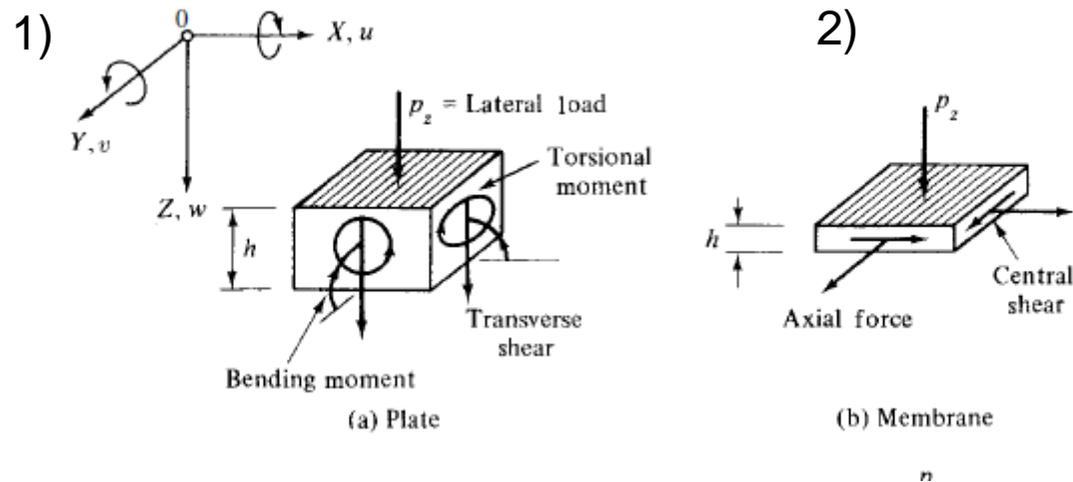


# Le piastre

1. piastre sottili  $h/L = 1/50 - 1/10$  : piastre sottili con rigidezza flessionale che portano distribuzioni di carico bidimensionale prevalentemente attraverso momenti flettenti, momenti torcenti e taglio in una maniera simile alle travi
2. Membrane  $h/L < 1/50$ : piastre molto sottili senza rigidezza flessionale che portano carichi prevalentemente attraverso azioni membranali assiali e taglio centrale . La capacità di sopportare i carichi può essere assimilata ad una reticolo di cavi in trazione dal momento che il momento resistente è trascurabile



# Le piastre: classificazione

3. piastre moderatamente spesse  $h/L = 1/10 - 1/5$  : piastre in cui l'effetto del taglio sulle componenti normali viene messo in conto
4. Piastre spesse  $h/L > 1/5$ : stato di sollecitazione tridimensionale

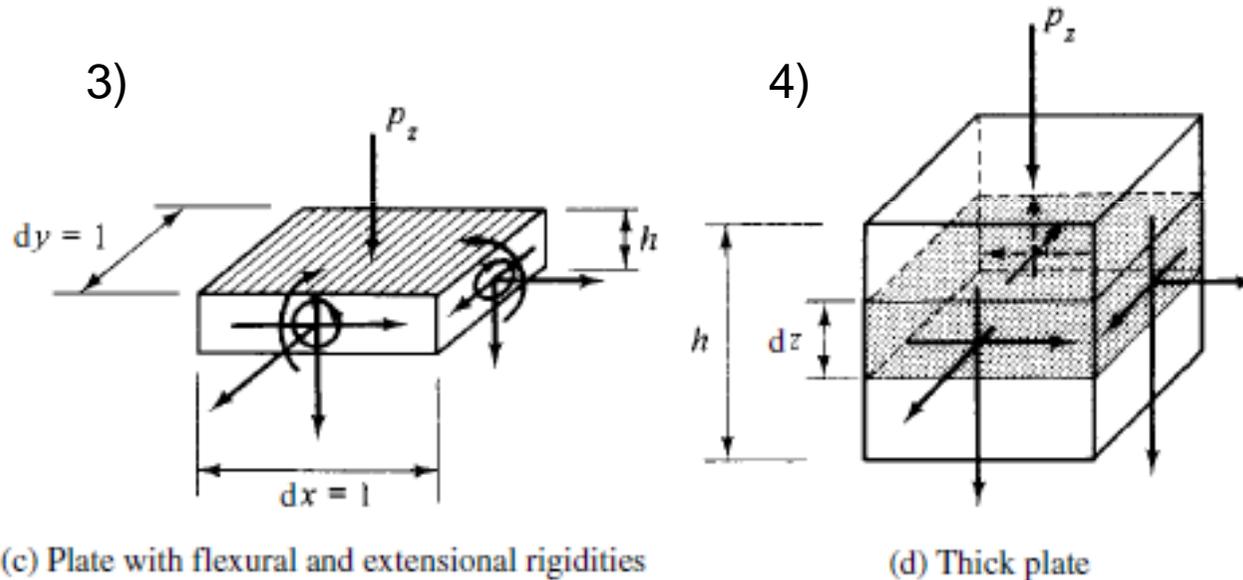
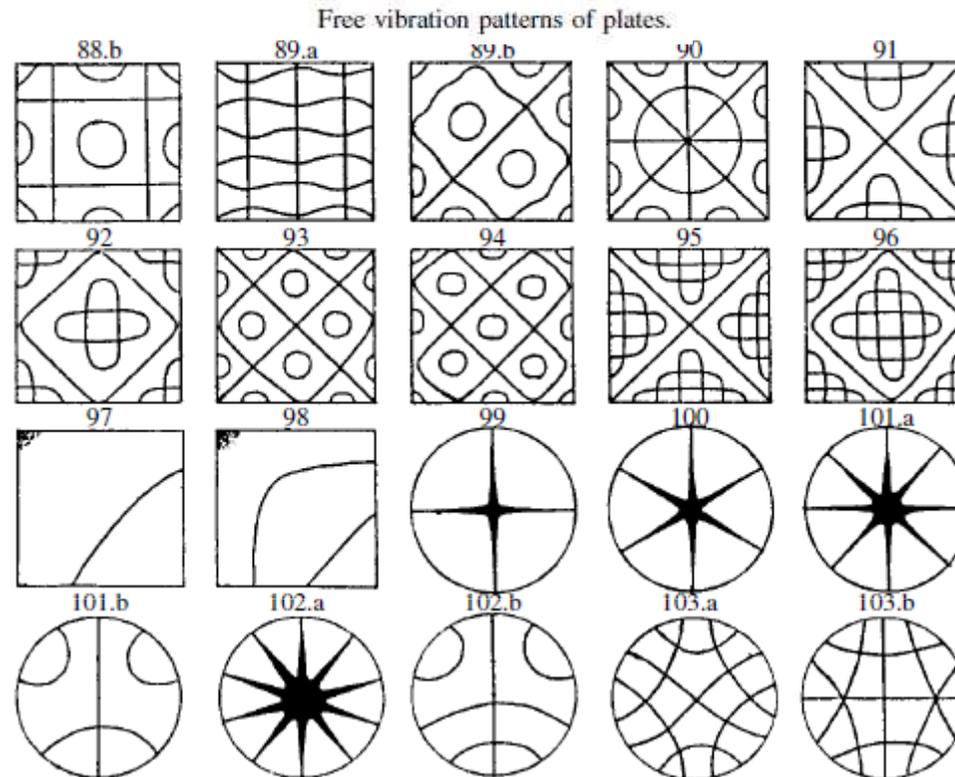


Figure I.3 Internal forces in various types of plate elements.

# Studio delle piastre: breve storia

The first mathematical approach to the membrane theory of very thin plates was formulated by L. Euler (1707–1783) in 1766. Euler solved the problems of free vibrations of rectangular, triangular and circular elastic membranes by using the analogy of two systems of stretched strings perpendicular to each other [II.2]. His student, Jacques Bernoulli (1759–1789), extended Euler's analogy to plates by replacing the net of strings with a gridwork of beams [II.3] having only bending rigidity. Since the torsional resistance of the beams was not included in the so-obtained differential equation of plates, he found only general *resemblance* between his theory and experiments but no close agreement.

# Studio delle piastre: breve storia



A real impetus to the research of plate vibrations, however, was given by the German physicist E. F. F. Chladni (1756–1827). In his book on acoustics [II.4], he described diverse experiments with vibrating plates. Chladni discovered various modes of free vibrations. In his experiments he used evenly distributed powder that formed regular patterns after introducing vibrations (Fig. II.1). The powder accumulated along the nodal lines, where no displacement occurred. In addition, he was able to determine the frequencies corresponding to these vibration patterns.

# Studio delle piastre: breve storia

Chladni



Invited by the French Academy of Science in 1809, he demonstrated his experiments in Paris. Chladni's presentation was also attended by Emperor Napoleon, who was duly impressed by his demonstration. Following Napoleon's suggestion, the French Academy invited applications for a price essay dealing with the mathematical theory of plate vibrations substantiated by experimental verification of the theoretical results.<sup>†</sup> Since, at first, no papers were submitted, the delivery date had to be extended twice. Finally, in October 1811, on the closing day of the competition, the Academy received only one paper, entitled "Reserches sur la théorie des surfaces élastiques," written by the mathematician Mlle. Germain (S. Germain, "L'état des sciences et des Lettres," Paris, 1833).



# Studio delle piastre: breve storia

In her first work on the theory of plate vibration, she used (following Euler's previous work on elastic curves) a strain energy approach. But in evaluating the strain energy using the virtual work technique, she made a mistake and obtained an erroneous differential equation for the free vibration of plate in the following form:

$$\frac{\partial^2 z}{\partial t^2} + \lambda^2 \left( \frac{\partial^6 z}{\partial x^4 \partial y^2} + \frac{\partial^6 z}{\partial x^2 \partial y^4} \right) = 0, \quad (\text{II.1})$$

where  $z(x, y, t)$  represents the middle surface of the plate in motion expressed in an  $X, Y, Z$  Cartesian coordinate system,  $t$  is the time and  $\lambda^2$  denotes a constant containing physical properties of the vibrating plate. This constant was, however, not clearly defined in her paper. Lagrange, who was one of the judges, noticed this mathematical error and corrected it. The so-obtained differential equation now correctly describing the free vibrations of plates reads

$$\frac{\partial^2 z}{\partial t^2} + k^2 \left( \frac{\partial^4 z}{\partial x^4} + 2 \frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \right) = 0. \quad (\text{II.2})$$

Since the judges were not entirely satisfied with Germain's work, they proposed the subject again. In October 1813, Mlle. Germain entered the now-corrected equation (II.2) but left out the precise definition of the constant  $k^2$ . Thus, she was again unsuccessful. Finally, at her third approach, she won the prize in 1816. But the judges criticized anew her definition of the constant  $k^2$  since she had thought that it contains the fourth power of the plate thickness instead of the correct value of  $h^3$ . Although her original works are very hard to read and contain some dubious mathematical and physical reasonings, she must, nevertheless, be admired for her courage, devotion and persistence. The claim of priority for writing the first valid differential equation describing free plate vibrations belongs—without any doubt—to her!

# Studio delle piastre: breve storia

Next, the mathematician L. D. Poisson (1781–1840) made an attempt to determine the correct value of the constant  $k^2$  in the differential equation of plate vibration (II.2).

Navier applied Bernoulli's hypotheses,<sup>‡</sup> which were already successfully used for treating bending of beams, adding to them the two-dimensional actions of strains and stresses, respectively. In his paper on this subject (published in 1823), he correctly defined the governing differential equation of plates subjected to static, lateral loads  $p_z$  as

$$D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = p_z(x, y). \quad (\text{II.3})$$

In this equation  $D$  denotes the flexural rigidity of the plate, which is now proportional to the cube of the plate thickness, whereas  $w(x, y)$  represents the deflected middle surface.

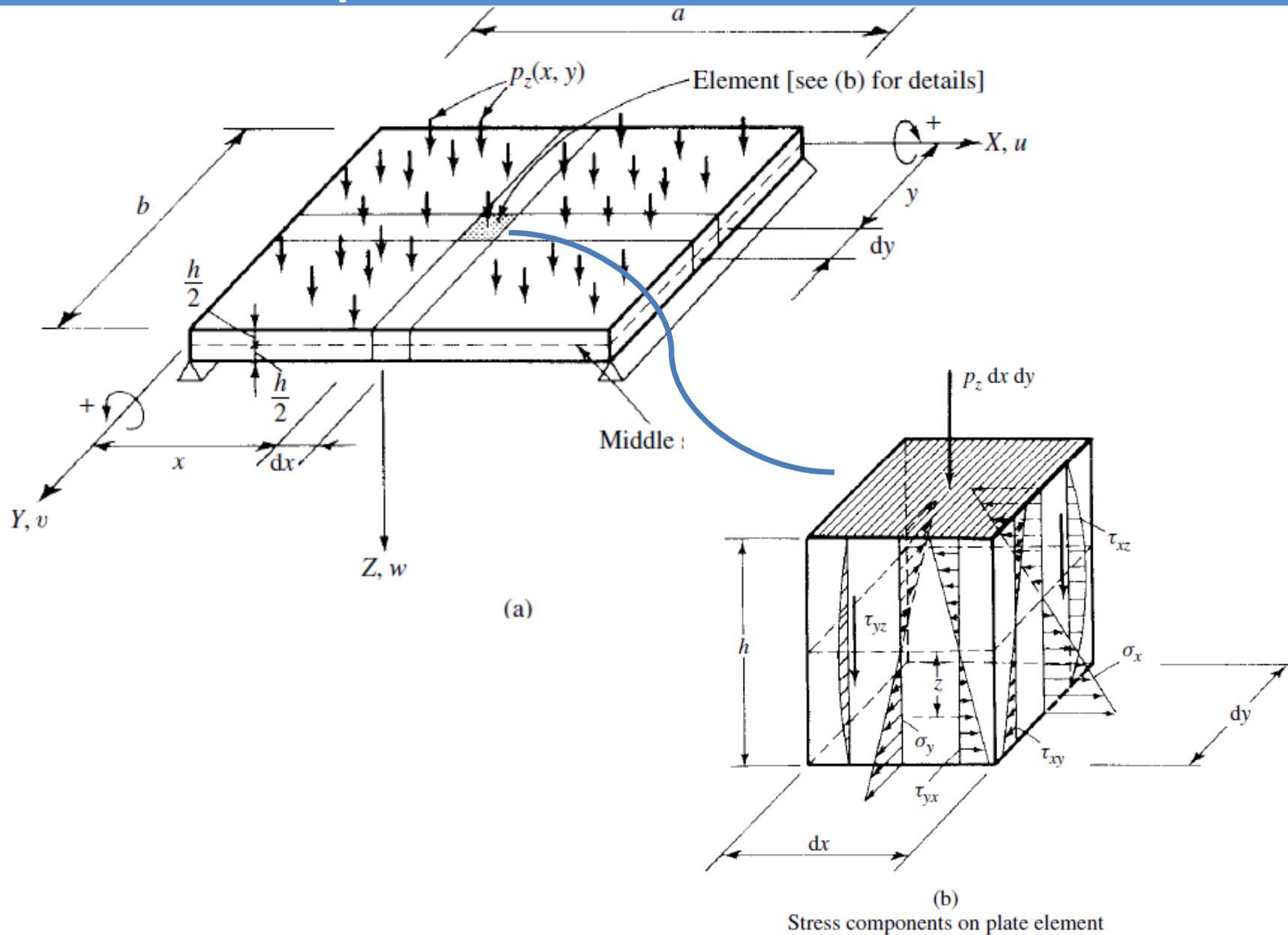
# Studio delle piastre: breve storia

In Germany, publication of Kirchhoff's book entitled *Lectures on Mathematical Physics, Mechanics* (in German) [II.7] created a similar impact on engineering science as that of Navier's *Leçons* in France. Gustav R. Kirchhoff (1824–1887), whose picture is shown in Fig. II.3, developed the first *complete* theory of plate bending. In his earlier paper on this subject, published in 1850, he summarized, first, the previous works done by French scientists in this field, but he failed to mention Navier's above-discussed achievements. Based on Bernoulli's hypotheses for beams, Kirchhoff derived the same differential equation for plate bending (II.3) as Navier, however, using a different energy approach. His very important contribution to plate theory was the introduction of supplementary boundary forces. These "equivalent shear forces"<sup>†</sup> replace, in fact, the torsional moments at the plate boundaries. Consequently, all boundary conditions could now be stated in functions of displacements and their derivatives with respect to  $x$  or  $y$ . Furthermore, Kirchhoff is considered to be the founder of the extended plate theory, which takes into account the combined bending and stretching. In anal  
could no longer be ne  
ment of a frequency e



found that nonlinear terms  
contributions are the develop-  
tion of *virtual displacement*

# Le piastre: stato tensionale



# Teoria di Kirchhoff delle piastre sottili

## Notazioni

Spostamenti  $\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$

Deformazioni  $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ v_{,z} + w_{,y} \\ u_{,z} + w_{,x} \\ u_{,y} + v_{,x} \end{Bmatrix}$

Tensioni  $\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$

## Legame costitutivo

$$\begin{aligned} \varepsilon_x &= \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] & \gamma_{yz} &= \frac{2(1+\nu)}{E} \tau_{yz} \\ \varepsilon_y &= \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right] & \gamma_{xz} &= \frac{2(1+\nu)}{E} \tau_{xz} \\ \varepsilon_z &= \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] & \gamma_{xy} &= \frac{2(1+\nu)}{E} \tau_{xy} \end{aligned}$$

# Teoria di Kirchhoff delle piastre sottili

**1) Ipotesi sulla tensione: la tensione normale in direzione dello spessore della piastra è trascurabile**

$$\sigma_z = 0$$

**2) Ipotesi sulle deformazioni: la dilatazione lineare in direzione dello spessore della piastra è trascurabile**

$$\varepsilon_z = 0$$

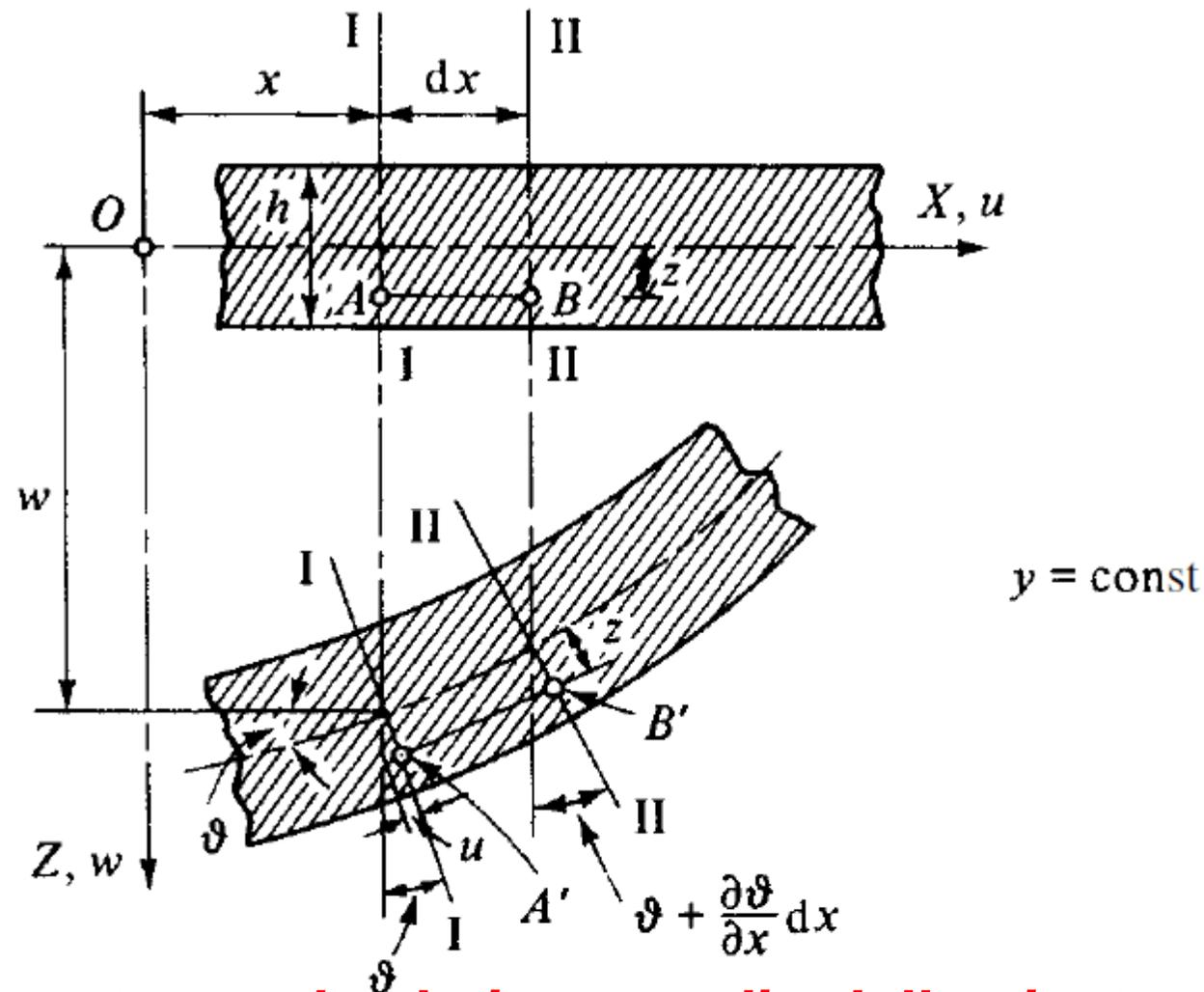
**3) Gli scorrimenti angolari tra il piano della piastra e la fibra ortogonale a tale piano sono trascurabili**

$$\gamma_{yz} = \gamma_{xz} = 0$$



**Ipotesi valida solo nella teoria di Kirchhoff e che non influenza le equazioni di equilibrio in termini di sforzi generalizzati**

# Teoria di Kirchhoff delle piastre sottili



**la fibra ortogonale al piano medio della piastra risulta ortogonale alla superficie media della piastra a deformazione avvenuta**

# Teoria di Kirchhoff delle piastre sottili

1. Trascurabilità della tensione normale nello spessore, i.e.  $\sigma_z = 0$ :

$$\begin{aligned}\varepsilon_x &= \frac{1}{E} [\sigma_x - \nu\sigma_y] & \gamma_{yz} &= \frac{2(1+\nu)}{E} \tau_{yz} \\ \varepsilon_y &= \frac{1}{E} [\sigma_y - \nu\sigma_x] & \gamma_{xz} &= \frac{2(1+\nu)}{E} \tau_{xz} \\ \varepsilon_z &= -\nu \frac{1}{E} [\sigma_x + \sigma_y] & \gamma_{xy} &= \frac{2(1+\nu)}{E} \tau_{xy}\end{aligned}$$

Invertendo la relazione di legame costitutivo si ottiene:

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2} \varepsilon_x + \frac{\nu E}{1-\nu^2} \varepsilon_y & \tau_{yz} &= \frac{E}{2(1+\nu)} \gamma_{yz} \\ \sigma_y &= \frac{\nu E}{1-\nu^2} \varepsilon_x + \frac{E}{1-\nu^2} \varepsilon_y & \tau_{xz} &= \frac{E}{2(1+\nu)} \gamma_{xz} \\ & & \tau_{xy} &= \frac{E}{2(1+\nu)} \gamma_{xy}\end{aligned}$$

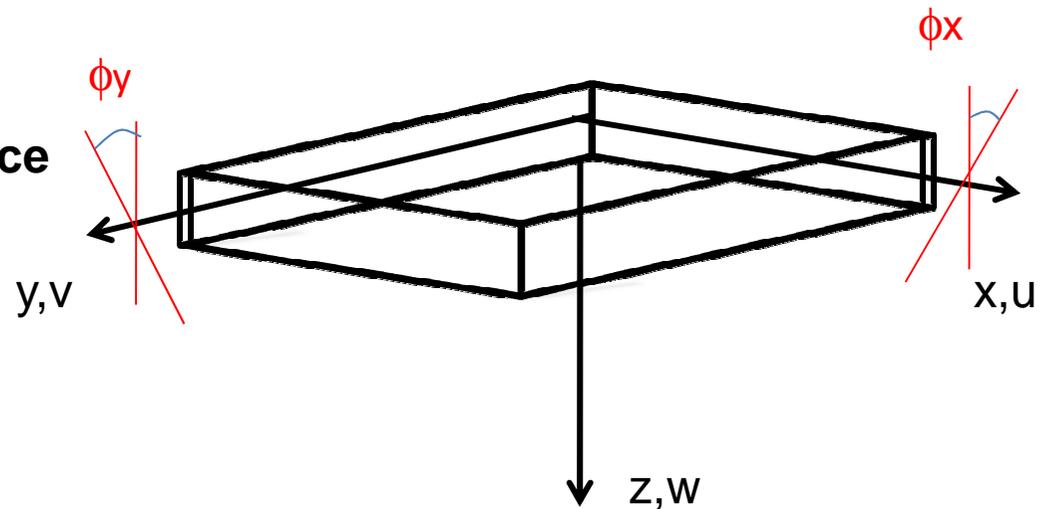
# Ipotesi cinematiche: dilatazione lineare in z e scorrimenti angolari in z trascurabili

**Ipotesi sulla dilatazione lineare**

$$\varepsilon_z = 0 \quad \Rightarrow \quad w_{,z} = 0$$

**Che integrata nello spessore fornisce**

$$w = w(x, y)$$



**Ipotesi sugli scorrimenti angolari**

$$\gamma_{yz} = 0 \quad \Rightarrow \quad v_{,z} + w_{,y} = 0$$

$$\gamma_{xz} = 0 \quad \Rightarrow \quad u_{,z} + w_{,x} = 0$$

**Che integrate nello spessore forniscono**

$$u = u_0 - zw_{,x}$$

$$v = v_0 - zw_{,y}$$

# Ipotesi cinematiche: dilatazione lineare in z e scorrimenti angolari in z trascurabili

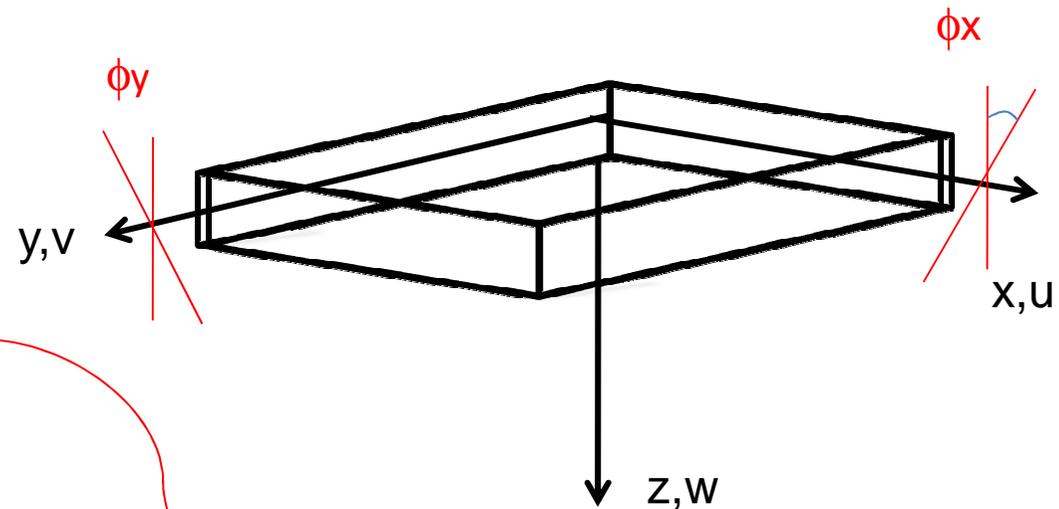
Il campo di spostamenti diventa

$$u(x, y) = u_0(x, y) - zw_{,x}(x, y)$$

$$v(x, y) = v_0(x, y) - zw_{,y}(x, y)$$

$$w = w(x, y)$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} + z \begin{Bmatrix} -w_{,x} \\ -w_{,y} \end{Bmatrix}$$



**Rotazione  $\phi_x$  positiva se  
antioraria nel piano xz  
attorno asse y**

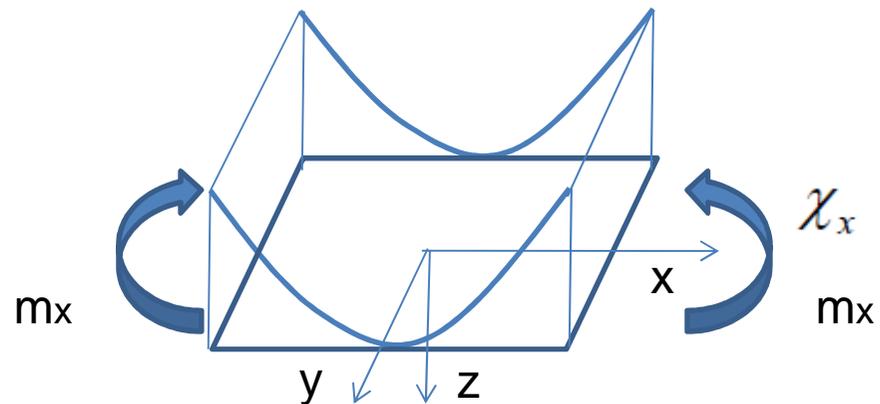
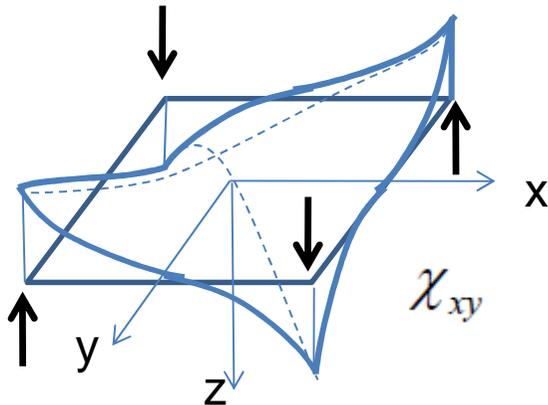
**Rotazione  $\phi_y$  positiva se  
antioraria nel piano yz  
attorno asse x**

# Teoria di Kirchhoff delle piastre sottili

Introduciamo le curvatures come le derivate delle rotazioni  
cambiate di segno

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{Bmatrix} \begin{Bmatrix} -w_{,x} \\ -w_{,y} \end{Bmatrix} = S \begin{Bmatrix} -w_{,x} \\ -w_{,y} \end{Bmatrix}$$

dove  $\chi_x$  : Curvature flessionali  
 $\chi_y$   
 $\chi_{xy}$  : Curvatura torsionale



# Teoria di Kirchhoff delle piastre sottili

## Deformazioni

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} + z \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \quad \text{dove} \quad \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} \begin{Bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{Bmatrix} \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix}$$

## Tensioni

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \mathbf{D} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

**Dove in uno stato piano di tensione il tensore costitutivo è**

$$\mathbf{D} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix}$$

# Sforzi specifici

Integrando sullo spessore si ottengono gli sforzi specifici per unità di lunghezza N/m

$$\begin{aligned} \mathbf{N} &= \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \boldsymbol{\sigma} dz = \int_{-h/2}^{h/2} \mathbf{D} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} dz = \int_{-h/2}^{h/2} \mathbf{D} \left( \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} + z \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \right) dz \\ &= \int_{-h/2}^{h/2} \mathbf{D} dz \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} + \int_{-h/2}^{h/2} z \mathbf{D} dz \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \mathbf{D}^m \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} \\ \mathbf{D}^m &= \int_{-h/2}^{h/2} \mathbf{D} dz = h \mathbf{D} \end{aligned}$$

Si annulla in quanto integrale di funzione dispari su dominio pari

# Momenti specifici

Integrando sullo spessore si ottengono i momenti specifici per unità di lunghezza Nm/m

$$\begin{aligned}\mathbf{M} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= \int_{-h/2}^{h/2} z \boldsymbol{\sigma} dz = \int_{-h/2}^{h/2} z \mathbf{D} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} dz = \int_{-h/2}^{h/2} z \mathbf{D} \left( \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} + z \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \right) dz \\ &= \int_{-h/2}^{h/2} z \mathbf{D} dz \begin{Bmatrix} e_x \\ e_y \\ e_{xy} \end{Bmatrix} + \int_{-h/2}^{h/2} z^2 \mathbf{D} dz \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \mathbf{D}^f \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \\ \mathbf{D}^f &= \int_{-h/2}^{h/2} z^2 \mathbf{D} dz = \frac{h^3}{12} \mathbf{D}\end{aligned}$$

# Sforzi membranali generalizzati

$$N_x = \frac{Eh}{1-\nu^2} e_x + \frac{\nu Eh}{1-\nu^2} e_y = \frac{Eh}{1-\nu^2} u_{o,x} + \frac{\nu Eh}{1-\nu^2} v_{o,y}$$

$$N_y = \frac{\nu Eh}{1-\nu^2} e_x + \frac{Eh}{1-\nu^2} e_y = \frac{\nu Eh}{1-\nu^2} u_{o,x} + \frac{Eh}{1-\nu^2} v_{o,y}$$

$$N_{xy} = \frac{Eh}{2(1+\nu)} e_{xy} = \frac{Eh}{2(1+\nu)} (u_{o,y} + v_{o,x})$$

# Sforzi flessionali : momenti specifici o generalizzati

$$M_x = \frac{E h^3}{12(1-\nu^2)} \chi_x + \frac{\nu E h^3}{12(1-\nu^2)} \chi_y = -\frac{E h^3}{12(1-\nu^2)} w_{,xx} - \frac{\nu E h^3}{12(1-\nu^2)} w_{,yy}$$

$$M_y = \frac{\nu E h^3}{12(1-\nu^2)} \chi_x + \frac{E h^3}{12(1-\nu^2)} \chi_y = -\frac{\nu E h^3}{12(1-\nu^2)} w_{,xx} - \frac{E h^3}{12(1-\nu^2)} w_{,yy}$$

$$M_{xy} = \frac{E h^3}{12 \cdot 2(1+\nu)} \chi_{xy} = -\frac{E h^3}{12(1+\nu)} w_{,xy}$$

# Sforzi membranali : equazioni di equilibrio

**Equilibrio alla traslazione lungo x e y**

$$(N_x + \Delta N_x - N_x) \Delta y + (N_{xy} + \Delta N_{xy} - N_{xy}) \Delta x + f_x \Delta x \Delta y = 0$$

$$(N_y + \Delta N_y - N_y) \Delta x + (N_{xy} + \Delta N_{xy} - N_{xy}) \Delta y + f_y \Delta x \Delta y = 0$$

**Semplificando e dividendo per  $\Delta x \Delta y$**

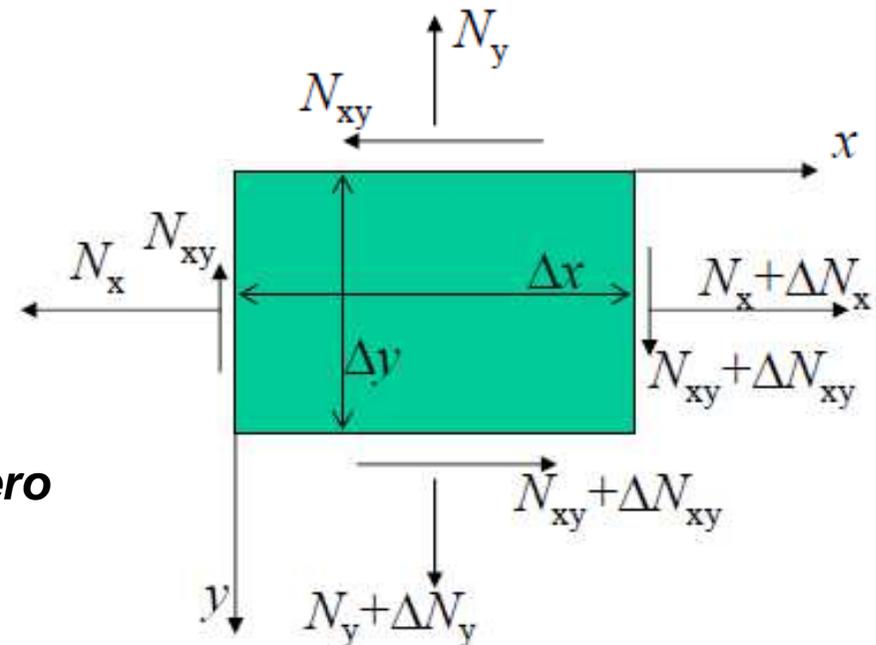
$$\frac{\Delta N_x}{\Delta x} + \frac{\Delta N_{xy}}{\Delta y} + f_x = 0$$

$$\frac{\Delta N_{xy}}{\Delta x} + \frac{\Delta N_y}{\Delta y} + f_y = 0$$

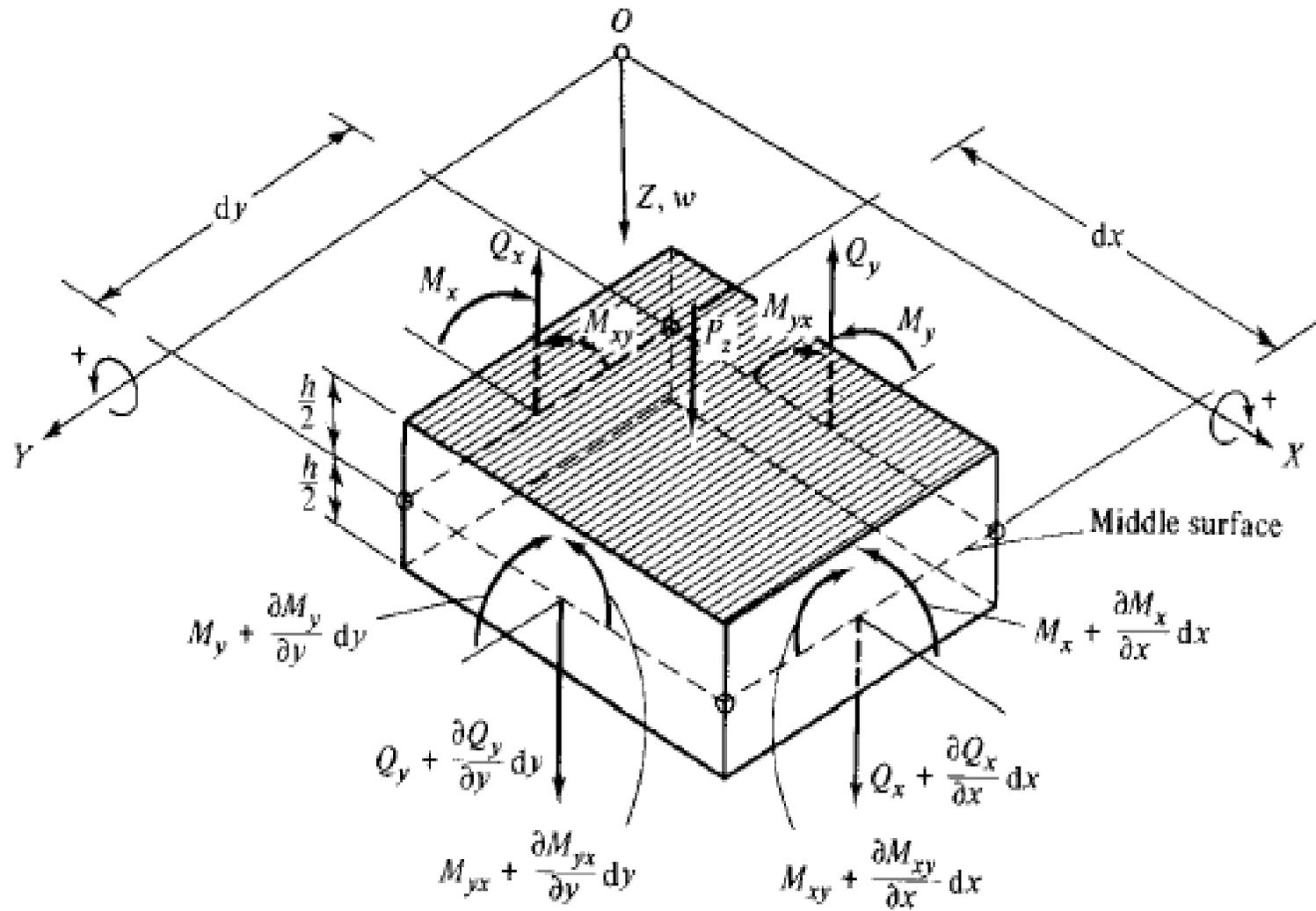
**nel limite per  $\Delta x$  e  $\Delta y$  che tendono a zero**

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + f_y = 0$$



# Sforzi flessionali



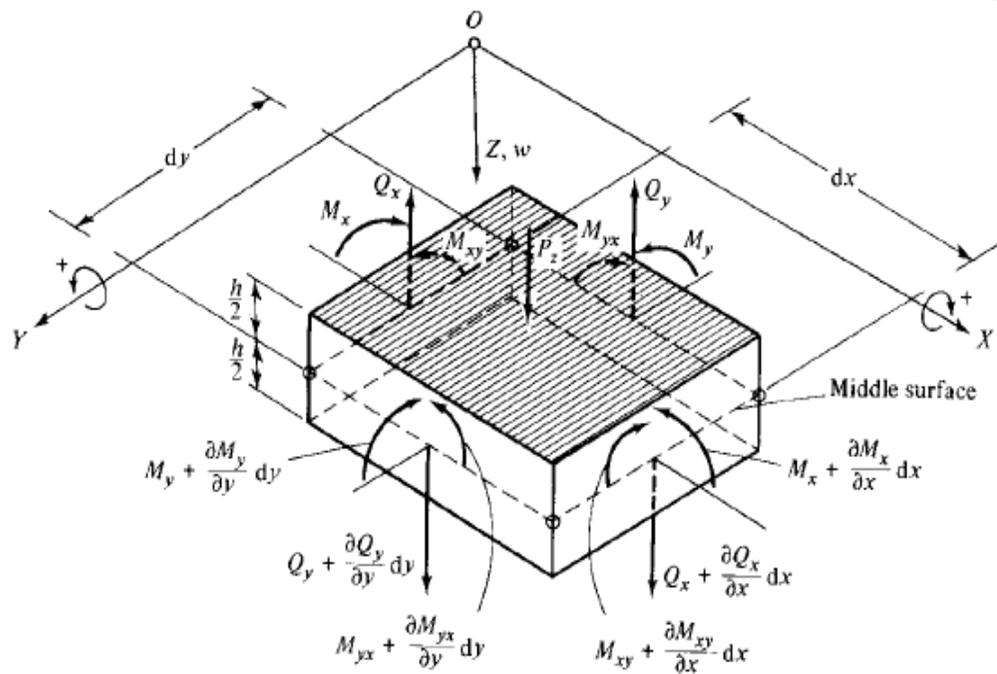
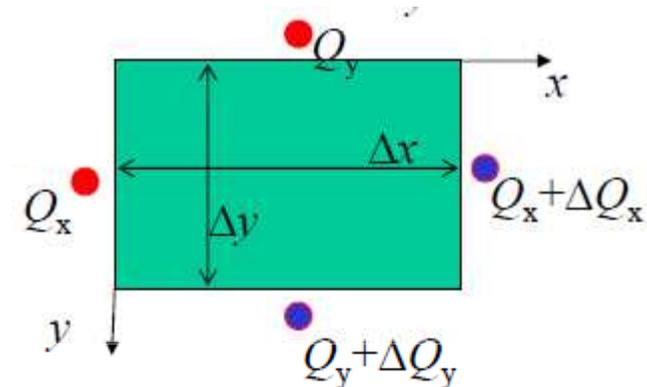
(a) Detailed illustration

# Sforzi flessionali: Equazioni di equilibrio

Equilibrio alla traslazione lungo z.

$$(\mathcal{Q}_x + \Delta\mathcal{Q}_x - \mathcal{Q}_x)\Delta y + (\mathcal{Q}_y + \Delta\mathcal{Q}_y - \mathcal{Q}_y)\Delta x + q\Delta x\Delta y = 0$$

$$\frac{\partial \mathcal{Q}_x}{\partial x} + \frac{\partial \mathcal{Q}_y}{\partial y} + q = 0$$



(a) Detailed illustration

# Equilibrio alla rotazione attorno ad y ed x

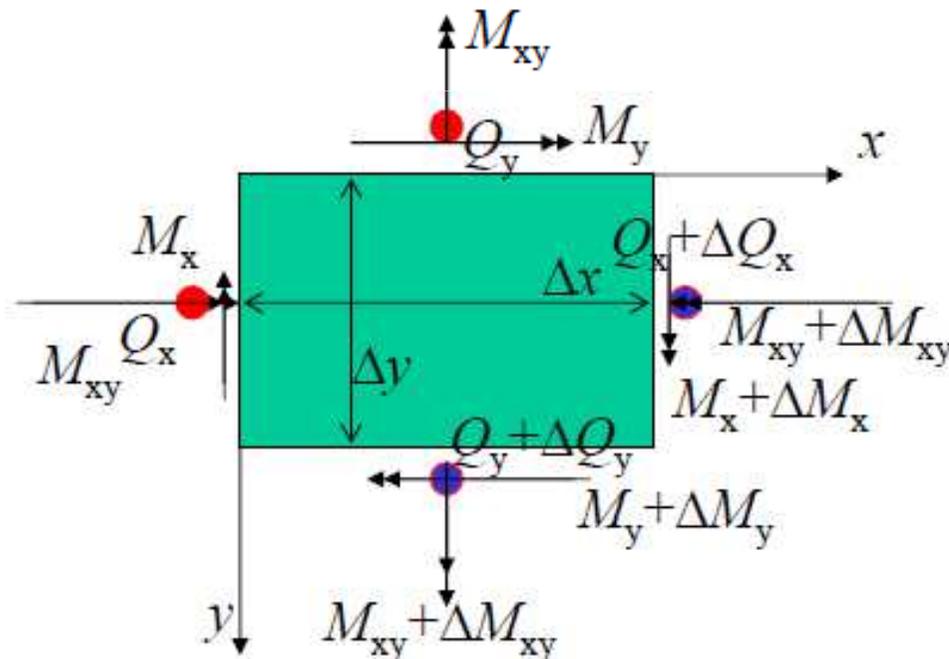
$$(M_x + \Delta M_x - M_x) \Delta y + (M_{xy} + \Delta M_{xy} - M_{xy}) \Delta x - (Q_x + \Delta Q_x) \Delta x \Delta y - q \frac{\Delta x^2 \Delta y}{2} = 0$$

$$(M_y + \Delta M_y - M_y) \Delta x + (M_{xy} + \Delta M_{xy} - M_{xy}) \Delta y - (Q_y + \Delta Q_y) \Delta x \Delta y - q \frac{\Delta x \Delta y^2}{2} = 0$$

Semplificando, dividendo per  $\Delta x \Delta y$ , ed eseguendo il limite per  $\Delta x$  e  $\Delta y$  che tendono a 0 si ottiene

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0$$



# Teoria di Kirchhoff delle piastre sottili

In definitiva le equazioni di equilibrio sono

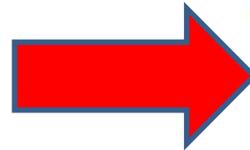
$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + f_y = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0$$



$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + f_y = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

# Equazione di Sophie Germain Lagrange

Sostituendo gli spostamenti

$$\frac{Eh}{1-\nu^2} u_{0,xx} + \frac{\nu Eh}{1-\nu^2} v_{0,xy} + \frac{Eh}{2(1+\nu)} (u_{0,yy} + v_{0,xy}) + f_x = 0$$

$$\frac{Eh}{2(1+\nu)} (u_{0,xy} + v_{0,xx}) + \frac{\nu Eh}{1-\nu^2} u_{0,xy} + \frac{Eh}{1-\nu^2} v_{0,yy} + f_y = 0$$

$$\frac{Eh^3}{12} \left[ \frac{1}{1-\nu^2} w_{,xxxx} + \frac{\nu}{1-\nu^2} w_{,xxyy} + \frac{2}{1+\nu} w_{,xxyy} + \frac{\nu}{1-\nu^2} w_{,xxyy} + \frac{1}{1-\nu^2} w_{,yyyy} \right] = q$$

In particolare l'ultima equazione diventa

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} = \frac{q}{D} \quad \text{dove} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

**Rigidezza  
flessionale  
della piastra  
inflessa**

Detta equazione di Sophie Germain Lagrange  
Anche scritta in forma compatta come

$$\nabla^4 w = \frac{q}{D}$$

# Teoria di Kirchhoff delle piastre sottili

$$\nabla^4 w = \frac{q}{D}$$

**Una soluzione esatta del problema governato dall'equazione di Sophie-Germain-Lagrange deve soddisfare l'equazione stessa sotto le opportune condizioni al contorno. Essendo un'equazione del IV ordine (8 costanti da determinare), occorrono 2 condizioni al contorno su ogni bordo:**

**-Condizioni al contorno di tipo CINEMATICO**

**-Condizioni al contorno di tipo STATICO: momento flettente, momento torcente e taglio**

# Teoria di Kirchhoff delle piastre sottili

Osserviamo che i tagli  $Q_x$  e  $Q_y$  non possono essere introdotti come sforzi generalizzati a causa del fatto che  $\gamma_{xz}$  e  $\gamma_{yz}$  sono nulli

Essi sono introdotti via equilibrio come gli sforzi staticamente equivalenti ai momenti flettente e torcente

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0$$

# Teoria di Kirchhoff delle piastre sottili

Kirchhoff ha dimostrato che la condizione al bordo relativa alla componente tagliante deve essere

$$w = \bar{w} \quad \text{oppure} \quad T_n + \frac{\partial M_{sn}}{\partial s} = V - \frac{\partial W_{sn}}{\partial s} \quad \text{su } \Gamma$$

**Dove :**

$T_n$  è lo sforzo trasversale relativo al bordo  $\Gamma$  di normale  $n$

$M_{sn}$  è il momento torcente generalizzato relativo al bordo  $\Gamma$  di normale  $n$  in direzione  $s$

$V$  rappresenta un'azione tagliante **esterna** applicata sul bordo  $\Gamma$  di normale  $n$  nota

$W_{sn}$  rappresenta un momento torcente **esterno** applicato sul bordo  $\Gamma$  di normale  $n$  in direzione  $s$

# Teoria di Kirchhoff delle piastre sottili

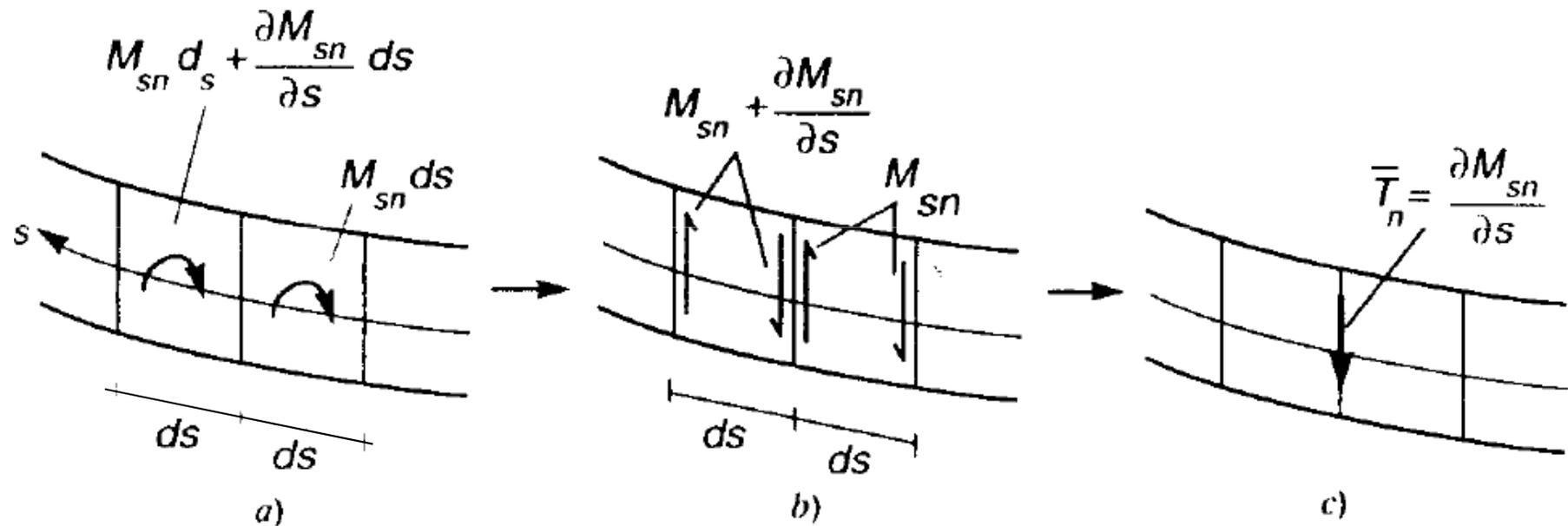


Figura 9.14

# Teoria di Kirchhoff delle piastre sottili

**Il taglio di Kirchhoff risulta essere una misura globale di azione trasversale interna comprensiva di un contributo staticamente equivalente al momento torcente**

$$\mathbf{T}_n^K = \mathbf{T}_n + \overline{\mathbf{T}} = \mathbf{T}_n + \frac{\partial M_{sn}}{\partial s}$$

**Il metodo seguito rappresenta solo un'interpretazione meccanica del taglio di Kirchhoff dovuta a Lord Kelvin e Tait alla fine dell '800**

# Teoria di Kirchhoff delle piastre sottili

I parametri cinematici indipendenti sul contorno sono

**w: inflessione**

**$\partial w / \partial n$  : rotazione normale**

Infatti la rotazione tangente  $\partial w / \partial s$  risulta nota una volta assegnato lo spostamento  $w$  sul tratto di contorno

Analogamente si dimostra che possono essere assegnate solo 2 condizioni al contorno di tipo statico.

Tali sollecitazioni devono essere coniugate nel senso del principio dei lavori virtuali all'inflessione  $w$  ed alla derivata normale

# Teoria di Kirchhoff delle piastre sottili

Il lavoro virtuale delle caratteristiche di sollecitazione lungo un tratto del contorno vale

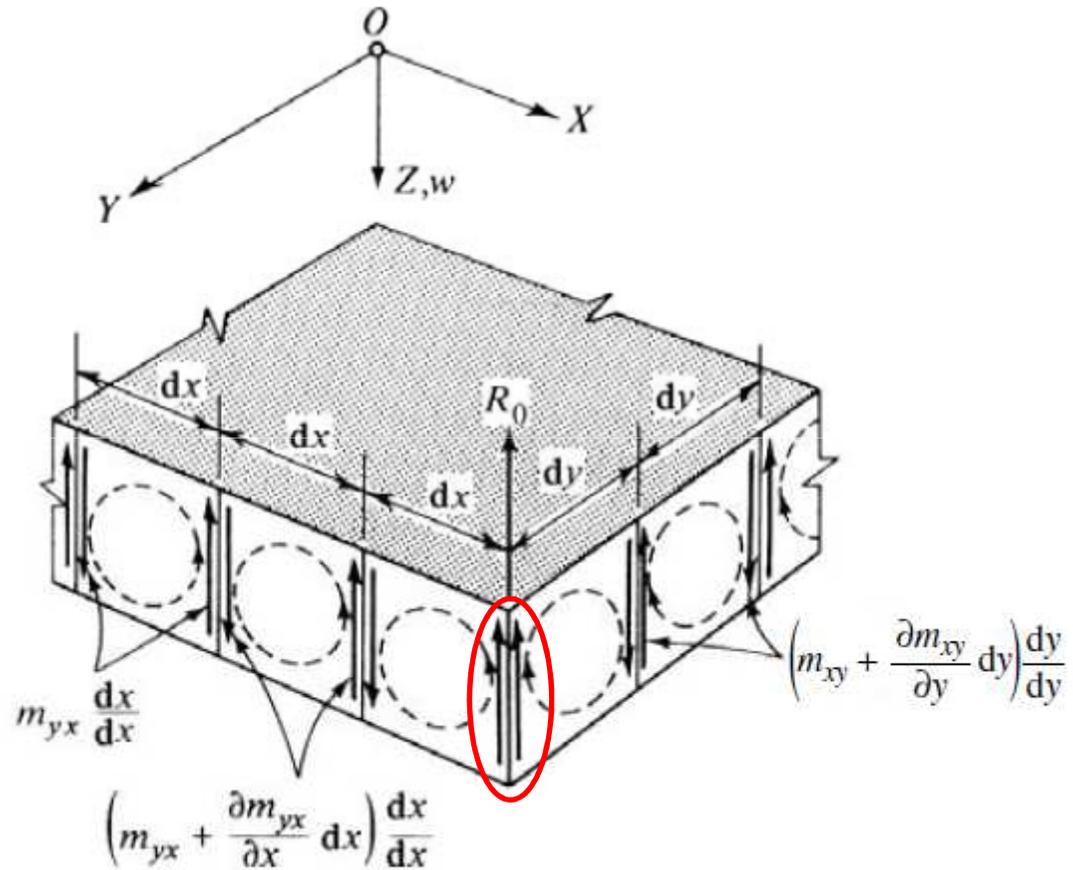
$$L = - \int_c M_n \frac{\partial w}{\partial n} ds + \int_c \left( T_n + \frac{\partial M_{ns}}{\partial s} \right) w ds$$

Per cui le caratteristiche di sollecitazione da assegnare sul contorno sono

$M_n$  momento flettente normale

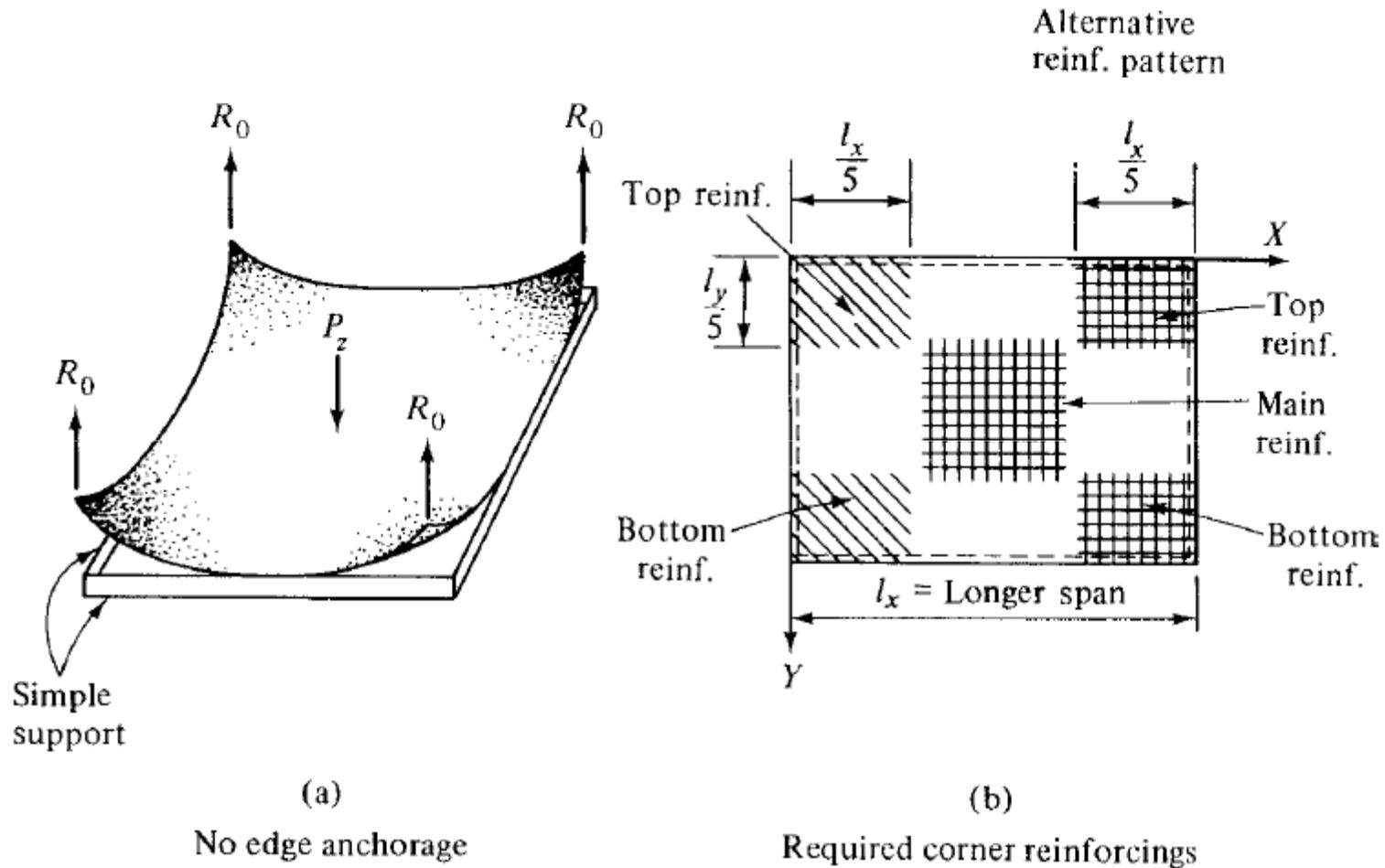
$T_n + \frac{\partial M_{ns}}{\partial s}$  taglio di Kirchhoff

# Taglio di Kirchhoff



**Effetti di Bordo dei momenti torcenti**

# Taglio di Kirchhoff



**Sollevamento degli spigoli**

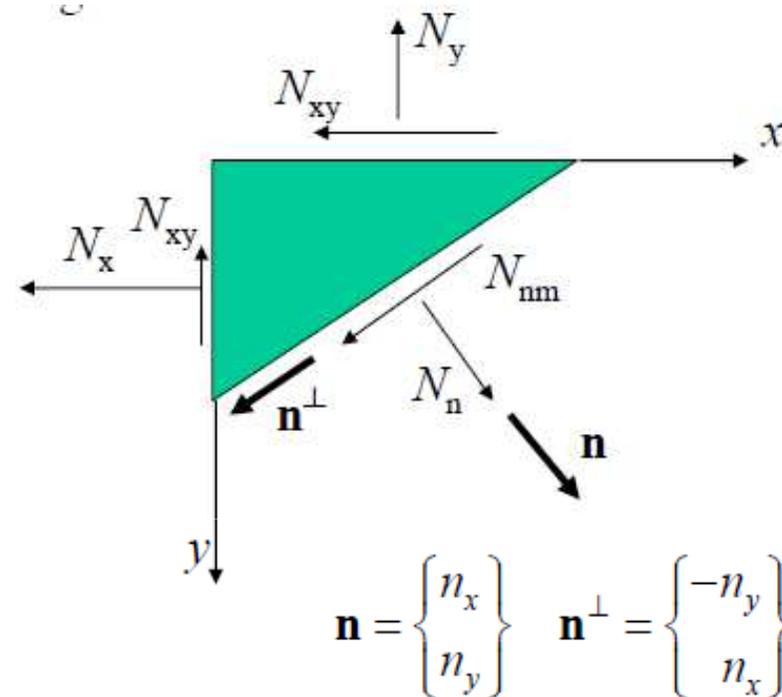
# Teoria di Kirchhoff : risultanti rispetto ad una direzione generica

$$\mathbf{N}^x = \begin{Bmatrix} N_x \\ N_{xy} \end{Bmatrix} \quad \mathbf{N}^y = \begin{Bmatrix} N_{xy} \\ N_y \end{Bmatrix}$$

$$N_n = \mathbf{N}^n \bullet \mathbf{n} \quad N_{nm} = \mathbf{N}^n \bullet \mathbf{n}^\perp$$

$$\mathbf{N}^n \Delta l_n - N_x \Delta y - N_y \Delta x = 0$$

$$\mathbf{N}^n - N_x \frac{\Delta y}{\Delta l_n} - N_y \frac{\Delta x}{\Delta l_n} = 0$$



$$\mathbf{n} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} \quad \mathbf{n}^\perp = \begin{Bmatrix} -n_y \\ n_x \end{Bmatrix}$$

$$\mathbf{N}^n = N_x n_x + N_y n_y \quad N_n = \mathbf{N}^n \bullet \mathbf{n} = (N_x n_x + N_y n_y) \bullet \mathbf{n}$$

$$N_{nm} = \mathbf{N}^n \bullet \mathbf{n}^\perp = (N_x n_x + N_y n_y) \bullet \mathbf{n}^\perp$$

$$N_n = (N_x n_x + N_y n_y) n_x + (N_{xy} n_x + N_{xy} n_y) n_y$$

$$N_{nm} = (-N_x n_y + N_{xy} n_x) n_x + (-N_{xy} n_y + N_y n_x) n_y$$

# Teoria di Kirchhoff : momenti generalizzati di asse generico

$$\mathbf{M}^x = \begin{Bmatrix} M_x \\ M_{xy} \end{Bmatrix} \quad \mathbf{M}^y = \begin{Bmatrix} M_{xy} \\ M_y \end{Bmatrix}$$

$$M_n = \mathbf{M}^n \bullet \mathbf{n} \quad M_{nm} = \mathbf{M}^n \bullet \mathbf{n}^\perp$$

$$\mathbf{M}^n \Delta l_n - \mathbf{M}^x \Delta y - \mathbf{M}^y \Delta x = 0$$

$$\mathbf{M}^n - \mathbf{M}^x \frac{\Delta y}{\Delta l_n} - \mathbf{M}^y \frac{\Delta x}{\Delta l_n} = 0$$

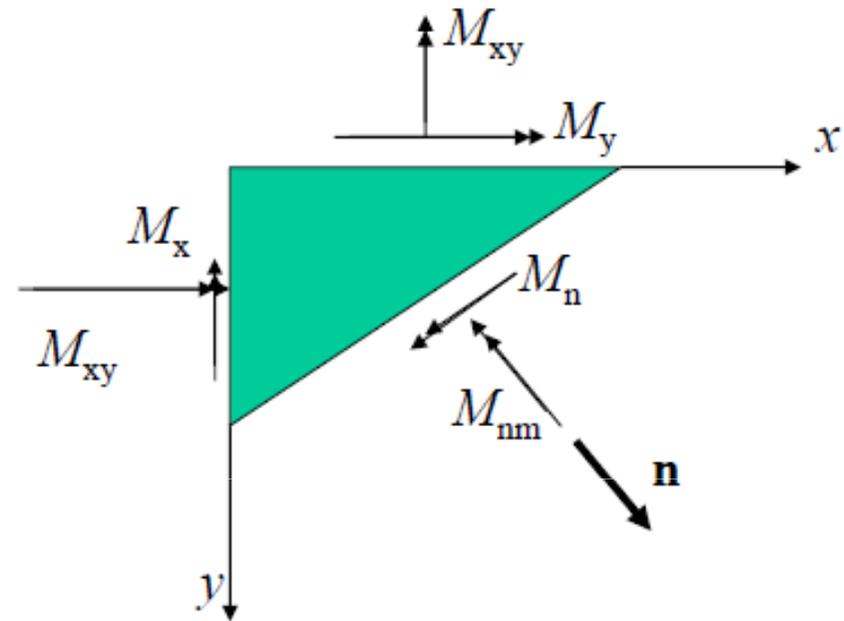
$$\mathbf{M}^n = \mathbf{M}^x n_x + \mathbf{M}^y n_y$$

$$M_n = \mathbf{M}^n \bullet \mathbf{n} = (\mathbf{M}^x n_x + \mathbf{M}^y n_y) \bullet \mathbf{n}$$

$$M_{nm} = \mathbf{M}^n \bullet \mathbf{n}^\perp = (\mathbf{M}^x n_x + \mathbf{M}^y n_y) \bullet \mathbf{n}^\perp$$

$$M_n = (M_x n_x + M_{xy} n_y) n_x + (M_{xy} n_x + M_y n_y) n_y$$

$$M_{nm} = (-M_x n_y + M_{xy} n_x) n_x + (-M_{xy} n_y + M_y n_x) n_y$$

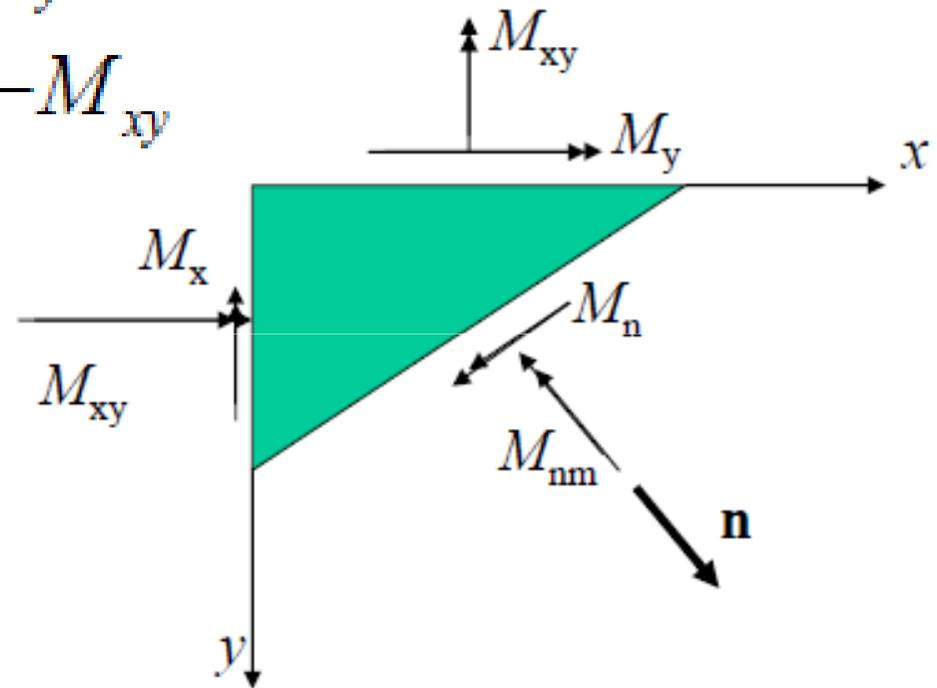


# Teoria di Kirchhoff delle piastre sottili

$$n_x = 0 \quad n_y = 1$$

$$M_n = M_y$$

$$M_{nm} = -M_{xy}$$



$$n_x = 1 \quad n_y = 0$$

$$M_n = M_x$$

$$M_{nm} = M_{xy}$$