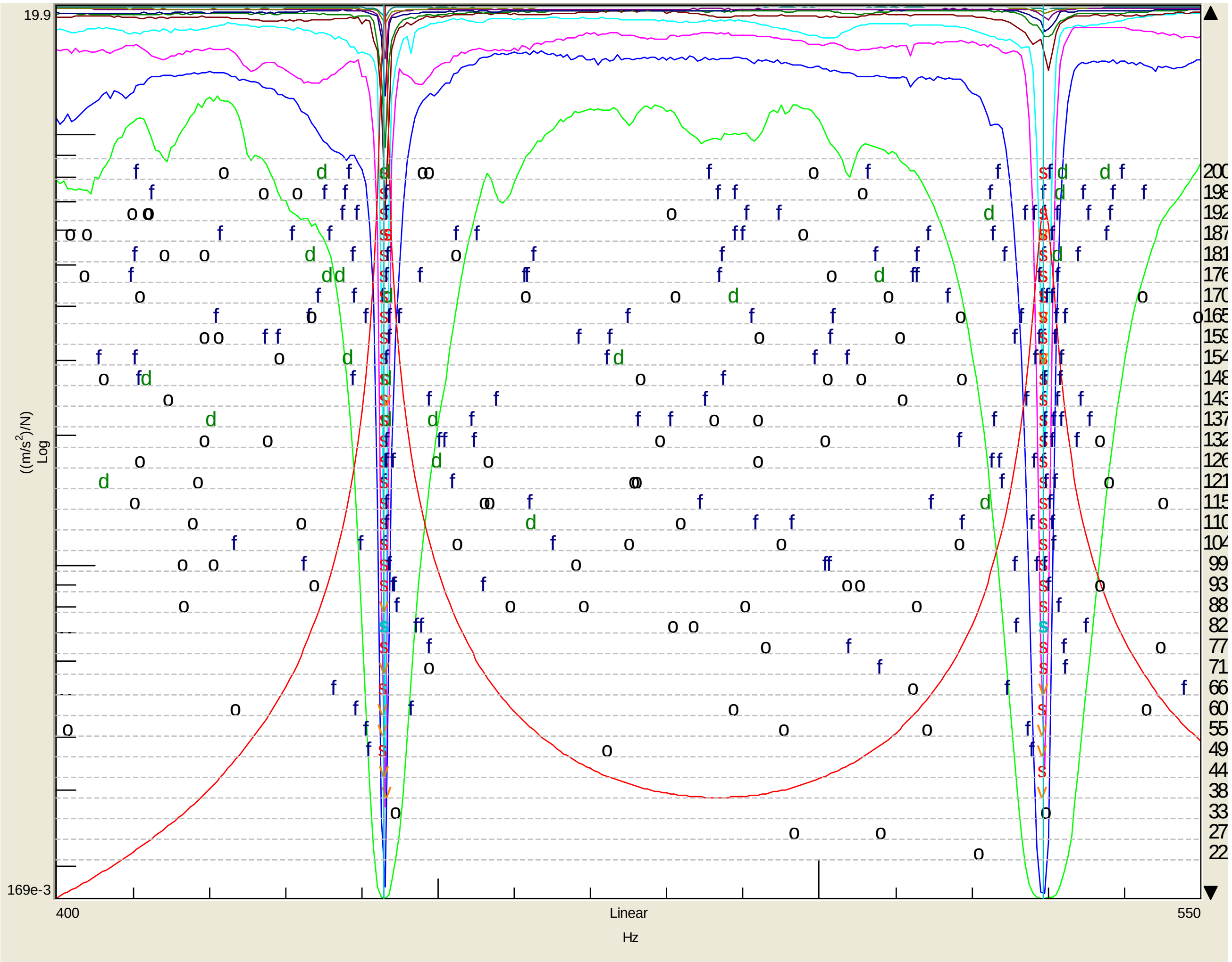
MultivariantMode Indicator Functions (MvMIF)



Multivariant MIF, *Complex MIF*, *Modified Real MIF*, *Coincident MIF* *Imaginary MIF*, *Real MIF*

SVD-based functions

Example of MIFs



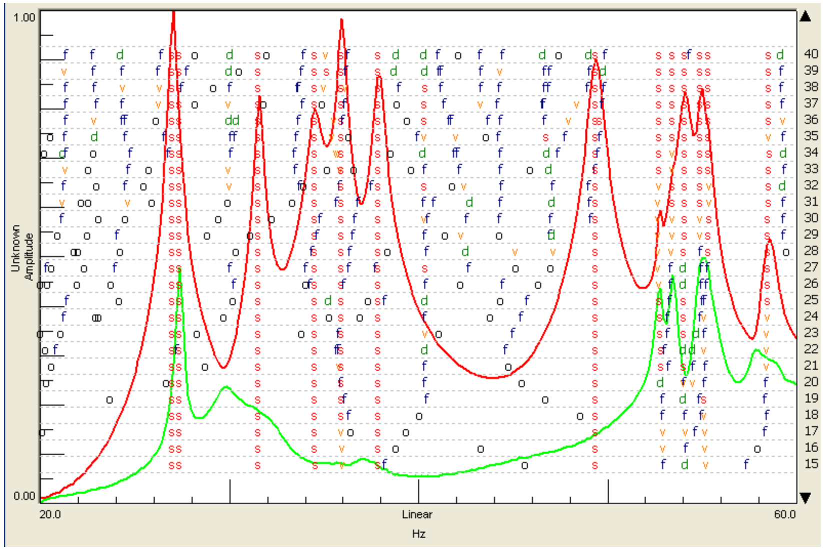
Parametric methods ("curve-fitting") – Stabilization diagram

- Major problem in modal parameter estimation

$$H(\omega)=\sum_{k=1}^{n}\frac{Q_k\psi_k\psi_k^{\,t}}{j\omega-\lambda_k}+\frac{Q^*_k\psi^*_k\psi_k^{\,H}}{j\omega-\lambda_k^*}$$

  - What is the model order?
  - How many modes to curve-fit?

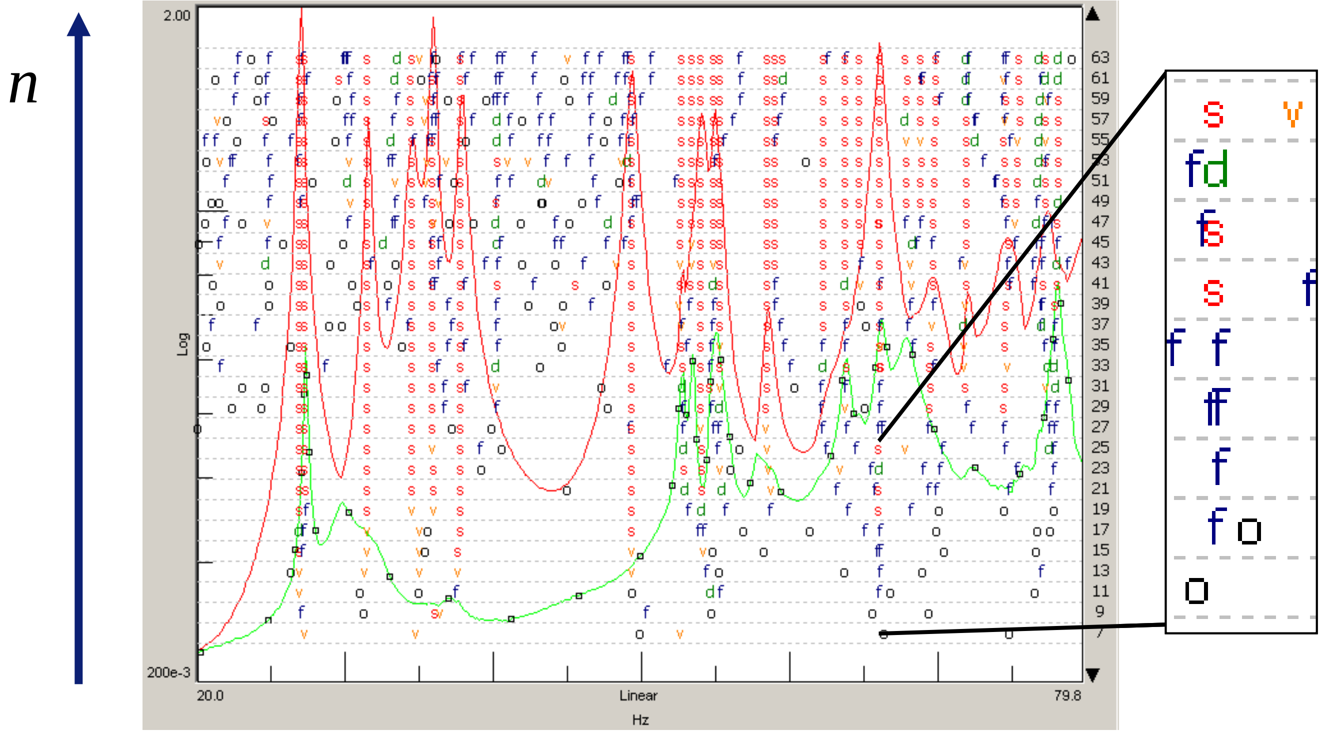
- Solutions
  - Mode indicator functions
  - Stabilization diagram



Stabilization diagram

$$H(\omega)=\sum_{k=1}^{n}\frac{Q_k\psi_k\psi_k^{\,t}}{j\omega-\lambda_k}+\frac{Q^*_k\psi^*_k\psi_k^{\,H}}{j\omega-\lambda^*_k}$$

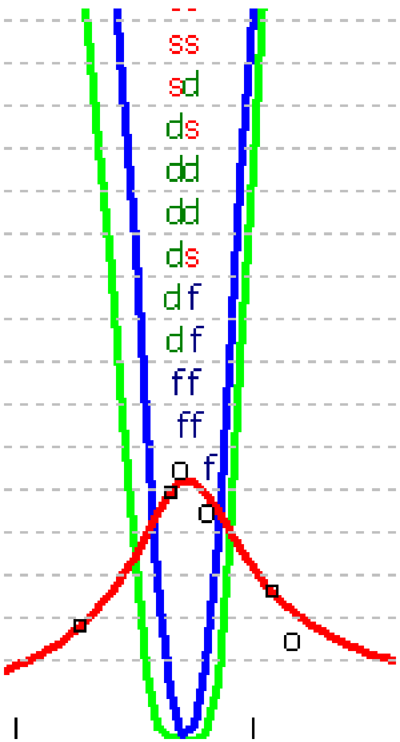
- Try a whole range of model orders
- Compare modal parameters at current order with previous order



Stability

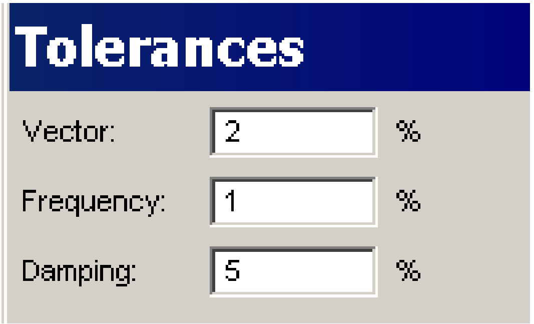
- o : new
- f : freq
- d : damp + freq
- v : vector
- s : all

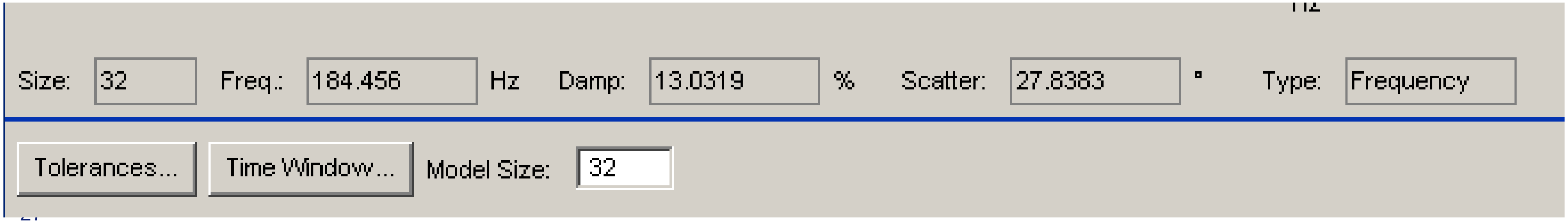
Stabilization Diagram –Pole Status



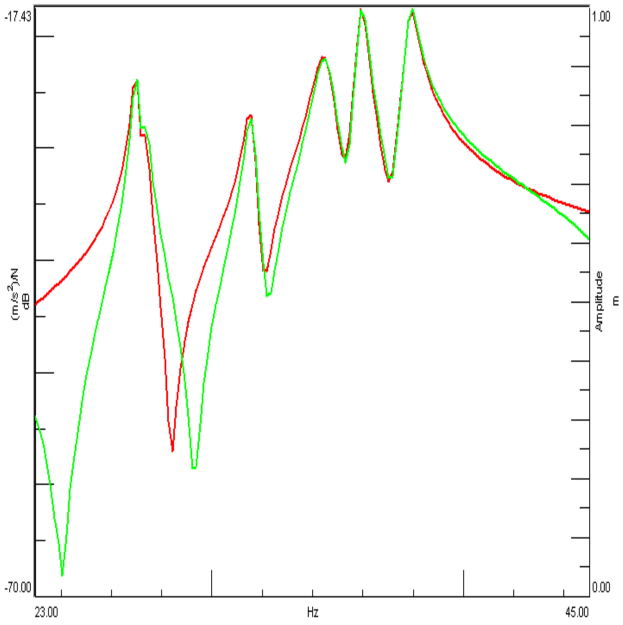
Pole status values

symbol	description
o	The pole is not stable.
f	The frequency of the pole does not change within the tolerances.
d	The damping and frequency of the pole does not change within the tolerances.
v	The pole vector does not change within the tolerances.
s	Both frequency, damping and vector are stable within the tolerances

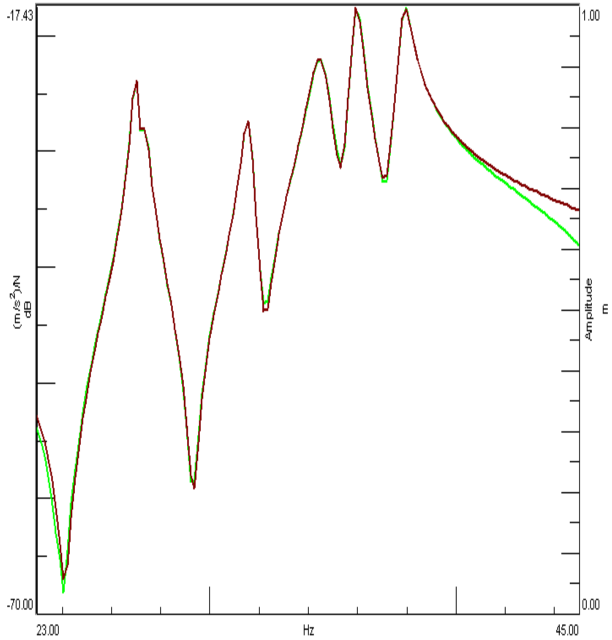




Synthesizing FRF's
Residuals (UR,LR)



without residual

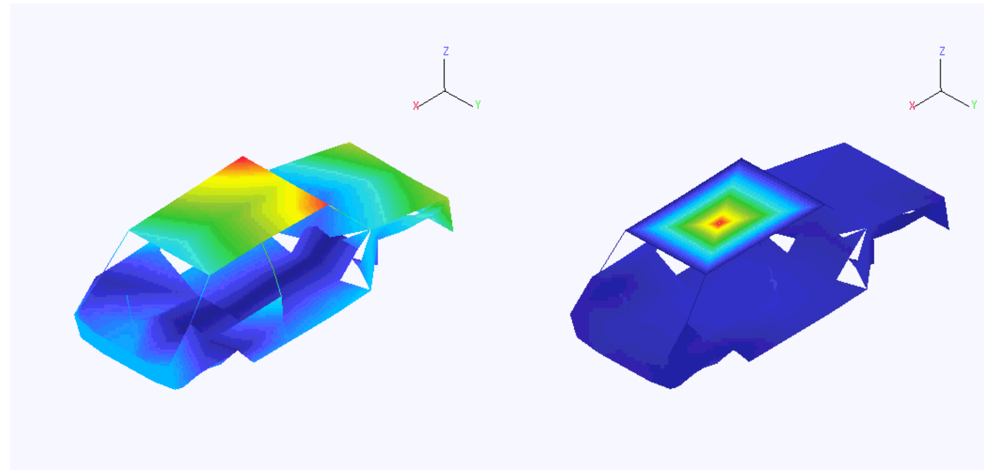


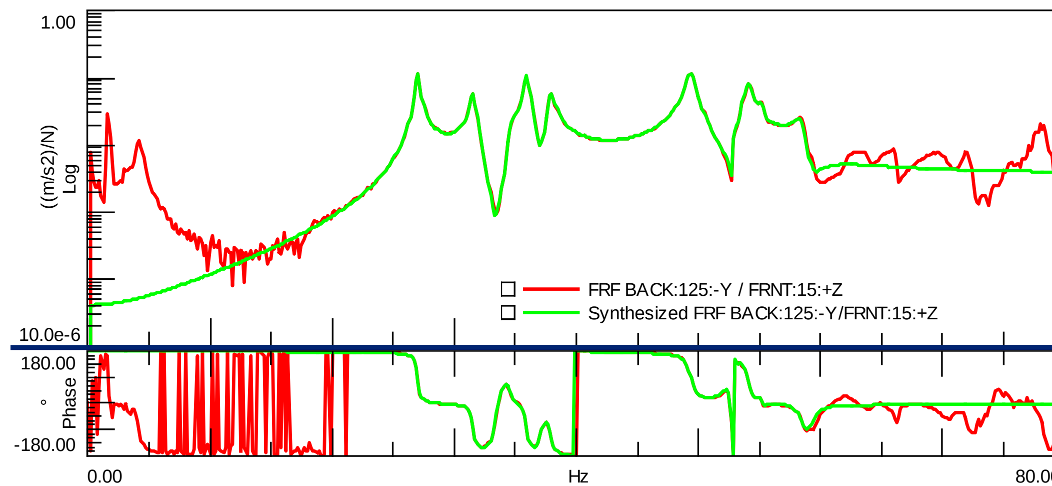
with residual

7. Validation methods: FRF synthesis, MAC, MOC, MPC.

Validation

- Plot and animate mode shapes
- Synthesis of FRFs
- MAC matrix
- Mode participation
- Mode colinearity
- …

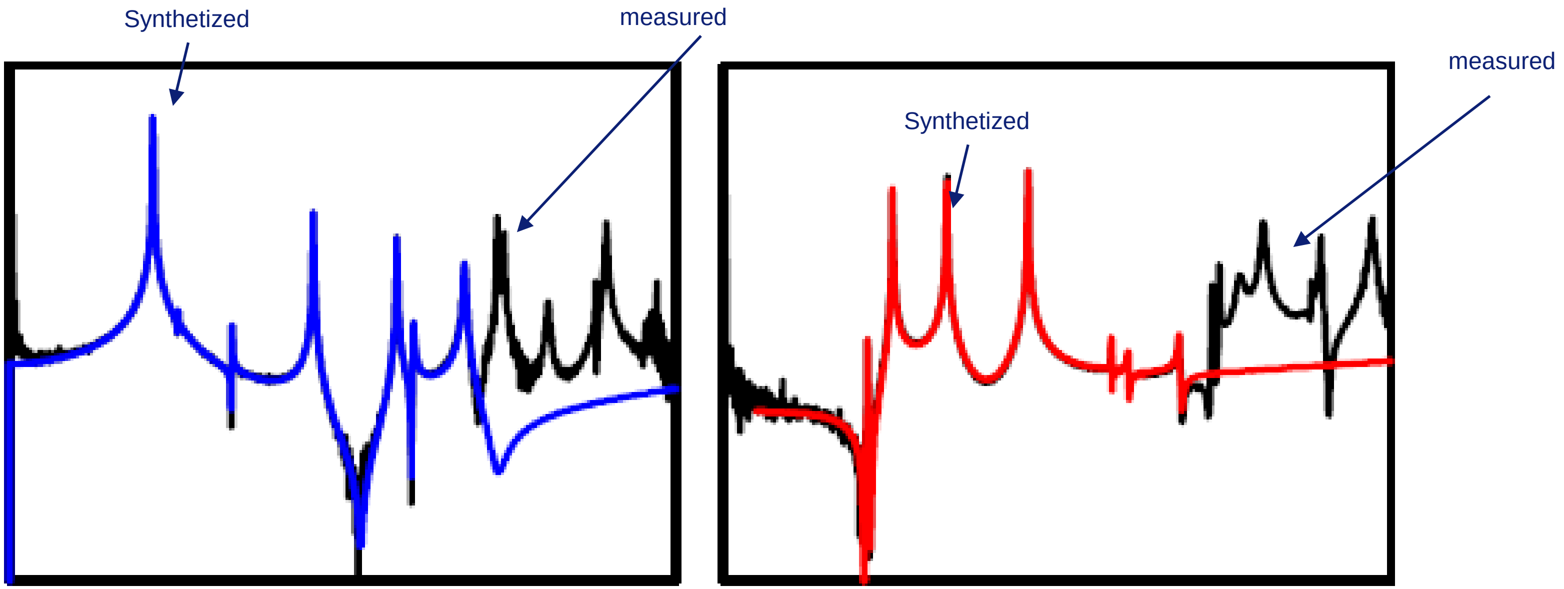




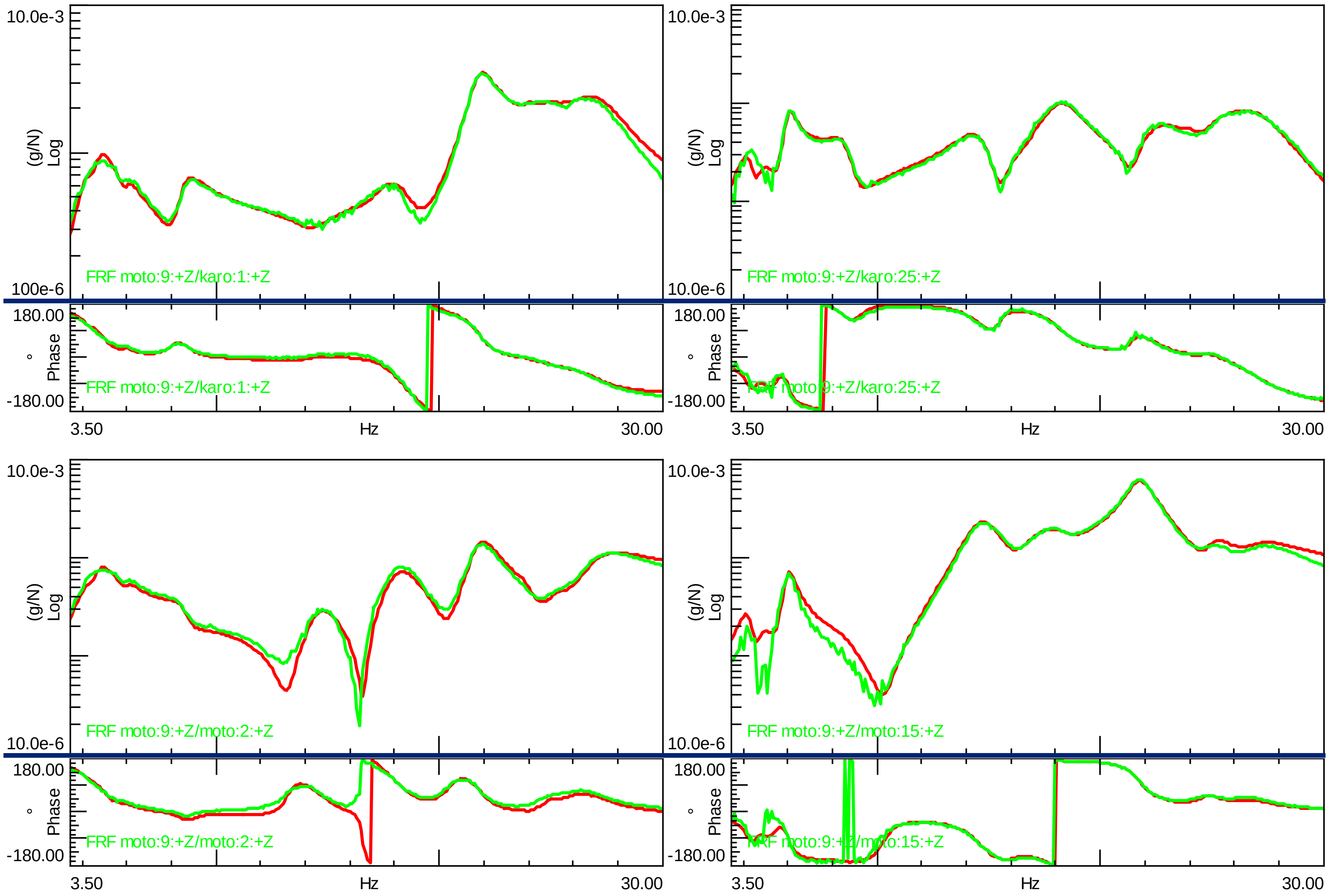
FRF Synthesis

The modal model can be used to synthesize any FRF measurement which can then be compared to the measured data

$$H_{ij,synthesis}(j\omega) = \sum_{k=1}^{n} \left(\frac{A_{ij,k}}{(j\omega - \lambda_k)} + \frac{A^*_{ij,k}}{(j\omega - \lambda^*_k)} \right) + UR + LR$$



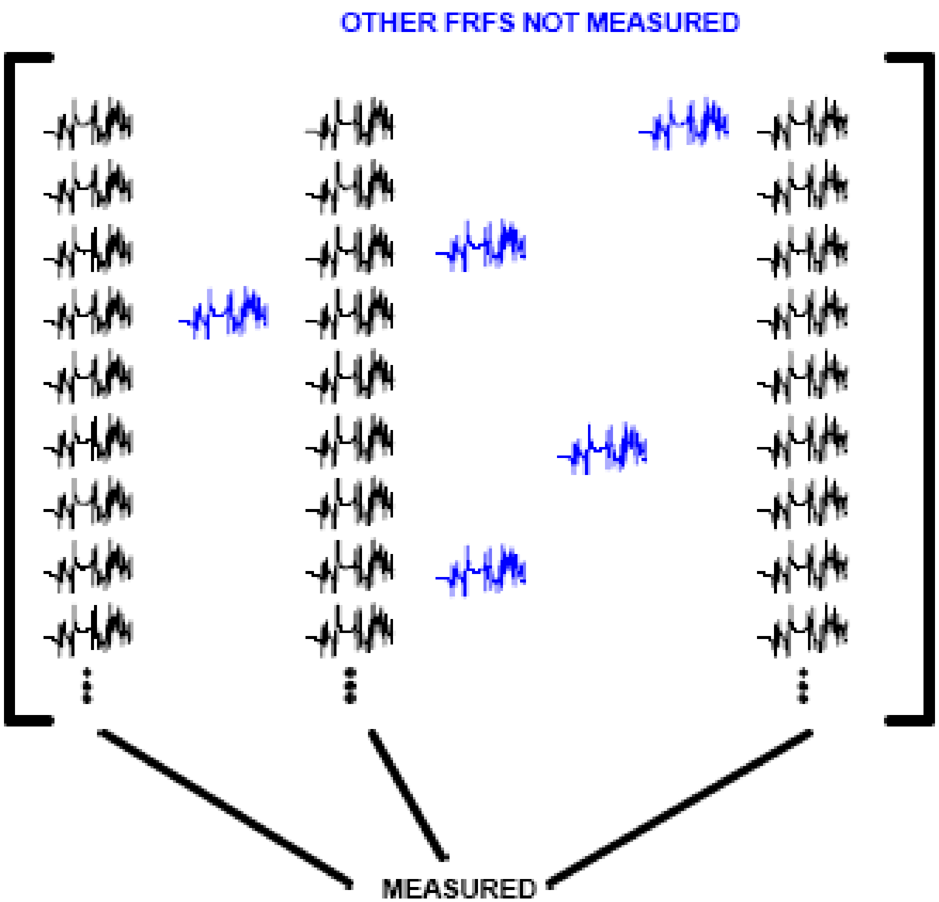
FRF Synthesis



FRF Synthesis

Any FRF can be synthesized - not just measured FRFs

Additional FRFs should be collected during the test phase for additional spot checks of the adequacy of the extracted dynamic modal model.



Modal Assurance Criteria - MAC

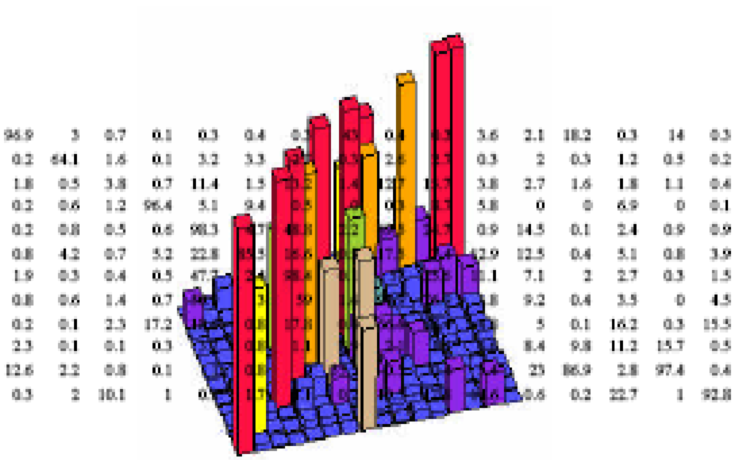
MAC provides an indication of the similarity (or lack thereof) between sets of different modal vectors.

$$MAC_{ij} = \frac{\left(\{V_i\}^T\{V_j\}\right)^2}{\left(\{V_i\}^T\{V_i\}\right)\left(\{V_j\}^T\{V_j\}\right)}$$

The MAC approaches 1 if the vectors are very similar

The MAC approaches 0 of the mode shapes are very dissimilar

MAC is very sensitive to spatial aliasing of the mode shapes

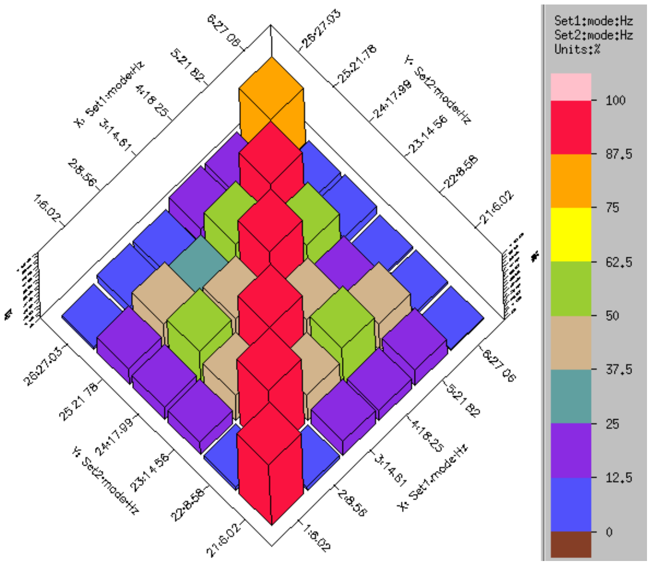


Since the eigenvectors belonging to the same system are orthogonal, MAC between modes should be 0 (AUTO-MAC)

Modal Parameters LSCE vs. PolyMAX

LSCE		PolyMAX	
f [Hz]	ξ [%]	*f* [Hz]	ξ [%]
		3.96	5.4
		4.24	9.6
		4.81	11.6
6.02	4.1	6.02	4.2
8.58	6.4	8.57	6.5
14.56	6.1	14.59	5.8
		15.74	6.3
		17.05	5.8
17.99	5.5	18.25	4.9
		20.91	2.7
21.78	2.9	21.81	2.7
		22.58	3.4
		23.98	0.9
		25.12	2.4
		25.23	3.1
		26.05	2.1
27.03	5.6	27.06	5.2

MAC



Complexity values

Modal participation factor – MP
Mode overcomplexity value -MOV
Mode Phase Collinearity – MPC
Mean Phase Deviation – MPD

They give information about the quality of the selected modes

Modal participation factor – MP

The **relative importance of different modes in a certain frequency** band can be investigated using the concept of modal participation. It define th eimportance of a mode with respect to the other.

Mode overcomplexity value -MOV

When a **mass is added** to a mechanical structure at a certain measurement point then the damped natural frequencies for all modes will shift downwards. This theoretical characteristic forms the basis of a criterion for the evaluation of estimated mode shape vectors. For each response station, the sensitivity of each natural frequency to a mass increase at that station can be calculated and should be negative. A quantity called the “Mode Overcomplexity Value” (MOV) is defined as the (weighted) percentage of the response stations for which a mass addition indeed decreases the natural frequency for a specific mode.

This MOV index should be high (near 100 %) for high quality modes. If this index is low the considered mode shape vector is either computational or wrongly estimated.
However if this MOV is low for all modes for a specific input station (say, below 10%), this might indicate that the excitation force direction was wrongly entered while measuring the FRFs for that input station. This error may be corrected by changing the signs of the modal participation factors for all modes for that particular input.

Mode Phase Collinearity – MPC

For lightly or proportionally damped structures, the estimated mode shapes should be purely normal (orthogonal). This means that the phase angle between two different complex mode shape coefficients of the same mode (i.e. for two different response stations) should be either 0_, 180_ or -180_. An indicator called the “Modal Phase Collinearity” (MPC) index expresses the linear functional relationship between the real and the imaginary parts of the unscaled mode shape vector. This index should be high (near 100%) for real normal modes. A low MPC index indicates a rather complex mode, either because of local damping elements in the tested structure or because of an erroneous measurement or analysis procedure.

Mean Phase Deviation – MPD

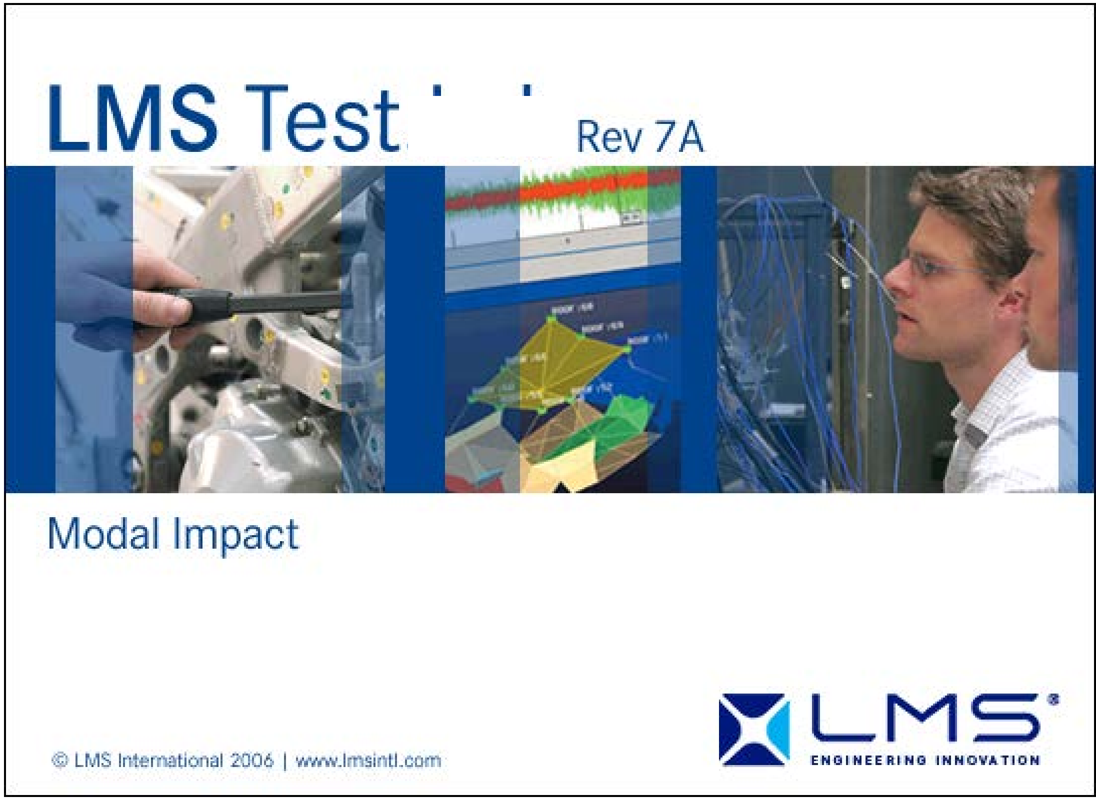
Another indicator for the complexity of unscaled mode shape vectors is the Mean Phase Deviation (MPD). This index is the statistical variance of the phase angles for each mode shape coefficient from their mean value, and indicates the phase scatter of a mode shape. This MPD value should be low (near 0_) for real normal modes.

8. Test Lab Overview

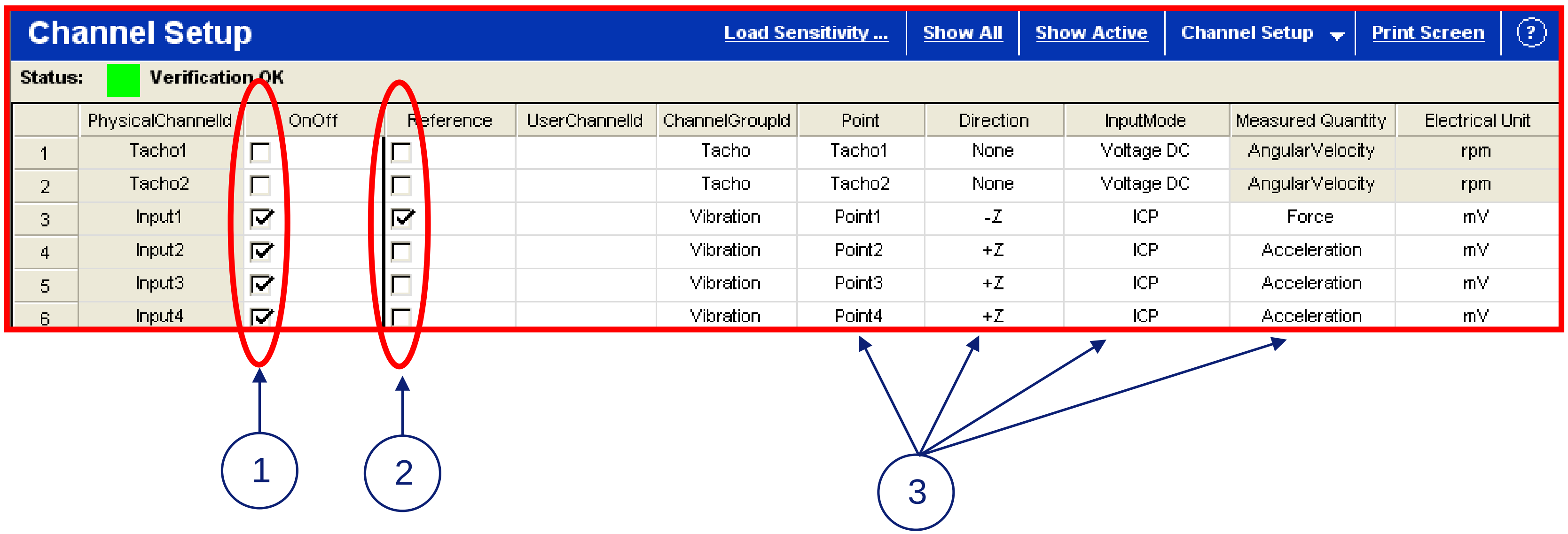
Emiliano Mucchi - *emiliano.mucchi@unife.it*

LMS Test.Lab Modal Impact

1. Layout of Test.Lab Modal Impact
2. Procedure
3. Channel Setup
4. Calibration
5. Impact Scope
6. Impact Setup:
   - trigger
   - bandwidth
   - windowing
   - driving point
7. Measure
8. Validate
9. Navigator



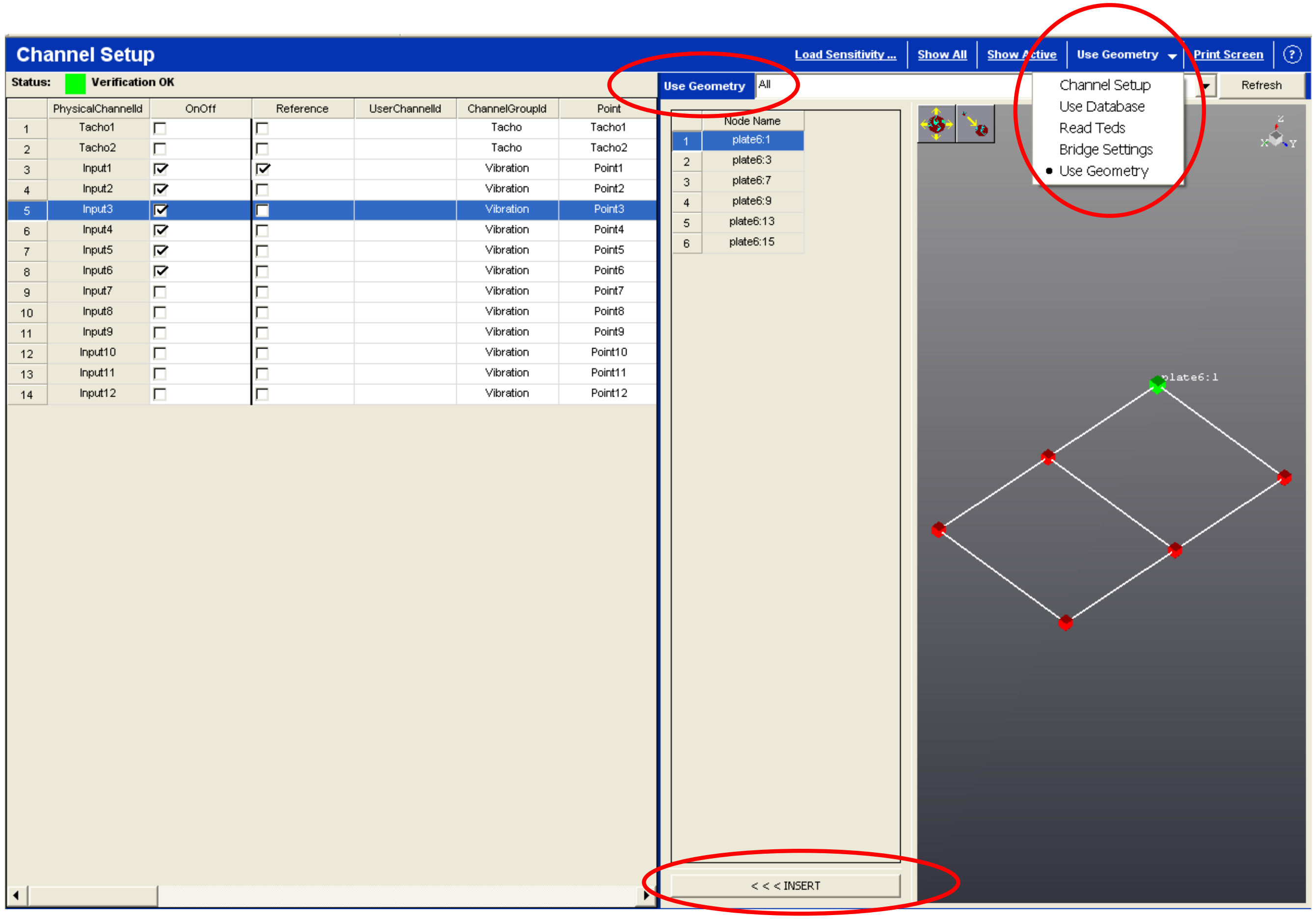
Channel Setup



	PhysicalChannelId	OnOff	Reference	UserChannelId	ChannelGroupId	Point	Direction	InputMode	Measured Quantity	Electrical Unit
1	Tacho1	☐	☐		Tacho	Tacho1	None	Voltage DC	AngularVelocity	rpm
2	Tacho2	☐	☐		Tacho	Tacho2	None	Voltage DC	AngularVelocity	rpm
3	Input1	☑	☑		Vibration	Point1	-Z	ICP	Force	mV
4	Input2	☑	☐		Vibration	Point2	+Z	ICP	Acceleration	mV
5	Input3	☑	☐		Vibration	Point3	+Z	ICP	Acceleration	mV
6	Input4	☑	☐		Vibration	Point4	+Z	ICP	Acceleration	mV

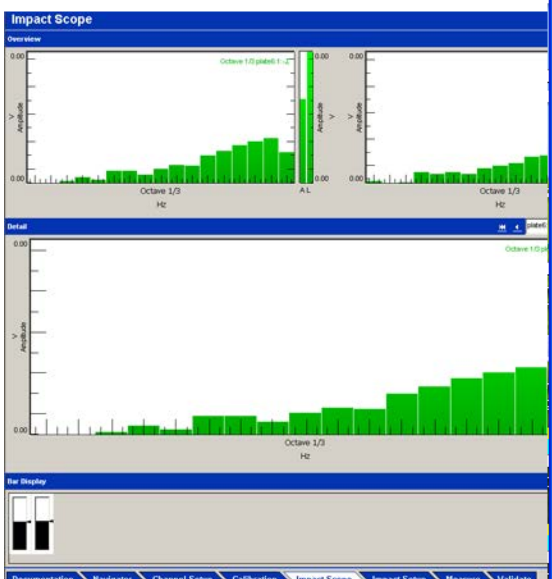
1. Select channels
2. Define the reference channel
3. Fill in channel setup. You must specify the Point ID and Direction.

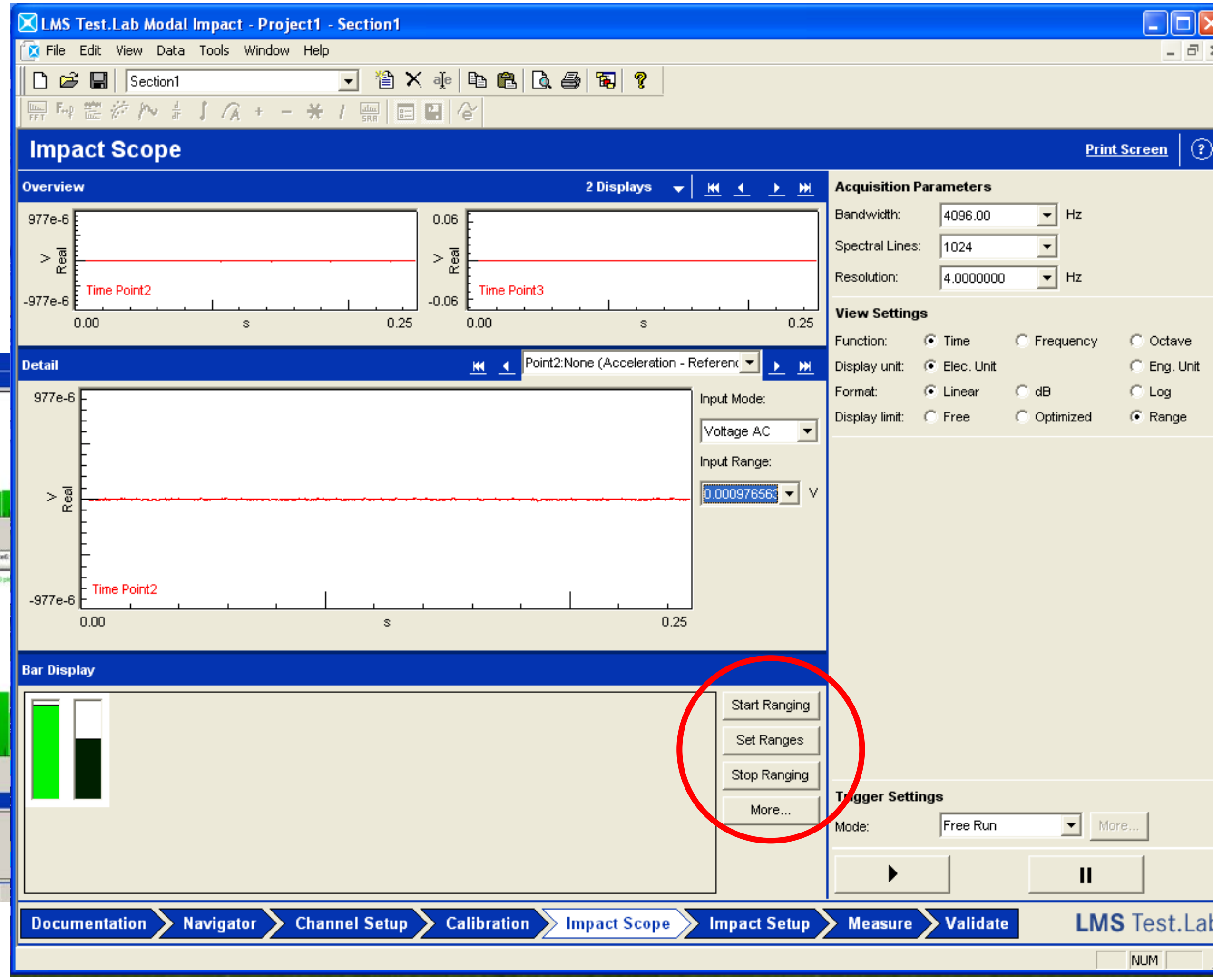
Channel names from Geometry



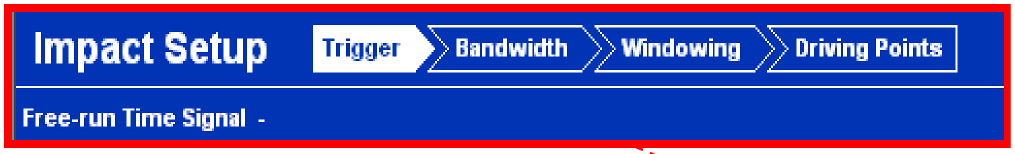
Impact Scope

The Impact Scope can be used to verify settings / instrumentation.

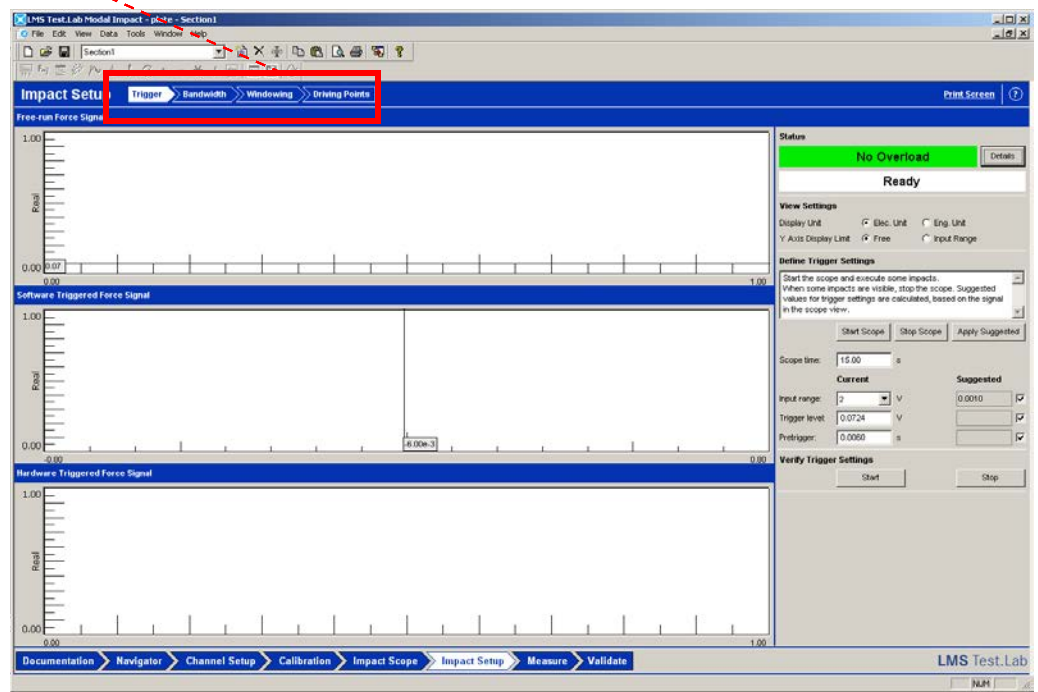




Impact Setup Procedure



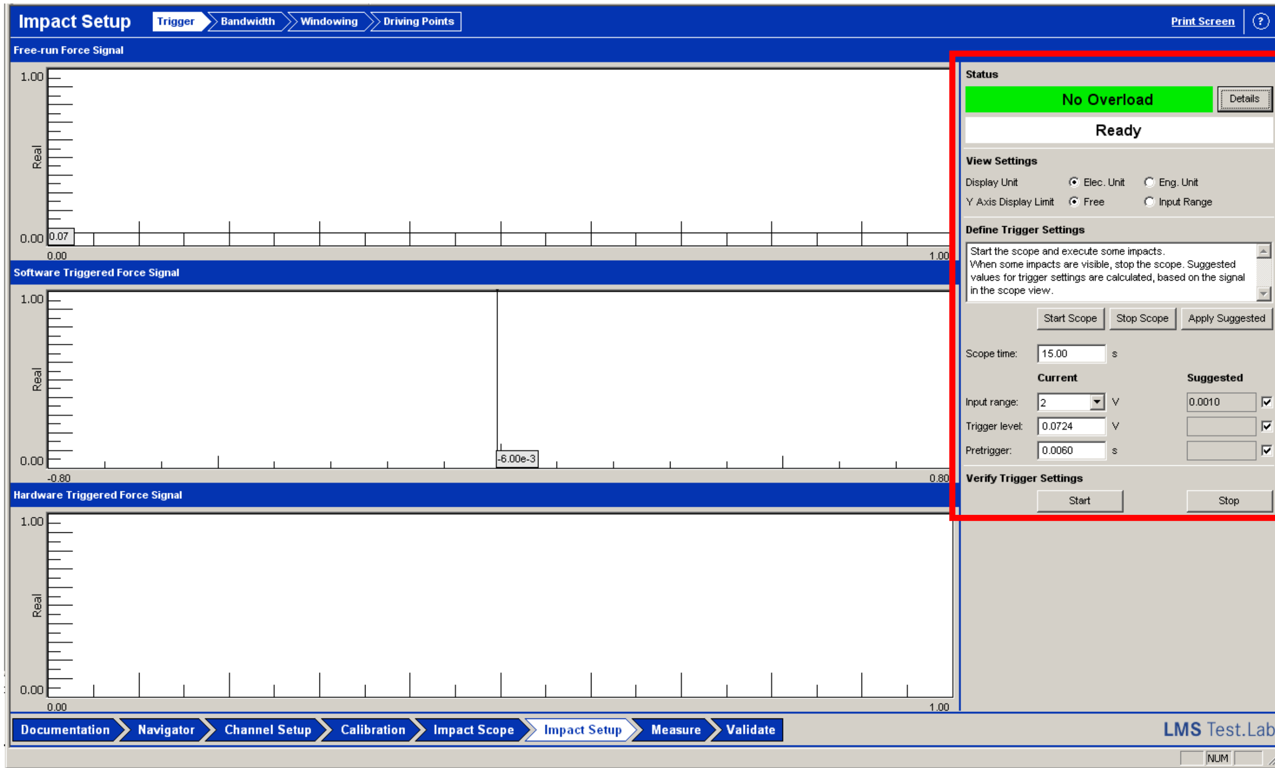
- ❑ Trigger
- ❑ Bandwidth
- ❑ Windowing
- ❑ Driving Points

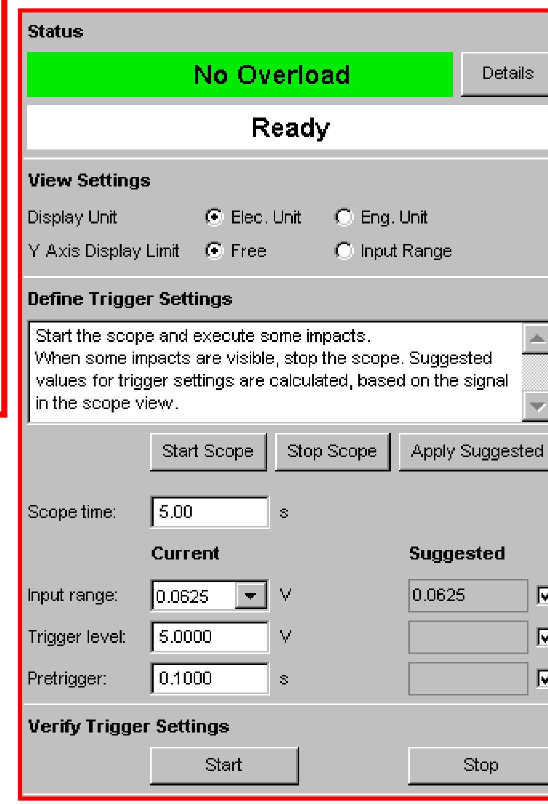


Impact Setup
Trigger

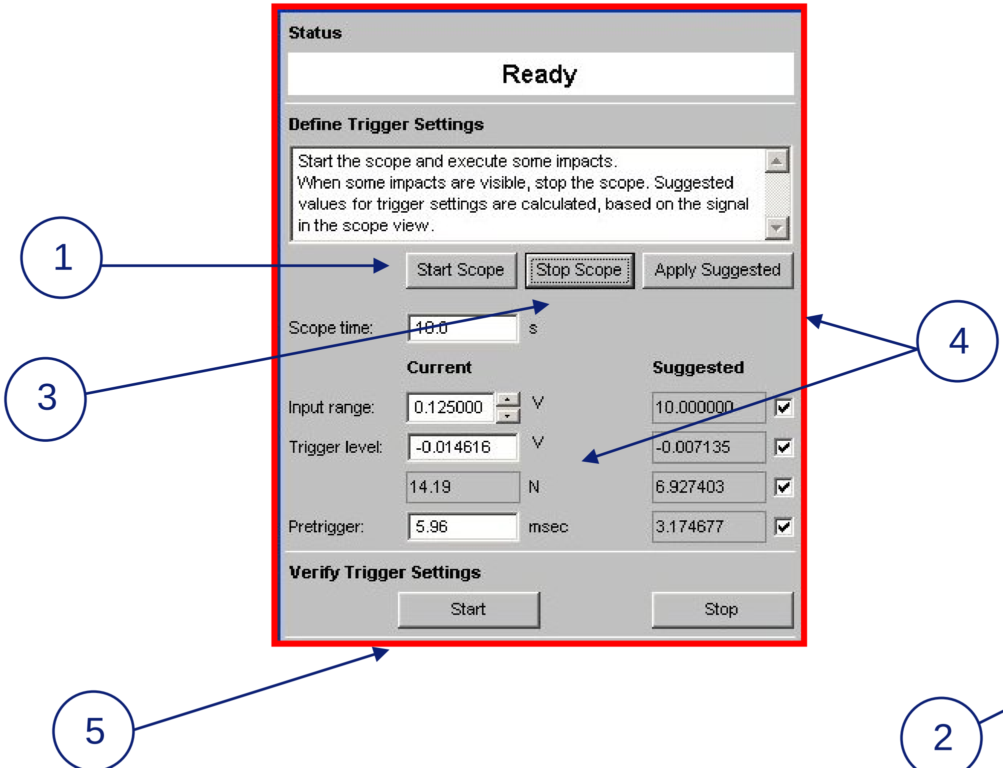
Display Area

Settings Panel





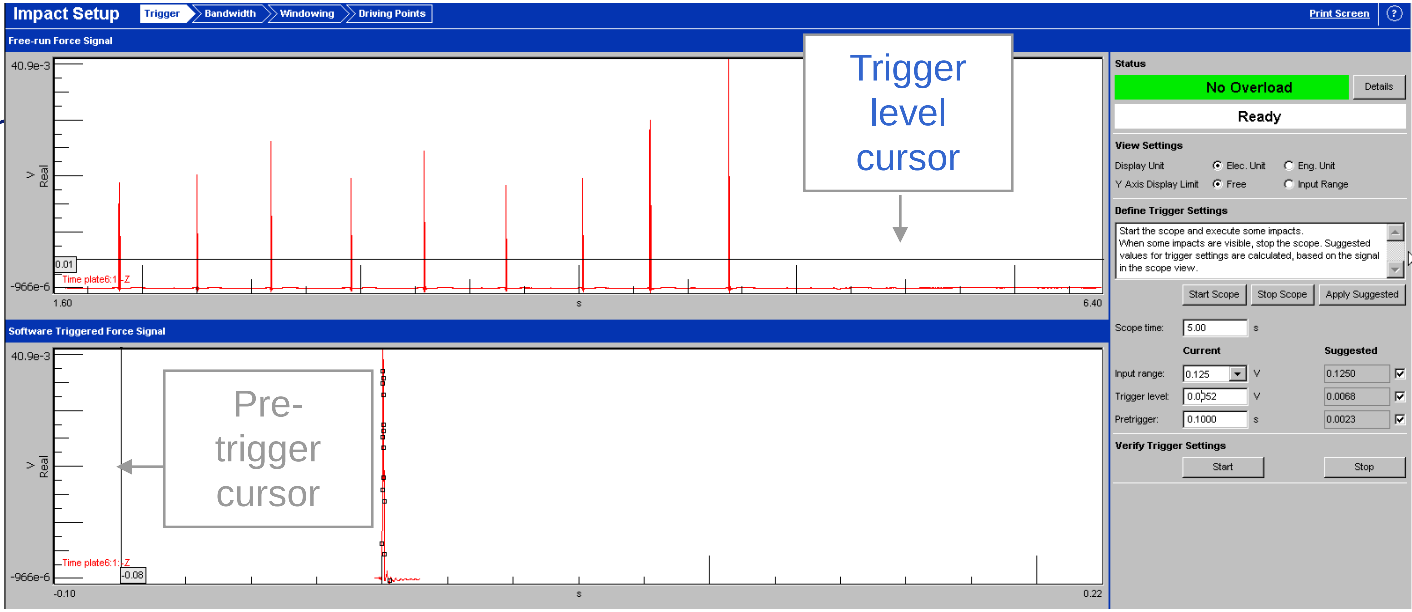
Impact Setup Trigger

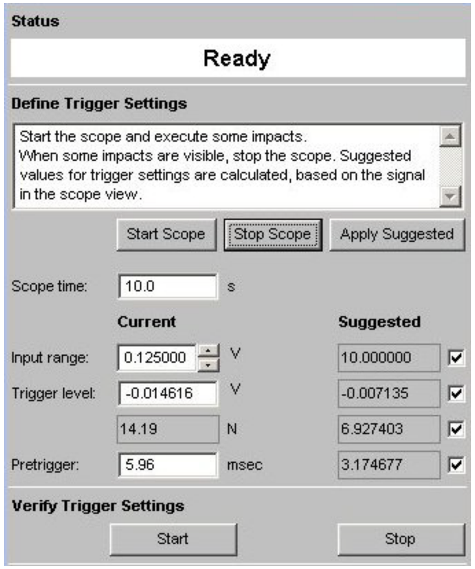


1. Start Scope
2. Hit the test object until you see some impacts.
3. Stop Scope
4. Press Apply Suggested to define Trigger Settings
5. Verify Trigger Settings

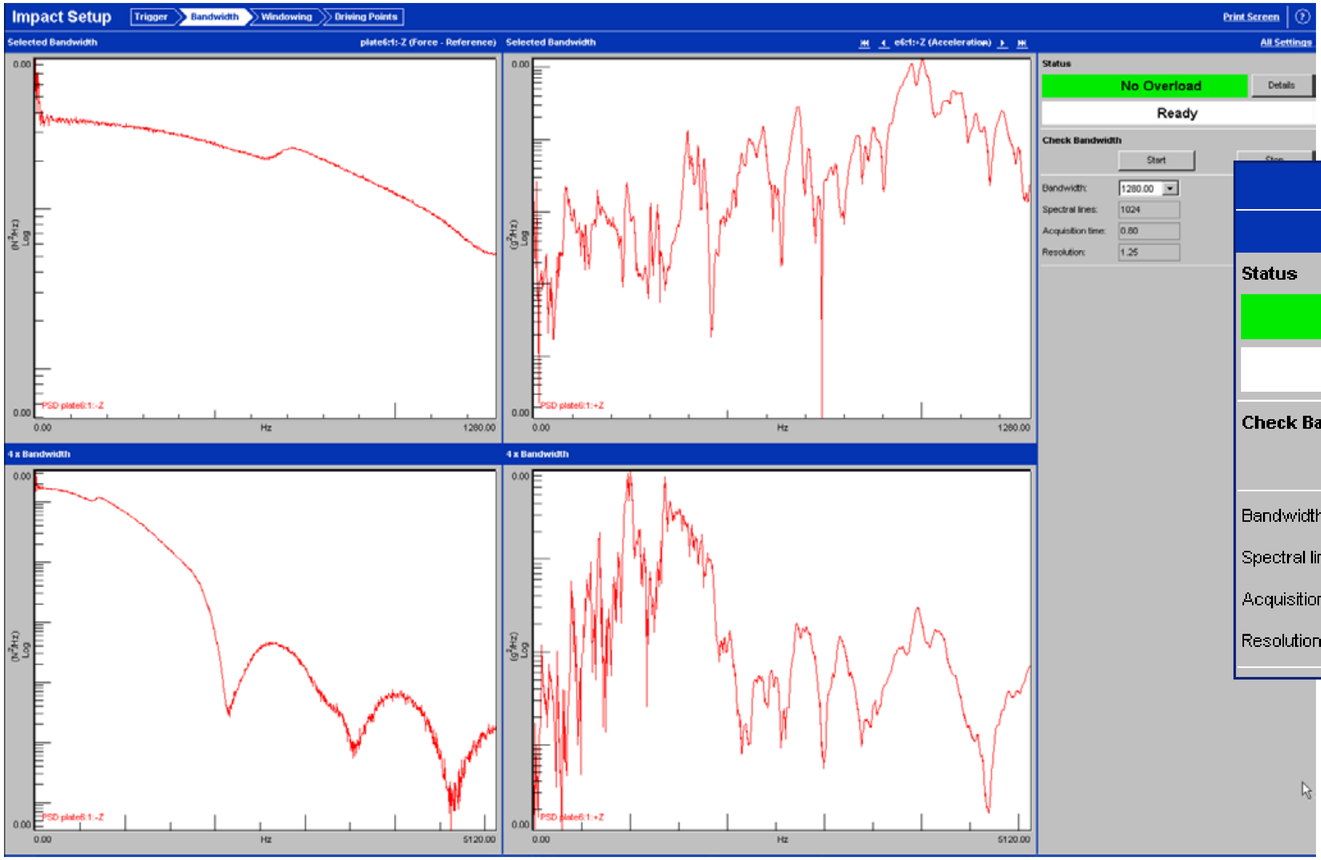
Impact setup
Trigger

Trigger Settir

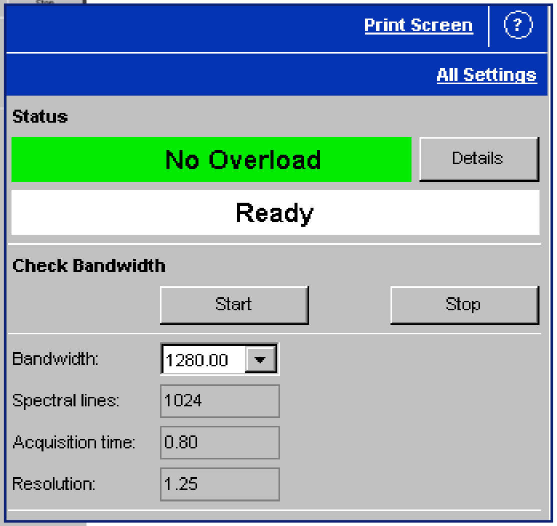




Impact Setup Bandwidth

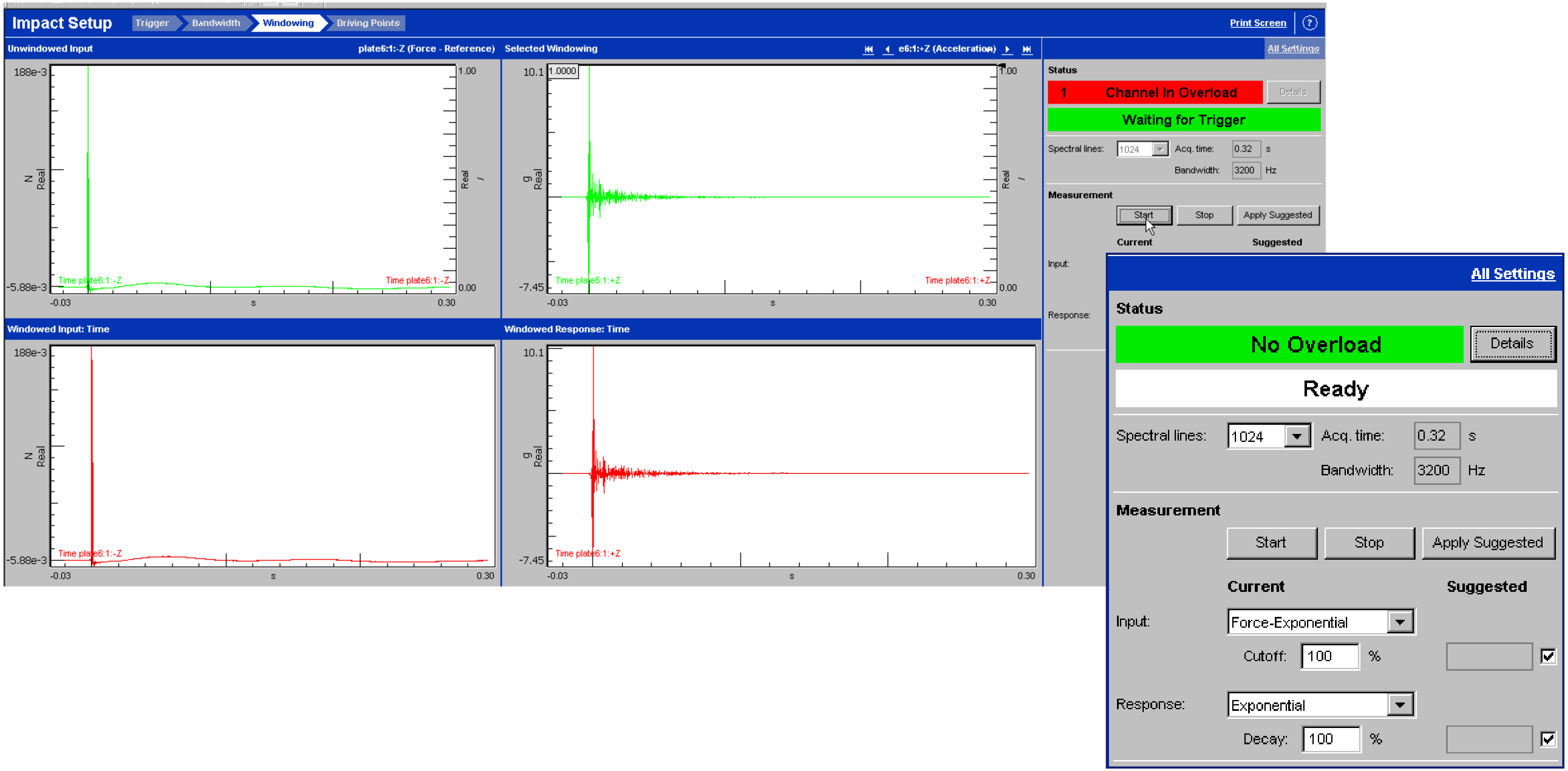


Settings Panel



Hit once to update the upper (1X BW) display and again to update lower (4X BW) displays.

Impact Setup
Windowing



Settings Panel

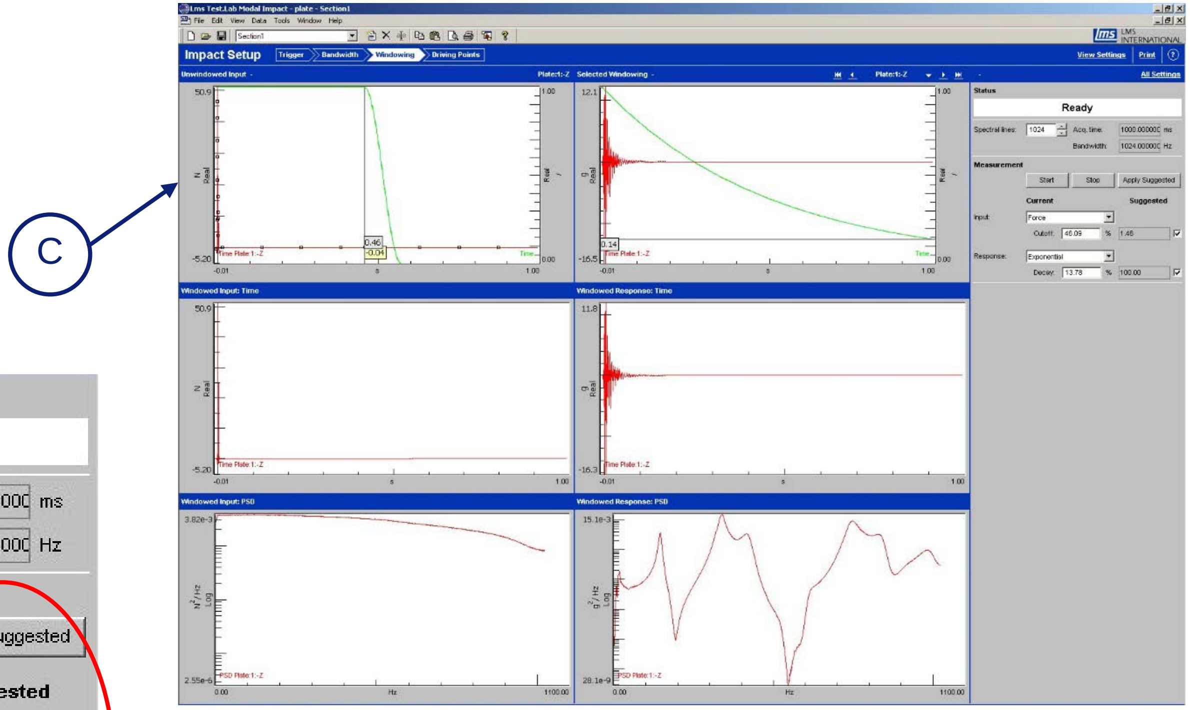
Impact Setup
Windowing

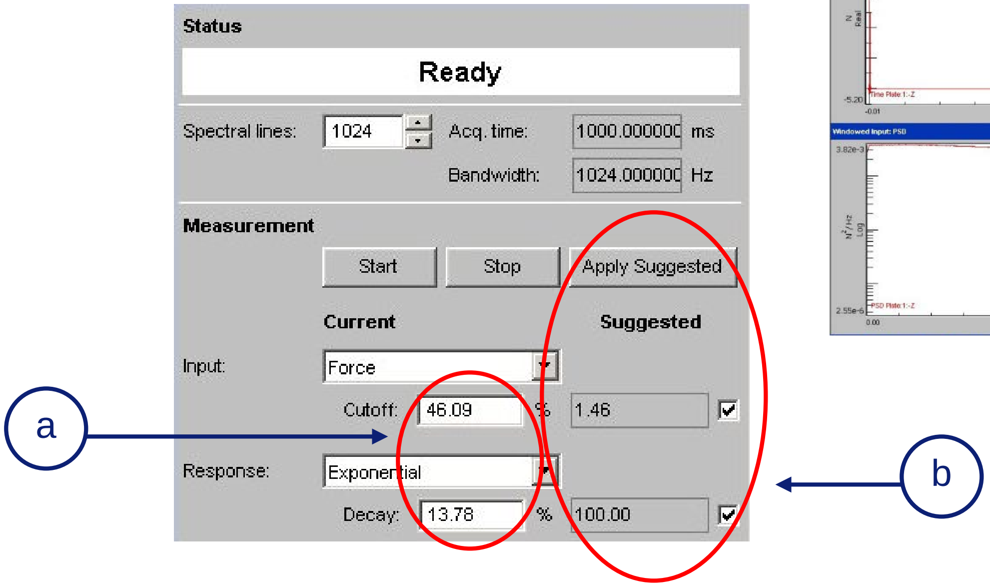
Windowing Settings:

(a) Manually

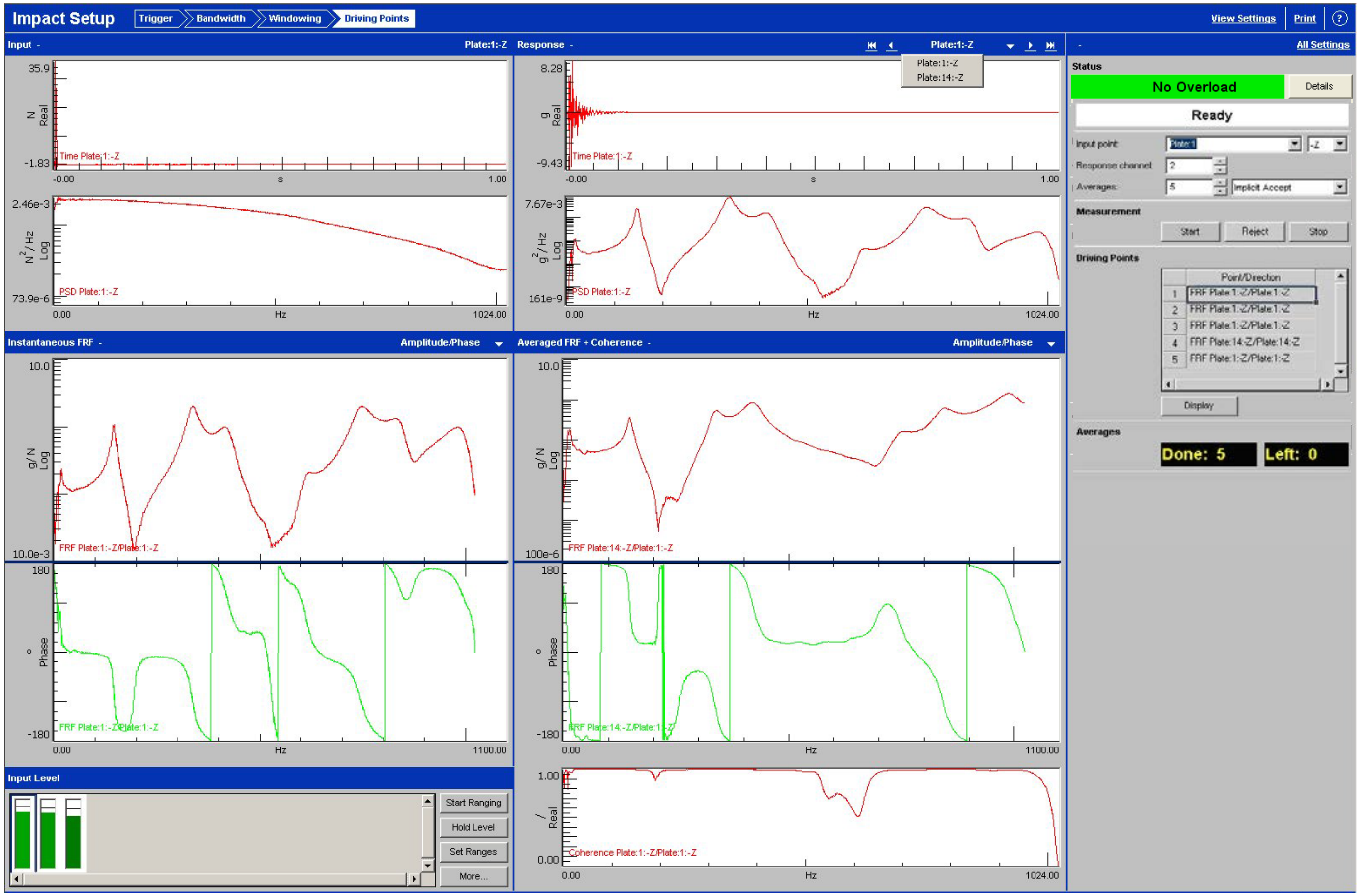
(b) Apply suggested

(c) Move cursor





Impact Setup
Driving Points



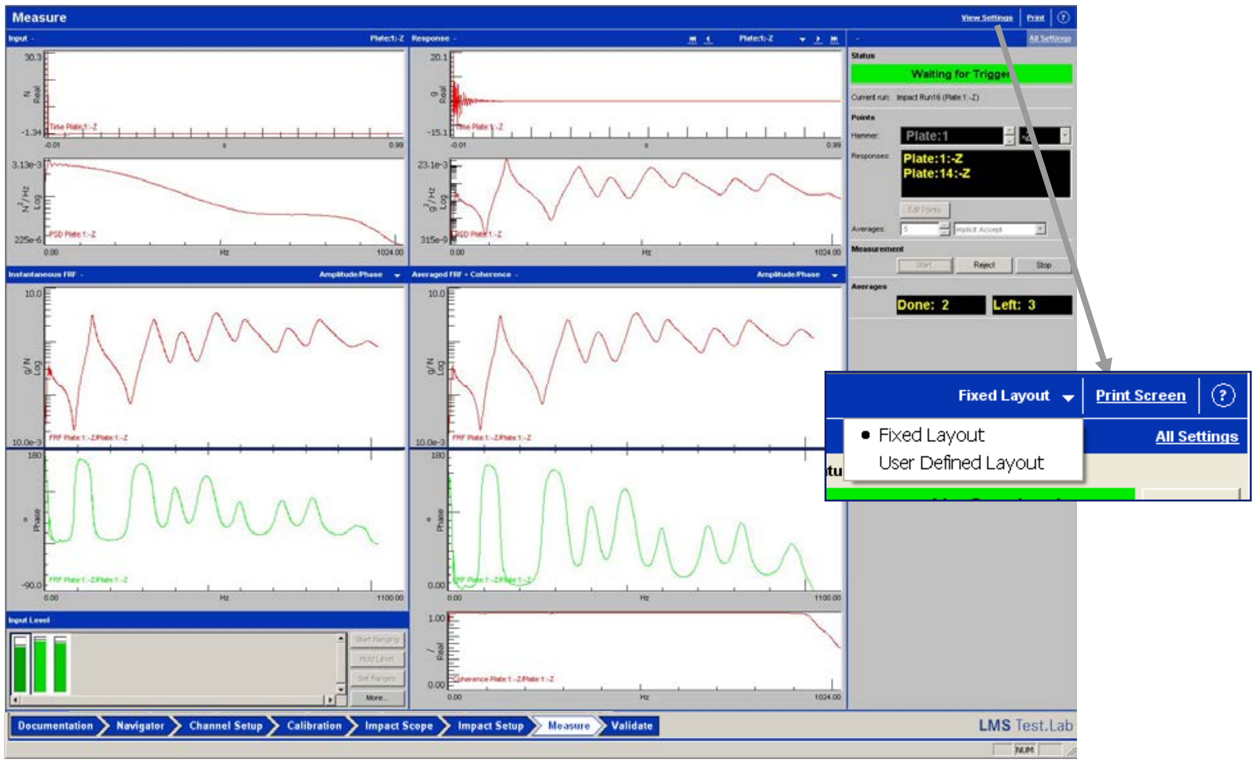
The Driving Point minor worksheet is used to quickly test several driving points and overlay them to determine the ideal impact location for your modal test.

You want to select an impact location that excites all of the resonances you are interested in.

Measure

Display Area

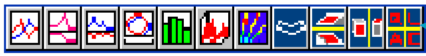
Settings Panel

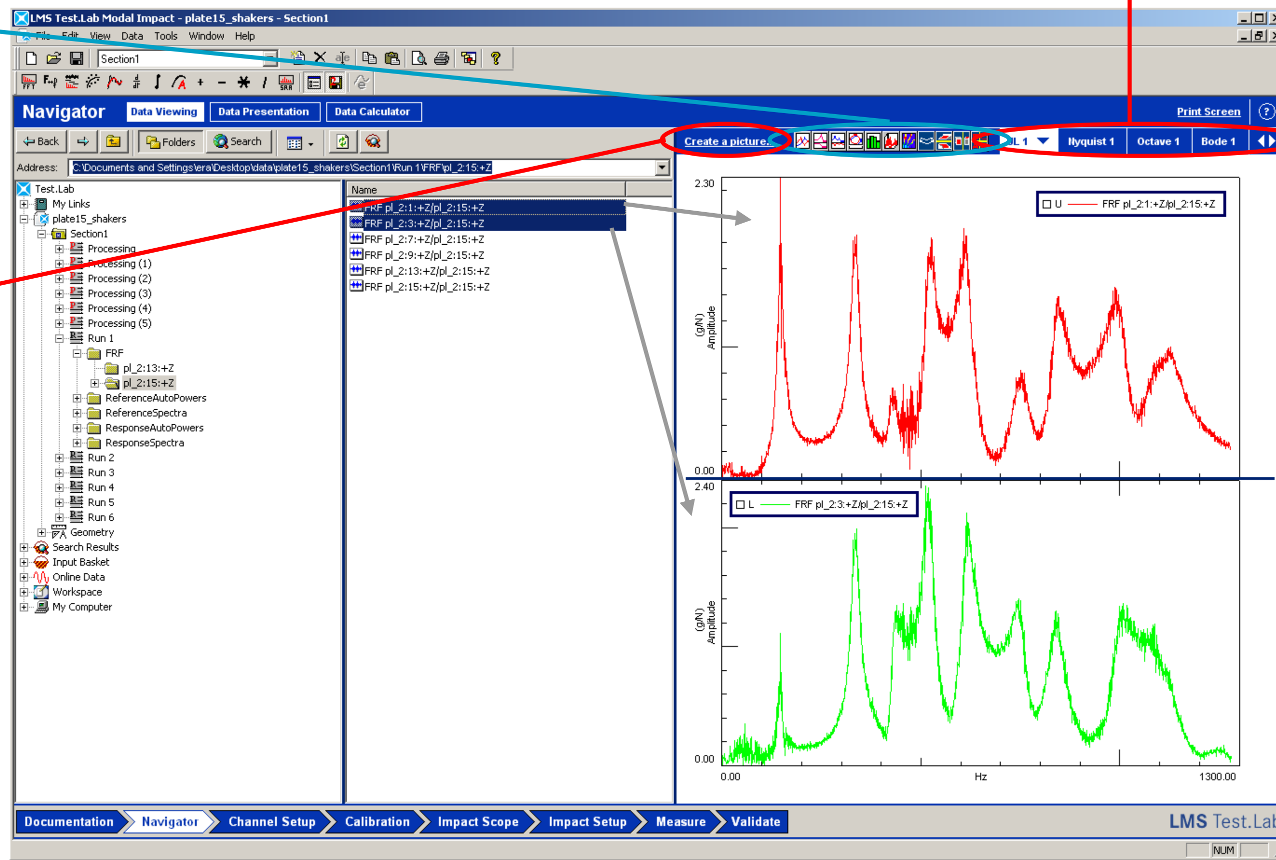


Now we are ready to take data, that is done on the Measure worksheet.

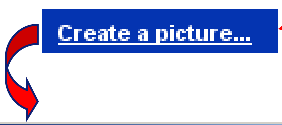
Navigator

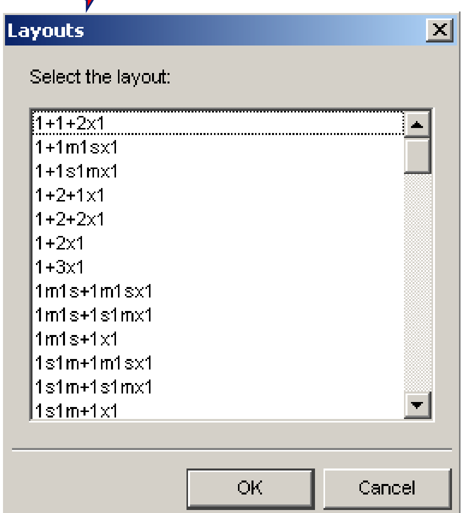
Favorite display layouts



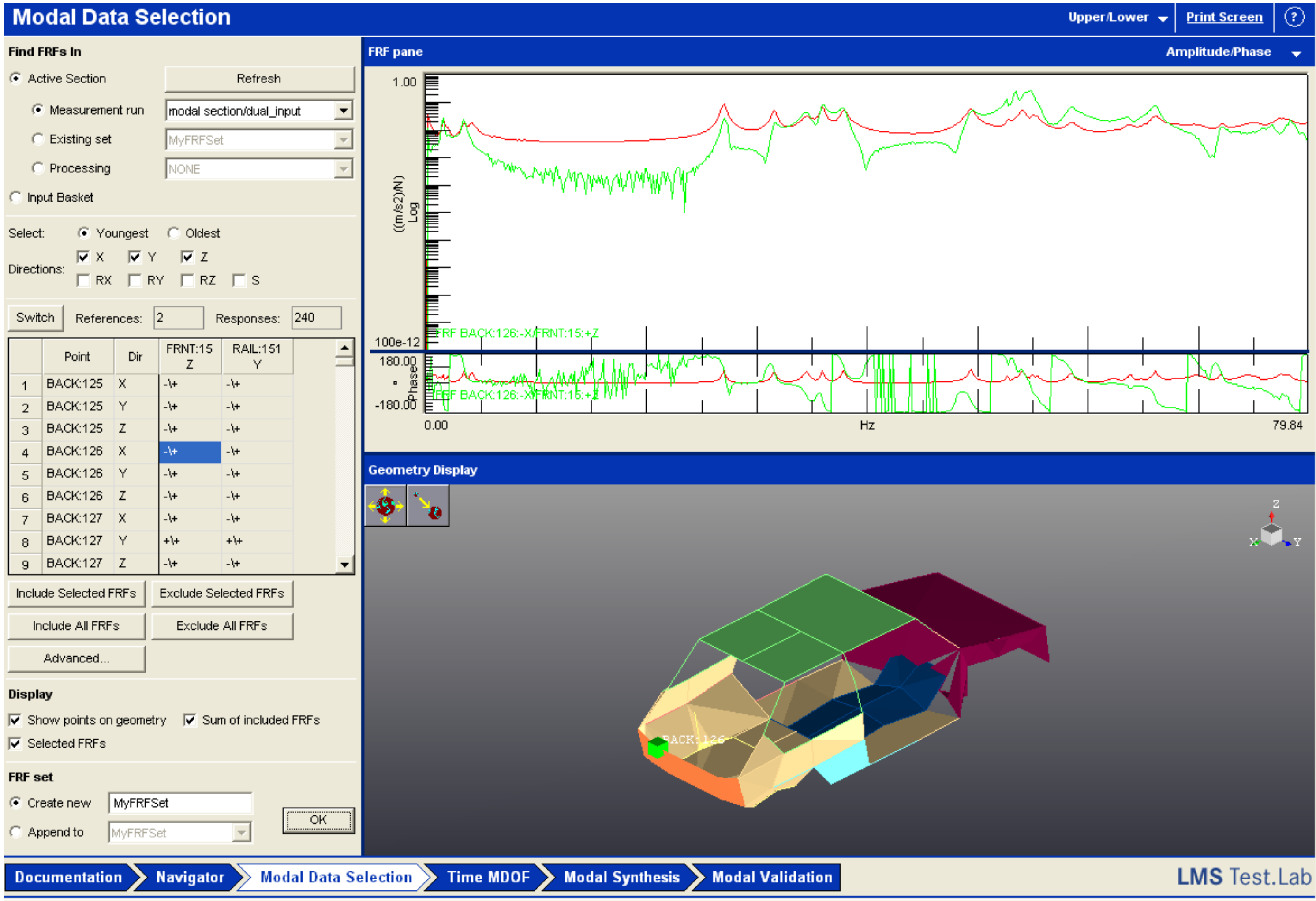


All display layouts





Modal data selection



Time MDOF/PolyMAX
1-2-3

- Modal analysis with Time MDOF or PolyMAX (add-in)
    - The same look and workflow (3 steps)
    - Based on different algorithm

Step 1 : select the analysis bandwidth

Step 2 : select poles

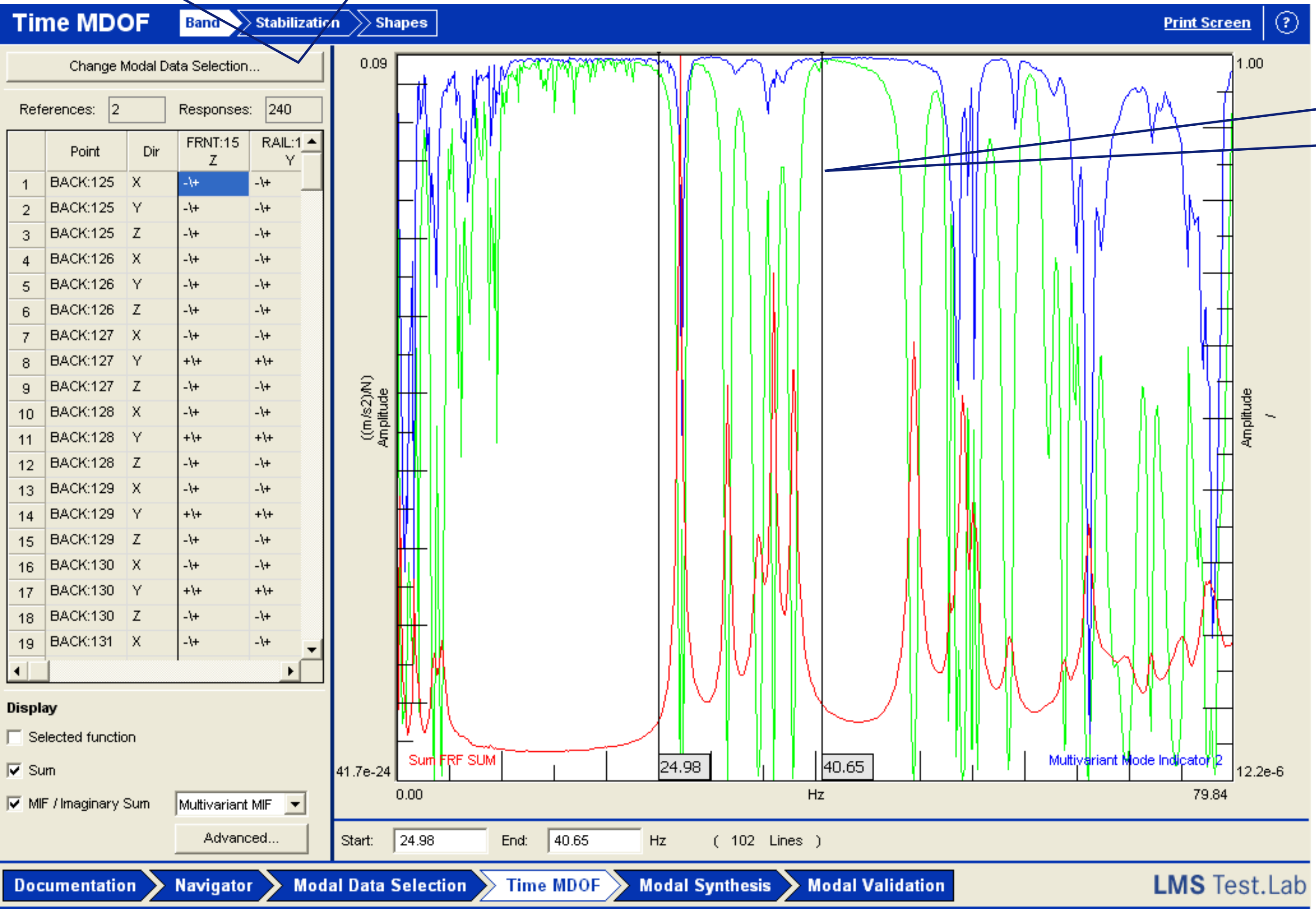
Step 3 : calculate mode shapes

Time MDOF/PolyMAX

Step 1 - Band

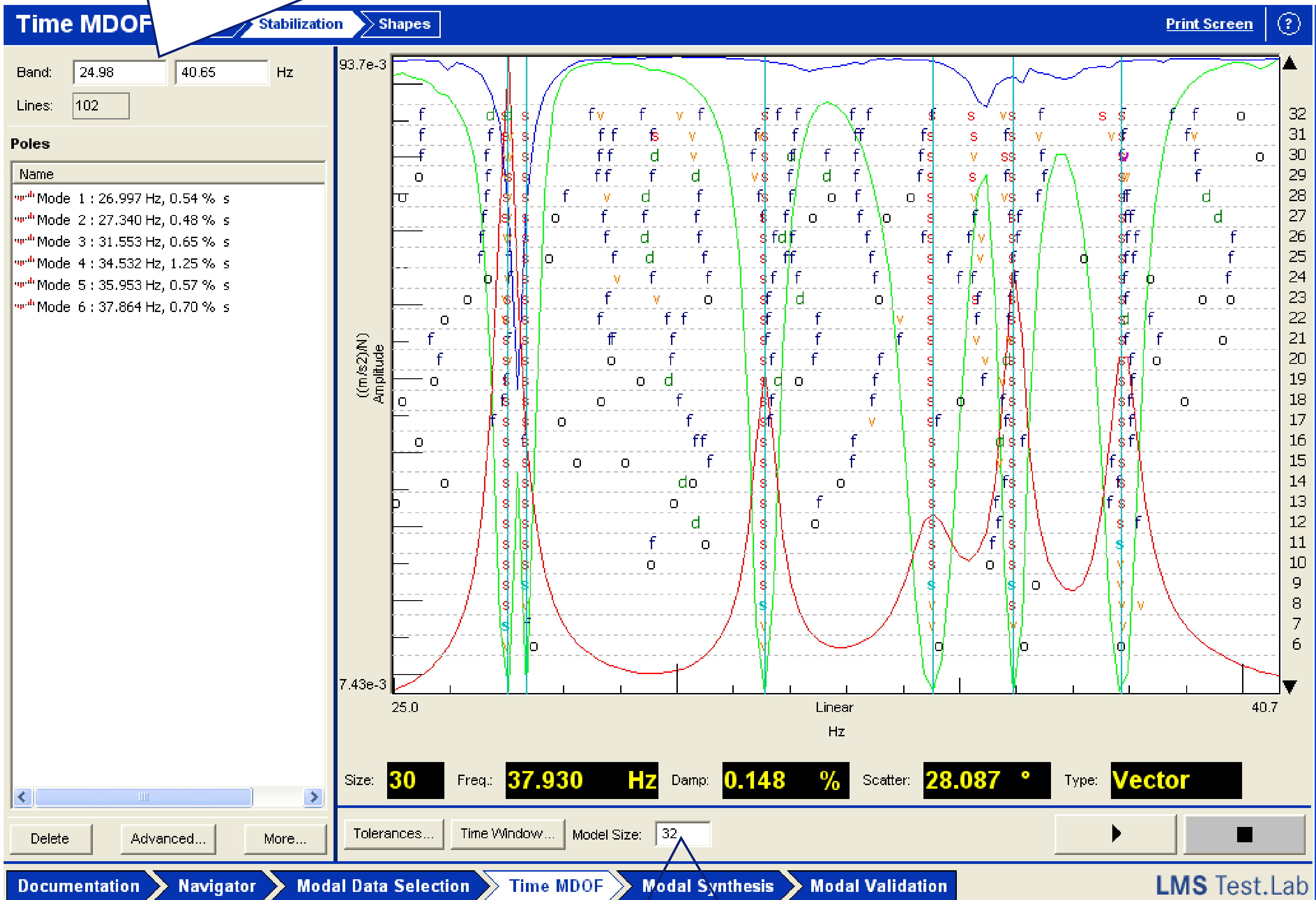
Redefine your data for modal analysis

Double cursor for defining analysis bandwidth



Time MDOF/PolyMAX
Step 2 - Stabilization

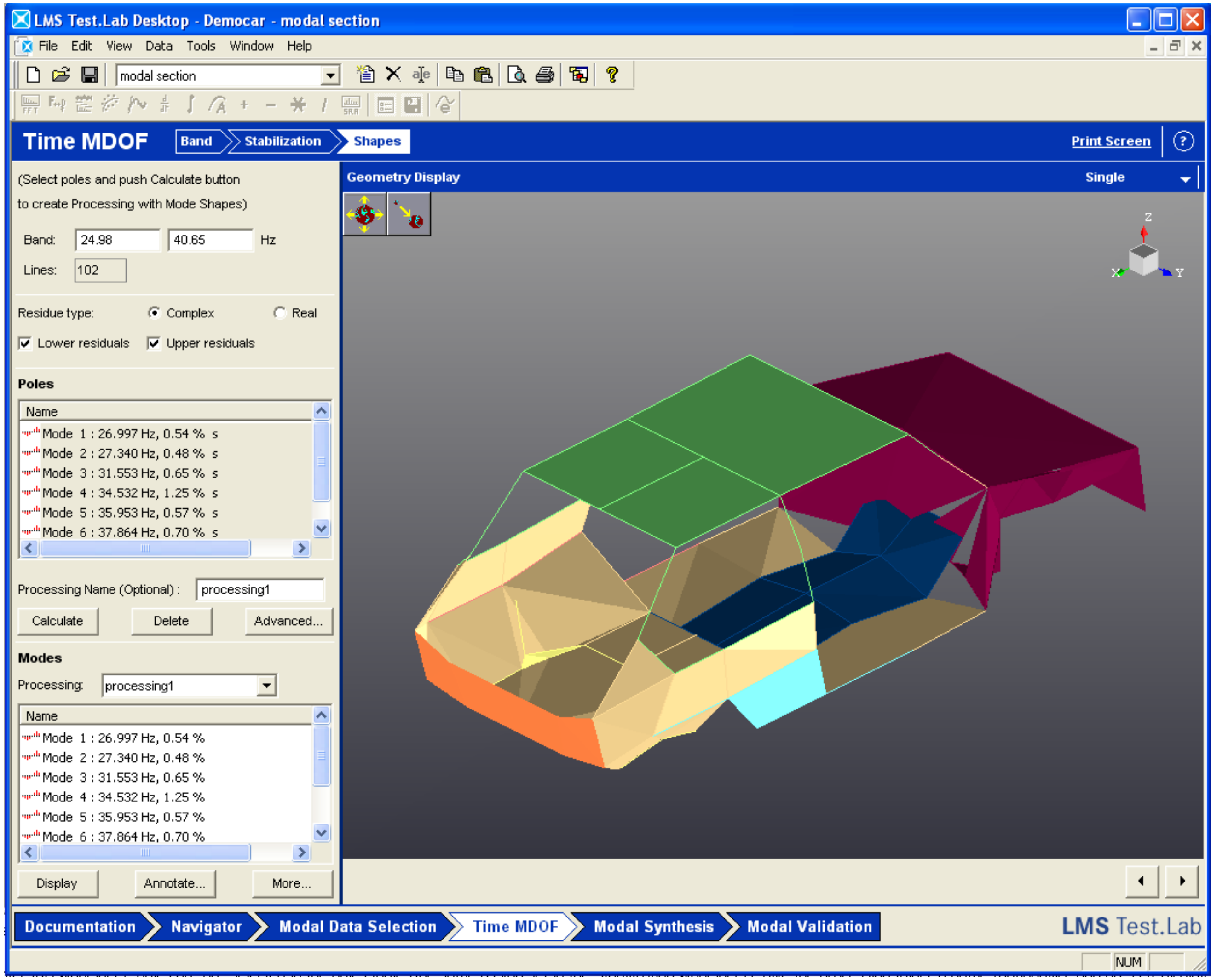
Redefine your analysis frequency if necessary



Adapt the model size

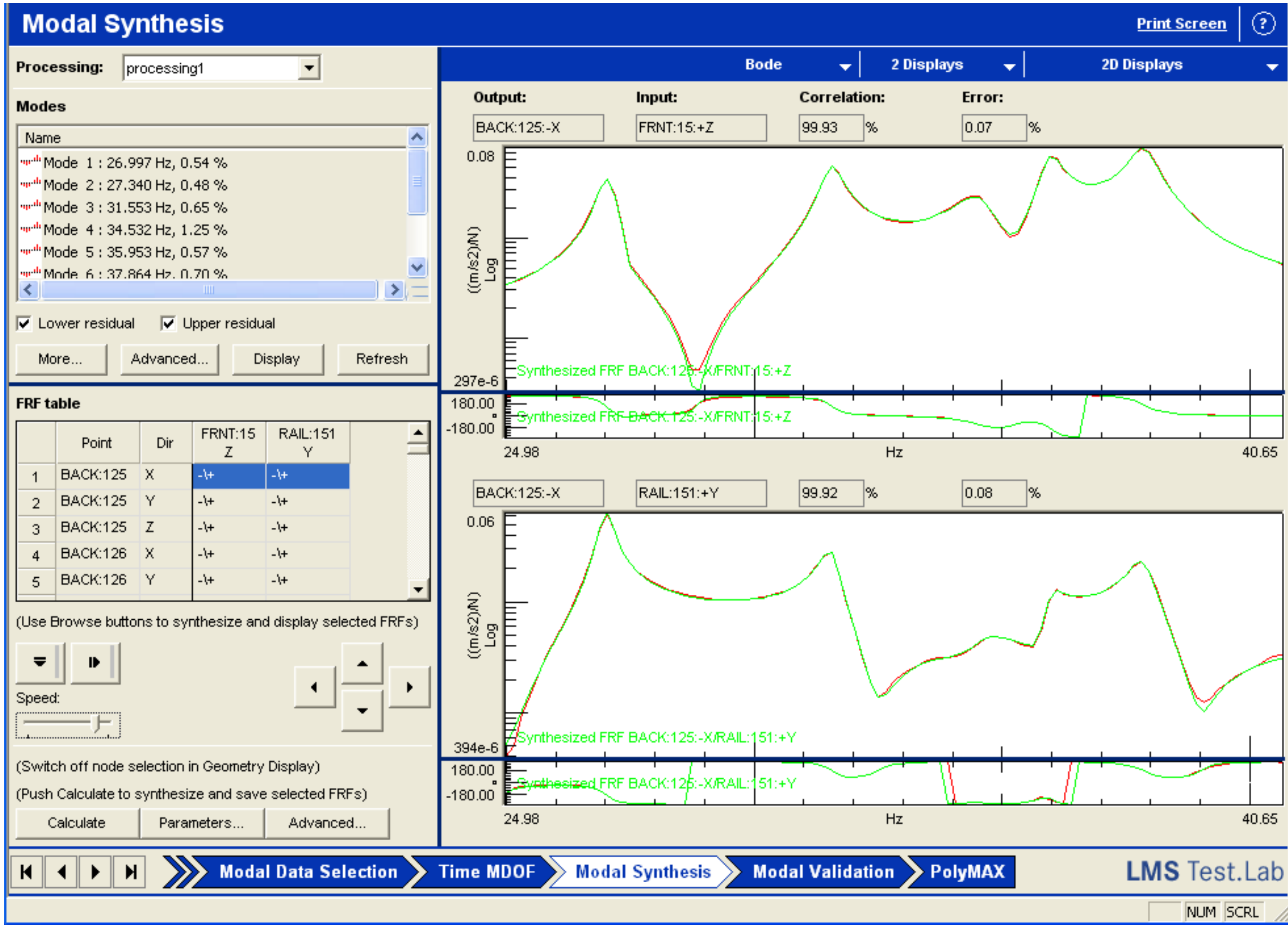
Time MDOF/PolyMAX

Step 3 - Shapes

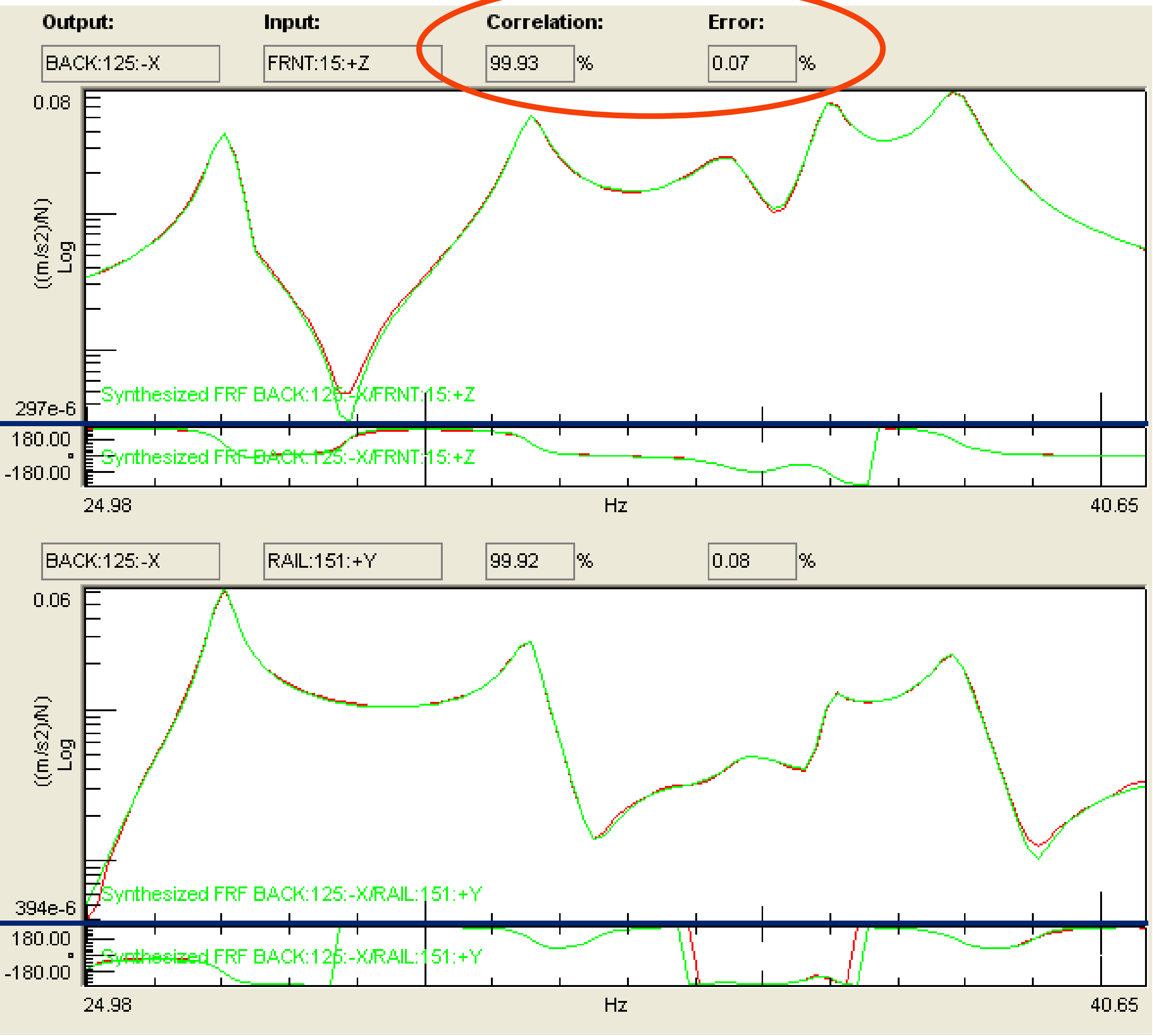


Modal Synthesis

FRF Synthesis

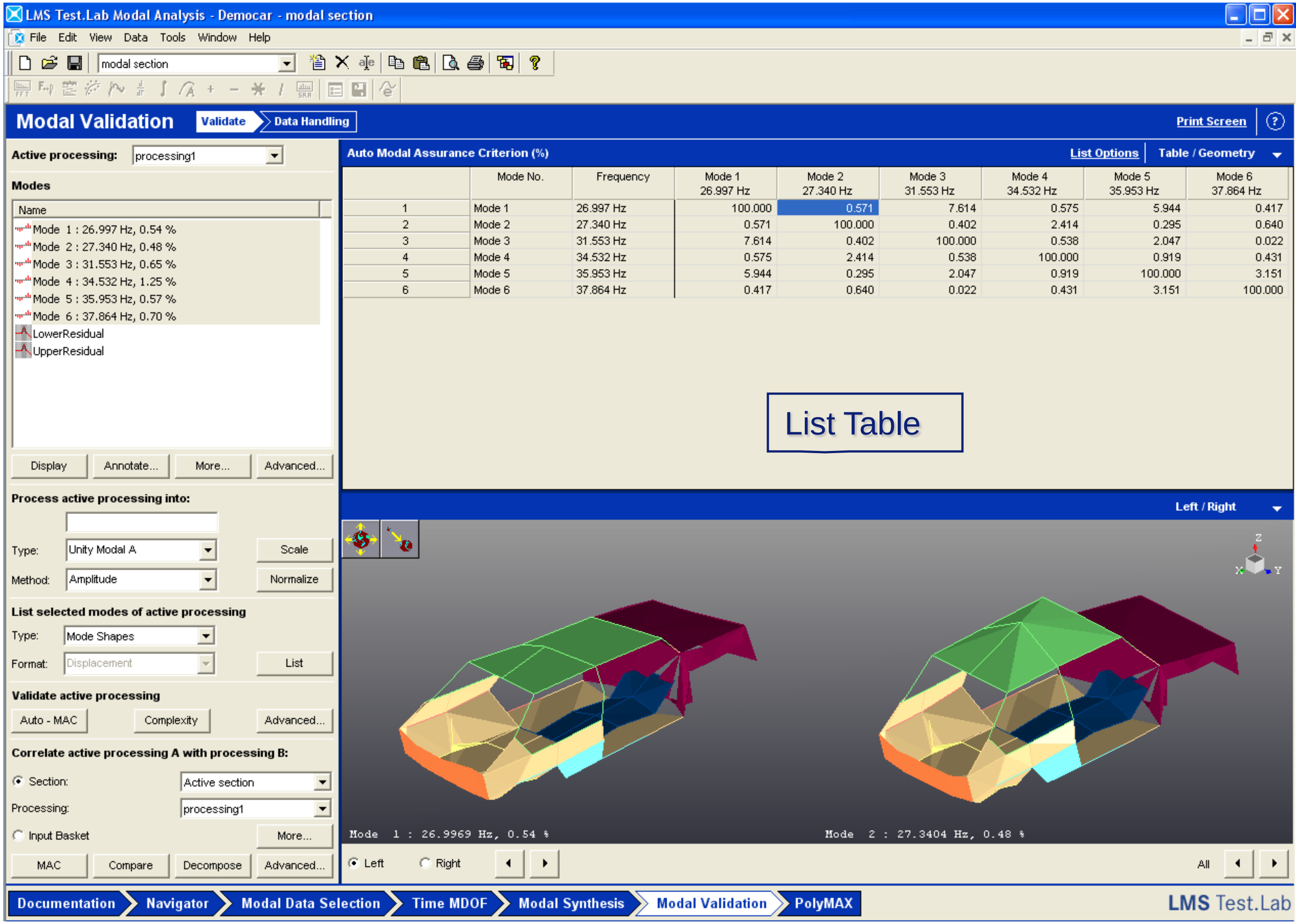


Modal Synthesis
Correlation and Error



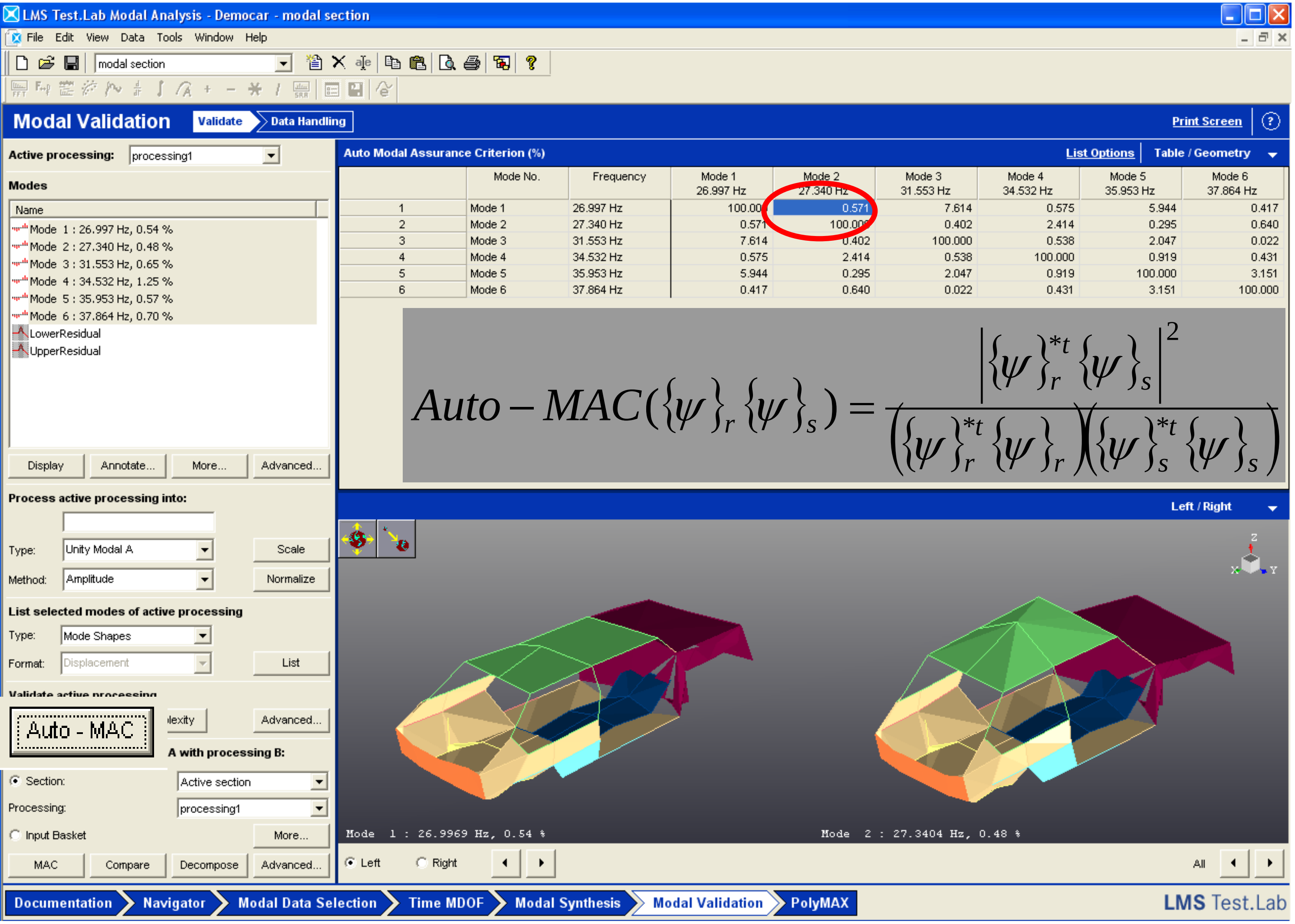
Modal Validation

Validation tools



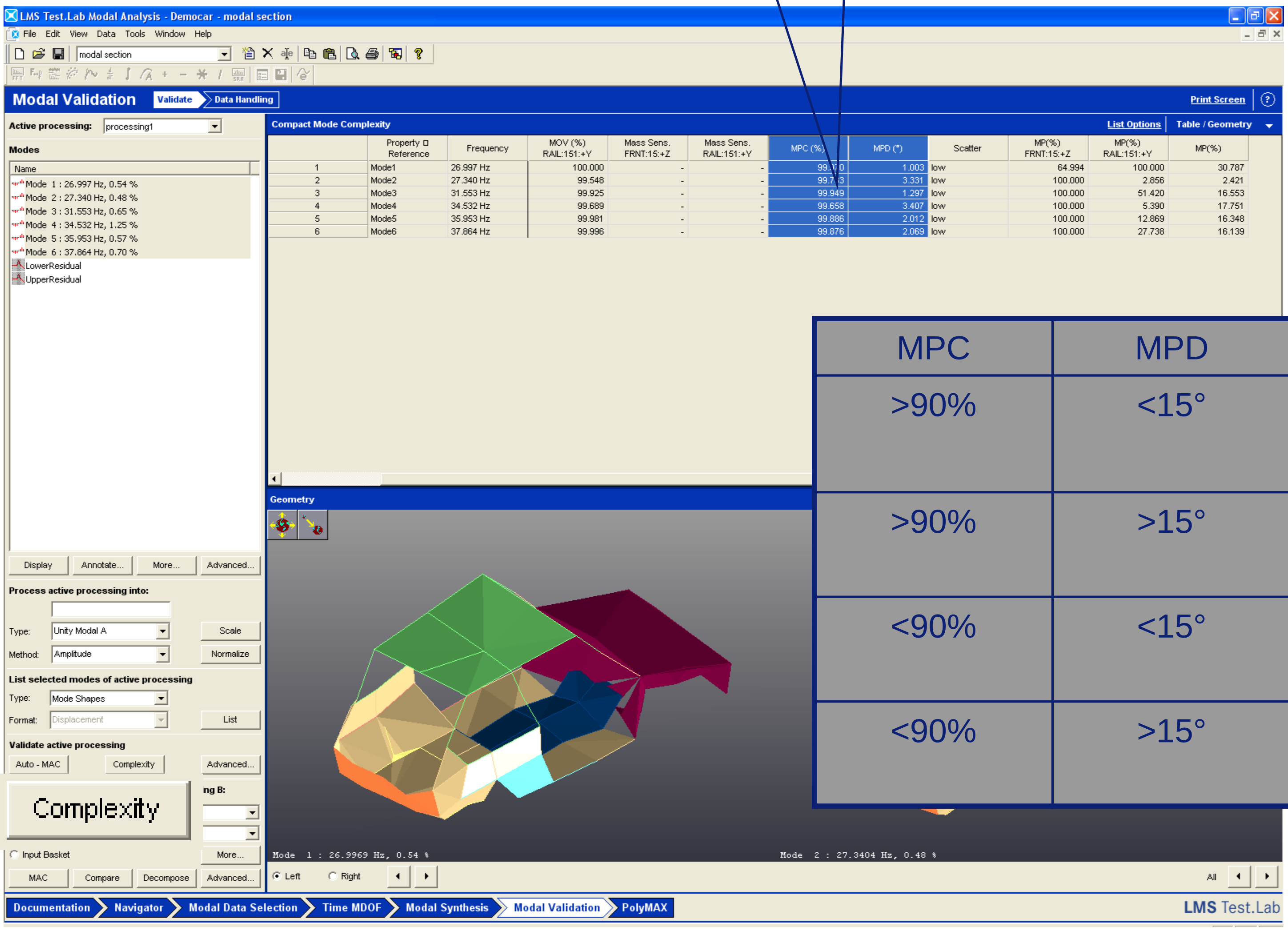
Modal Validation

Auto-MAC



Modal Validation
Complexity

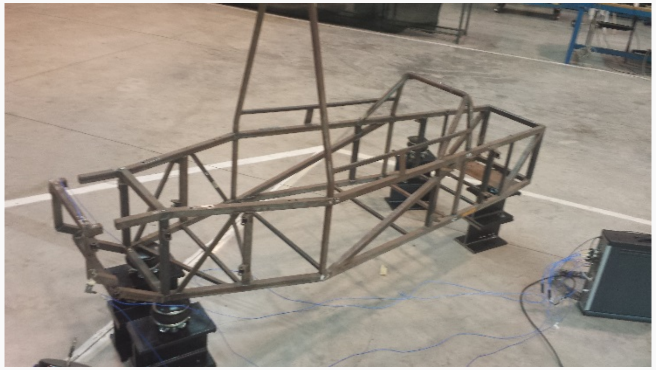
MPC and MPD



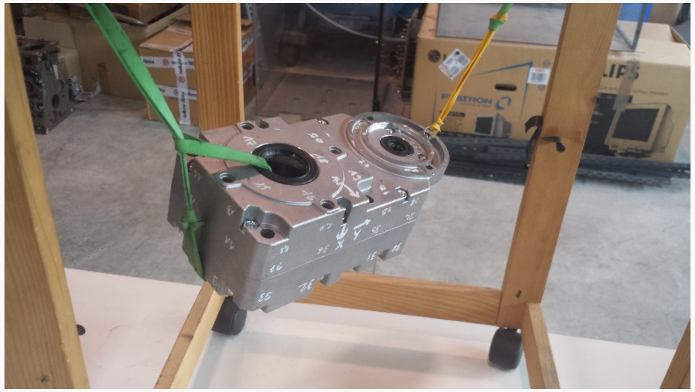
MPC	MPD	Scatter
>90%	<15°	Low
>90%	>15°	?
<90%	<15°	?
<90%	>15°	High

9. Practice

Modal analysis

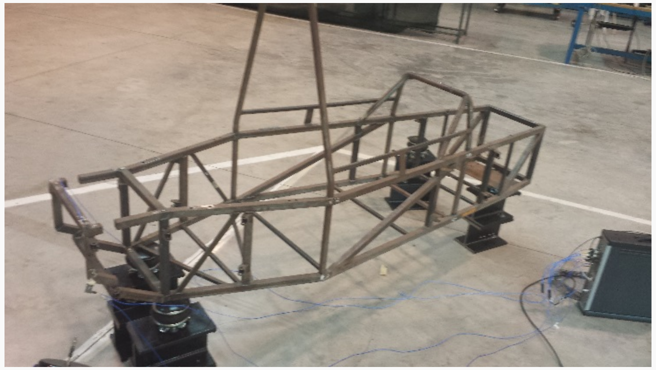


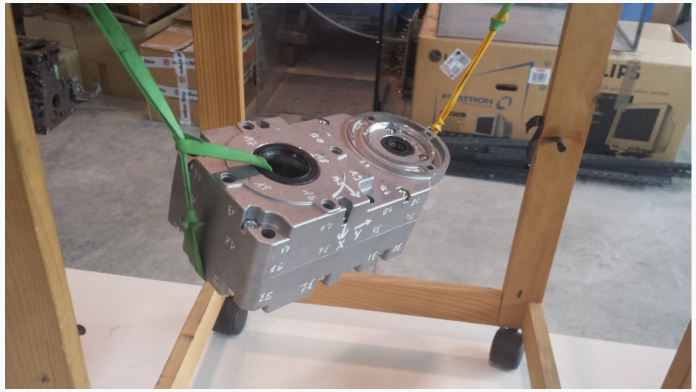
Free-free modal analysis of a car BIW by using shaker



Free-free modal analysis of cover of gearbox by using hammer

Geometry





Cad model is not available (as usual in practice)

Cad model is available (list of coordinates XYZ)

To do

- Build the geometry
- Mount accelerometers
- Define channel set up
- Define trigger/pretrigger/acquisition parameters/ averages/etc
- Acquire FRFs
- Perform modal analysis with LSCE and Polymax
- Estimate complexity values
- Evaluate synthetized FRF
- Estimate MAC between Polymax eigenvectors and LSCE eigenvectors