

**FIGURE 10.8** Linearity of voltage with displacement of core.

temperature and high output. The mass of the magnetic core restricts the use of the LVDT for high-frequency applications [10.4].

As long as the core is not moved very far from the center of the coil, the output voltage varies linearly with the displacement of the core, as shown in Fig. 10.8; hence the name linear variable differential transformer.

## 10.3 Vibration Pickups

When a transducer is used in conjunction with another device to measure vibrations, it is called a *vibration pickup*. The commonly used vibration pickups are known as seismic instruments. A seismic instrument consists of a mass-spring-damper system mounted on the vibrating body, as shown in Fig. 10.9. Then the vibratory motion is measured by finding the displacement of the mass relative to the base on which it is mounted.

The instrument consists of a mass  $m$ , a spring  $k$ , and a damper  $c$  inside a cage, which is fastened to the vibrating body. With this arrangement, the bottom ends of the spring and the dashpot will have the same motion as the cage (which is to be measured,  $y$ ) and their vibration excites the suspended mass into motion. Then the displacement of the mass relative to the cage,  $z = x - y$ , where  $x$  denotes the vertical displacement of the suspended mass, can be measured if we attach a pointer to the mass and a scale to the cage, as shown in Fig. 10.9.<sup>3</sup>

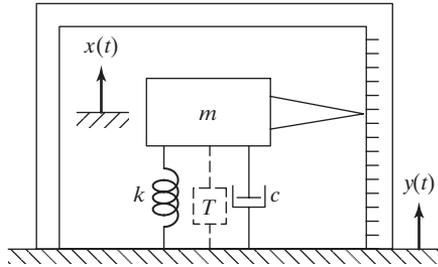
The vibrating body is assumed to have a harmonic motion:

$$y(t) = Y \sin \omega t \quad (10.13)$$

The equation of motion of the mass  $m$  can be written as

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \quad (10.14)$$

<sup>3</sup>The output of the instrument shown in Fig. 10.9 is the relative mechanical motion of the mass, as shown by the pointer and the graduated scale on the cage. For high-speed operation and convenience, the motion is often converted into an electrical signal by a transducer.


**FIGURE 10.9** Seismic instrument.

By defining the relative displacement  $z$  as

$$z = x - y \quad (10.15)$$

Eq. (10.14) can be written as

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \quad (10.16)$$

Equations (10.13) and (10.16) lead to

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin \omega t \quad (10.17)$$

This equation is identical to Eq. (3.75); hence the steady-state solution is given by

$$z(t) = Z \sin(\omega t - \phi) \quad (10.18)$$

where  $Z$  and  $\phi$  are given by (see Eqs. (3.76) and (3.77)):

$$Z = \frac{Y\omega^2}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}} = \frac{r^2 Y}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \quad (10.19)$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \quad (10.20)$$

$$r = \frac{\omega}{\omega_n} \quad (10.21)$$

and

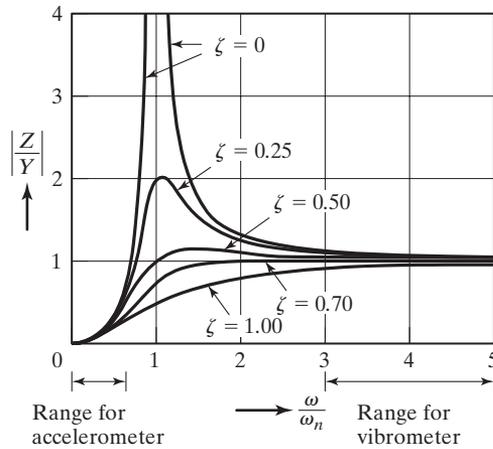
$$\zeta = \frac{c}{2m\omega_n} \quad (10.22)$$

The variations of  $Z$  and  $\phi$  with respect to  $r$  are shown in Figs. 10.10 and 10.11. As will be seen later, the type of instrument is determined by the useful range of the frequencies, indicated in Fig. 10.10.

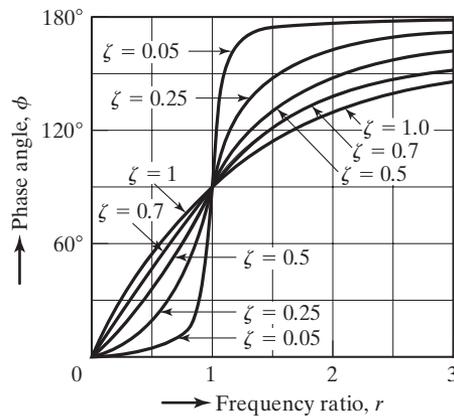
### 10.3.1 Vibrometer

A *vibrometer* or a *seismometer* is an instrument that measures the displacement of a vibrating body. It can be observed from Fig. 10.10 that  $Z/Y \approx 1$  when  $\omega/\omega_n \geq 3$  (range II). Thus the relative displacement between the mass and the base (sensed by the transducer) is essentially the same as the displacement of the base. For an exact analysis, we consider Eq. (10.19). We note that

$$z(t) \simeq Y \sin(\omega t - \phi) \tag{10.23}$$



**FIGURE 10.10** Response of a vibration-measuring instrument.



**FIGURE 10.11** Variation of  $\phi$  with  $r$ .

if

$$\frac{r^2}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \approx 1 \quad (10.24)$$

A comparison of Eq. (10.23) with  $y(t) = Y \sin \omega t$  shows that  $z(t)$  gives directly the motion  $y(t)$  except for the phase lag  $\phi$ . This phase lag can be seen to be equal to  $180^\circ$  for  $\zeta = 0$ . Thus the recorded displacement  $z(t)$  lags behind the displacement being measured  $y(t)$  by time  $t' = \phi/\omega$ . This time lag is not important if the base displacement  $y(t)$  consists of a single harmonic component.

Since  $r = \omega/\omega_n$  has to be large and the value of  $\omega$  is fixed, the natural frequency  $\omega_n = \sqrt{k/m}$  of the mass-spring-damper must be low. This means that the mass must be large and the spring must have a low stiffness. This results in a bulky instrument, which is not desirable in many applications. In practice, the vibrometer may not have a large value of  $r$  and hence the value of  $Z$  may not be equal to  $Y$  exactly. In such a case, the true value of  $Y$  can be computed by using Eq. (10.19), as indicated in the following example.

### EXAMPLE 10.2 Amplitude by Vibrometer

A vibrometer having a natural frequency of 4 rad/s and  $\zeta = 0.2$  is attached to a structure that performs a harmonic motion. If the difference between the maximum and the minimum recorded values is 8 mm, find the amplitude of motion of the vibrating structure when its frequency is 40 rad/s.

**Solution:** The amplitude of the recorded motion  $Z$  is 4 mm. For  $\zeta = 0.2$ ,  $\omega = 40.0$  rad/s, and  $\omega_n = 4$  rad/s,  $r = 10.0$ , and Eq. (10.19) gives

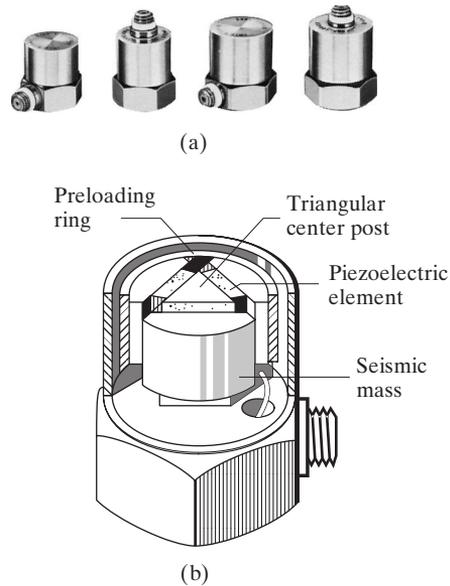
$$Z = \frac{Y(10)^2}{[(1 - 10^2)^2 + \{2(0.2)(10)\}^2]^{1/2}} = 1.0093Y$$

Thus the amplitude of vibration of the structure is  $Y = Z/1.0093 = 3.9631$  mm. ■

### 10.3.2 Accelerometer

An accelerometer is an instrument that measures the acceleration of a vibrating body (see Fig. 10.12). Accelerometers are widely used for vibration measurements [10.7] and also to record earthquakes. From the accelerometer record, the velocity and displacements are obtained by integration. Equations (10.18) and (10.19) yield

$$-z(t)\omega_n^2 = \frac{1}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \{-Y\omega^2 \sin(\omega t - \phi)\} \quad (10.25)$$



**FIGURE 10.12** Accelerometers.  
(Courtesy of Bruel and Kjaer Instruments, Inc.,  
Marlborough, MA.)

This shows that if

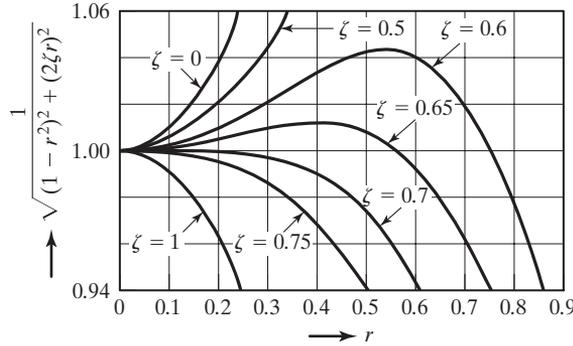
$$\frac{1}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \approx 1 \quad (10.26)$$

Eq. (10.25) becomes

$$-z(t)\omega_n^2 \approx -Y\omega^2 \sin(\omega t - \phi) \quad (10.27)$$

By comparing Eq. (10.27) with  $\ddot{y}(t) = -Y\omega^2 \sin \omega t$ , we find that the term  $z(t)\omega_n^2$  gives the acceleration of the base  $\ddot{y}$ , except for the phase lag  $\phi$ . Thus the instrument can be made to record (give) directly the value of  $\ddot{y} = -z(t)\omega_n^2$ . The time by which the record lags the acceleration is given by  $t' = \phi/\omega$ . If  $\ddot{y}$  consists of a single harmonic component, the time lag will not be of importance.

The value of the expression on the left-hand side of Eq. (10.26) is shown plotted in Fig. 10.13. It can be seen that the left-hand side of Eq. (10.26) lies between 0.96 and 1.04 for  $0 \leq r \leq 0.6$  if the value of  $\zeta$  lies between 0.65 and 0.7. Since  $r$  is small, the natural frequency of the instrument has to be large compared to the frequency of vibration to be measured. From the relation  $\omega_n = \sqrt{k/m}$ , we find that the mass needs to be small and the spring needs to have a large value of  $k$  (i.e., short spring), so the instrument will be small in size. Due



**FIGURE 10.13** Variation of lefthand side of Eq. (10.26) with  $r$ .

to their small size and high sensitivity, accelerometers are preferred in vibration measurements. In practice, Eq. (10.26) may not be satisfied exactly; in such cases the quantity

$$\frac{1}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}}$$

can be used to find the correct value of the acceleration measured, as illustrated in the following example.

**EXAMPLE 10.3** Design of an Accelerometer

An accelerometer has a suspended mass of 0.01 kg with a damped natural frequency of vibration of 150 Hz. When mounted on an engine undergoing an acceleration of 1g at an operating speed of 6000 rpm, the acceleration is recorded as 9.5 m/s<sup>2</sup> by the instrument. Find the damping constant and the spring stiffness of the accelerometer.

**Solution:** The ratio of measured to true accelerations is given by

$$\frac{1}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} = \frac{\text{Measured value}}{\text{True value}} = \frac{9.5}{9.81} = 0.9684 \tag{E.1}$$

which can be written as

$$[(1 - r^2)^2 + (2\zeta r)^2] = (1/0.9684)^2 = 1.0663 \tag{E.2}$$

The operating speed of the engine gives

$$\omega = \frac{6000(2\pi)}{60} = 628.32 \text{ rad/s}$$

The damped natural frequency of vibration of the accelerometer is

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n = 150(2\pi) = 942.48 \text{ rad/s}$$

Thus

$$\frac{\omega}{\omega_d} = \frac{\omega}{\sqrt{1 - \zeta^2} \omega_n} = \frac{r}{\sqrt{1 - \zeta^2}} = \frac{628.32}{942.48} = 0.6667 \quad (\text{E.3})$$

Equation (E.3) gives

$$r = 0.6667 \sqrt{1 - \zeta^2} \quad \text{or} \quad r^2 = 0.4444(1 - \zeta^2) \quad (\text{E.4})$$

Substitution of Eq. (E.4) into (E.2) leads to a quadratic equation in  $\zeta^2$  as

$$1.5801 \zeta^4 - 2.2714 \zeta^2 + 0.7576 = 0 \quad (\text{E.5})$$

The solution of Eq. (E.5) gives

$$\zeta^2 = 0.5260, 0.9115$$

or

$$\zeta = 0.7253, 0.9547$$

By choosing  $\zeta = 0.7253$  arbitrarily, the undamped natural frequency of the accelerometer can be found as

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}} = \frac{942.48}{\sqrt{1 - 0.7253^2}} = 1368.8889 \text{ rad/s}$$

Since  $\omega_n = \sqrt{k/m}$ , we have

$$k = m\omega_n^2 = (0.01)(1368.8889)^2 = 18738.5628 \text{ N/m}$$

The damping constant can be determined from

$$c = 2m\omega_n\zeta = 2(0.01)(1368.8889)(0.7253) = 19.8571 \text{ N-s/m}$$

■

### 10.3.3 Velometer

A velometer measures the velocity of a vibrating body. Equation (10.13) gives the velocity of the vibrating body

$$\dot{y}(t) = \omega Y \cos \omega t \quad (10.28)$$

and Eq. (10.18) gives

$$\dot{z}(t) = \frac{r^2 \omega Y}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \cos(\omega t - \phi) \quad (10.29)$$

If

$$\frac{r^2}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \simeq 1 \quad (10.30)$$

then

$$\dot{z}(t) \simeq \omega Y \cos(\omega t - \phi) \quad (10.31)$$

A comparison of Eqs. (10.28) and (10.31) shows that, except for the phase difference  $\phi$ ,  $\dot{z}(t)$  gives directly  $\dot{y}(t)$ , provided that Eq. (10.30) holds true. In order to satisfy Eq. (10.30),  $r$  must be very large. In case Eq. (10.30) is not satisfied, then the velocity of the vibrating body can be computed using Eq. (10.29).

### EXAMPLE 10.4 Design of a Velometer

Design a velometer if the maximum error is to be limited to 1 percent of the true velocity. The natural frequency of the velometer is to be 80 Hz and the suspended mass is to be 0.05 kg.

**Solution:** The ratio ( $R$ ) of the recorded and the true velocities is given by Eq. (10.29):

$$R = \frac{r^2}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} = \frac{\text{Recorded velocity}}{\text{True velocity}} \quad (E.1)$$

The maximum of (E.1) occurs when (see Eq. (3.82))

$$r = r^* = \frac{1}{\sqrt{1 - 2\zeta^2}} \quad (E.2)$$

Substitution of Eq. (E.2) into (E.1) gives

$$\frac{\left(\frac{1}{1-2\zeta^2}\right)}{\sqrt{\left[1-\left(\frac{1}{1-2\zeta^2}\right)\right]^2+4\zeta^2\left(\frac{1}{1-2\zeta^2}\right)}} = R$$

which can be simplified as

$$\frac{1}{\sqrt{4\zeta^2-4\zeta^4}} = R \quad (\text{E.3})$$

For an error of 1 percent,  $R = 1.01$  or  $0.99$ , and Eq. (E.3) leads to

$$\zeta^4 - \zeta^2 + 0.245075 = 0 \quad (\text{E.4})$$

and

$$\zeta^4 - \zeta^2 + 0.255075 = 0 \quad (\text{E.5})$$

Equation (E.5) gives imaginary roots and Eq. (E.4) gives

$$\zeta^2 = 0.570178, 0.429821$$

or

$$\zeta = 0.755101, 0.655607$$

We choose the value  $\zeta = 0.755101$  arbitrarily. The spring stiffness can be found as

$$k = m\omega_n^2 = 0.05(502.656)^2 = 12633.1527 \text{ N/m}$$

since

$$\omega_n = 80(2\pi) = 502.656 \text{ rad/s}$$

The damping constant can be determined from

$$c = 2\zeta\omega_n m = 2(0.755101)(502.656)(0.05) = 37.9556 \text{ N-s/m}$$

■

### 10.3.4 Phase Distortion

As shown by Eq. (10.18), all vibration-measuring instruments exhibit phase lag. Thus the response or output of the instrument lags behind the motion or input it measures. The time lag is given by the phase angle divided by the frequency  $\omega$ . The time lag is not important if we measure a single harmonic component. But, occasionally, the vibration to be recorded is not harmonic but consists of the sum of two or more harmonic components. In such a case, the recorded graph may not give an accurate picture of the vibration, because different harmonics may be amplified by different amounts and their phase shifts may also be different. The distortion in the waveform of the recorded signal is called the *phase distortion* or *phase-shift error*. To illustrate the nature of the phase-shift error, we consider a vibration signal of the form shown in Fig. 10.14(a) [10.10]:

$$y(t) = a_1 \sin \omega t + a_3 \sin 3\omega t \tag{10.32}$$

Let the phase shift be  $90^\circ$  for the first harmonic and  $180^\circ$  for the third harmonic of Eq. (10.32). The corresponding time lags are given by  $t_1 = \theta_1/\omega = 90^\circ/\omega$  and  $t_2 = \theta_2/(3\omega) = 180^\circ/(3\omega)$ . The output signal is shown in Fig. 10.14(b). It can be seen that the output signal is quite different from the input signal due to phase distortion.

As a general case, let the complex wave being measured be given by the sum of several harmonics as

$$y(t) = a_1 \sin \omega t + a_2 \sin 2\omega t + \dots \tag{10.33}$$

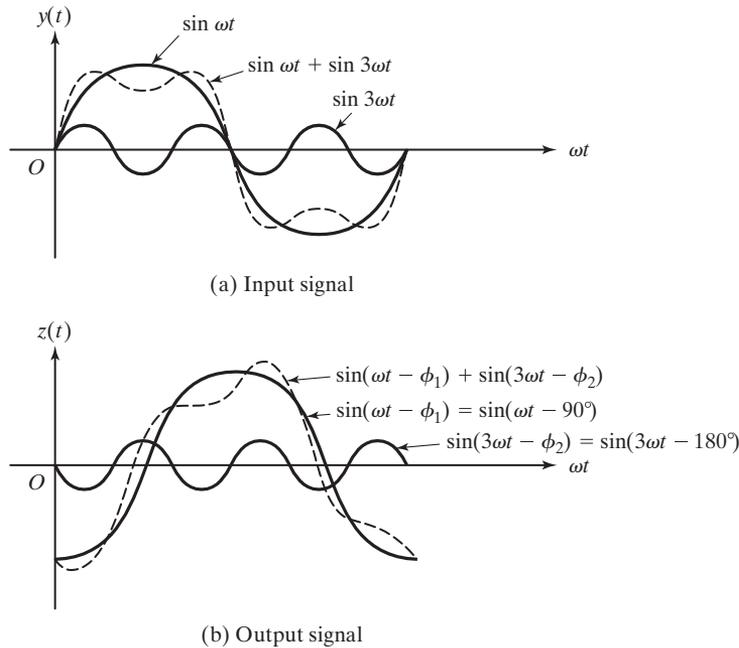


FIGURE 10.14 Phase-shift error.

If the displacement is measured using a vibrometer, its response to each component of the series is given by an equation similar to Eq. (10.18), so that the output of the vibrometer becomes

$$z(t) = a_1 \sin(\omega t - \phi_1) + a_2 \sin(2\omega t - \phi_2) + \dots \quad (10.34)$$

where

$$\tan \phi_j = \frac{2\zeta \left( j \frac{\omega}{\omega_n} \right)}{1 - \left( j \frac{\omega}{\omega_n} \right)^2}, \quad j = 1, 2, \dots \quad (10.35)$$

Since  $\omega/\omega_n$  is large for this instrument, we can find from Fig. 10.11 that  $\phi_j \simeq \pi$ ,  $j = 1, 2, \dots$ , and Eq. (10.34) becomes

$$z(t) \simeq -[a_1 \sin \omega t + a_2 \sin 2\omega t + \dots] \simeq -y(t) \quad (10.36)$$

Thus the output record will be simply opposite to the motion being measured. This is unimportant and can easily be corrected.

By using a similar reasoning, we can show, in the case of a velometer, that

$$\dot{z}(t) \simeq -\dot{y}(t) \quad (10.37)$$

for an input signal consisting of several harmonics. Next we consider the phase distortion for an accelerometer. Let the acceleration curve to be measured be expressed, using Eq. (10.33), as

$$\ddot{y}(t) = -a_1 \omega^2 \sin \omega t - a_2 (2\omega)^2 \sin 2\omega t - \dots \quad (10.38)$$

The response or output of the instrument to each component can be found as in Eq. (10.34), and so

$$\ddot{z}(t) = -a_1 \omega^2 \sin(\omega t - \phi_1) - a_2 (2\omega)^2 \sin(2\omega t - \phi_2) - \dots \quad (10.39)$$

where the phase lags  $\phi_j$  are different for different components of the series in Eq. (10.39). Since the phase lag  $\phi$  varies almost linearly from  $0^\circ$  at  $r = 0$  to  $90^\circ$  at  $r = 1$  for  $\zeta = 0.7$  (see Fig. 10.11), we can express  $\phi$  as

$$\phi \simeq \alpha r = \alpha \frac{\omega}{\omega_n} = \beta \omega \quad (10.40)$$

where  $\alpha$  and  $\beta = \alpha/\omega_n$  are constants. The time lag is given by

$$t' = \frac{\phi}{\omega} = \frac{\beta\omega}{\omega} = \beta \quad (10.41)$$

This shows that the time lag of the accelerometer is independent of the frequency for any component, provided that the frequency lies in the range  $0 \leq r \leq 1$ . Since each component of the signal has the same time delay or phase lag, we have, from Eq. (10.39),

$$\begin{aligned} -\omega^2 \ddot{z}(t) &= -a_1 \omega^2 \sin(\omega t - \omega\beta) - a_2 (2\omega)^2 \sin(2\omega t - 2\omega\beta) - \dots \\ &= -a_1 \omega^2 \sin \omega\tau - a_2 (2\omega)^2 \sin 2\omega\tau - \dots \end{aligned} \quad (10.42)$$

where  $\tau = t - \beta$ . Note that Eq. (10.42) assumes that  $0 \leq r \leq 1$ —that is, even the highest frequency involved,  $n\omega$ , is less than  $\omega_n$ . This may not be true in practice. Fortunately, no significant phase distortion occurs in the output signal, even when some of the higher-order frequencies are larger than  $\omega_n$ . The reason is that, generally, only the first few components are important to approximate even a complex waveform; the amplitudes of the higher harmonics are small and contribute very little to the total waveform. Thus the output record of the accelerometer represents a reasonably true acceleration being measured [10.7, 10.11].

## 10.4 Frequency-Measuring Instruments

Most frequency-measuring instruments are of the mechanical type and are based on the principle of resonance. Two kinds are discussed in the following paragraphs: the Fullarton tachometer and the Frahm tachometer.

**Single-Reed Instrument or Fullarton Tachometer.** This instrument consists of a variable-length cantilever strip with a mass attached at one of its ends. The other end of the strip is clamped, and its free length can be changed by means of a screw mechanism (see Fig. 10.15(a)). Since each length of the strip corresponds to a different natural frequency, the reed is marked along its length in terms of its natural frequency. In practice, the clamped end of the strip is pressed against the vibrating body, and the screw mechanism is manipulated to alter its free length until the free end shows the largest amplitude of vibration. At that instant, the excitation frequency is equal to the natural frequency of the cantilever; it can be read directly from the strip.

**Multireed-Instrument or Frahm Tachometer.** This instrument consists of a number of cantilevered reeds carrying small masses at their free ends (see Fig. 10.15(b)). Each reed has a different natural frequency and is marked accordingly. Using a number of reeds makes it possible to cover a wide frequency range. When the instrument is mounted on a vibrating body, the reed whose natural frequency is nearest the unknown frequency of the