8

'Standard' and 'hierarchical' element shape functions: some general families of C_0 continuity

8.1 Introduction

In Chapters 4, 5, and 6 the reader was shown in some detail how linear elasticity problems could be formulated and solved using very simple finite element forms. In Chapter 7 this process was repeated for the quasi-harmonic equation. Although the detailed algebra was concerned with shape functions which arose from triangular and tetrahedral shapes only it should by now be obvious that other element forms could equally well be used. Indeed, once the element and the corresponding shape functions are determined, subsequent operations follow a standard, well-defined path which could be entrusted to an algebraist not familiar with the physical aspects of the problem. It will be seen later that in fact it is possible to program a computer to deal with wide classes of problems by specifying the shape functions only. The choice of these is, however, a matter to which intelligence has to be applied and in which the human factor remains paramount. In this chapter some rules for the generation of several families of one-, two-, and three-dimensional elements will be presented.

In the problems of elasticity illustrated in Chapters 4, 5, and 6 the displacement variable was a vector with two or three components and the shape functions were written in matrix form. They were, however, derived for each component separately and in fact the matrix expressions in these were derived by multiplying a scalar function by an identity matrix [e.g., Eqs (4.7), (5.3), and (6.7)]. This scalar form was used directly in Chapter 7 for the quasi-harmonic equation. We shall therefore concentrate in this chapter on the scalar shape function forms, calling these simply N_i.

The shape functions used in the displacement formulation of elasticity problems were such that they satisfy the convergence criteria of Chapter 2:

(*a*) the continuity of the *unknown only* had to occur between elements (i.e., slope continuity is not required), or, in mathematical language, C_0 continuity was needed;
(*b*) the function has to allow any arbitrary linear form to be taken so that the constant strain (constant first derivative) criterion could be observed.

The shape functions described in this chapter will require the satisfaction of these two criteria. They will thus be applicable to all the problems of the preceding chapters and also to other problems which require these conditions to be obeyed. Indeed they are applicable to any situation where the functional Π or $\delta\Pi$ (see Chapter 3) is defined by derivatives of first order only.

The element families discussed will progressively have an increasing number of degrees of freedom. The question may well be asked as to whether any economic or other advantage is gained by thus increasing the complexity of an element. The answer here is not an easy one although it can be stated as a general rule that as the order of an element increases so the total number of unknowns in a problem can be reduced for a given accuracy of representation. Economic advantage requires, however, a reduction of total computation and data preparation effort, and this does not follow automatically for a reduced number of total variables because, though equation-solving times may be reduced, the time required for element formulation increases.

However, an overwhelming economic advantage in the case of three-dimensional analysis has already been hinted at in Chapters 6 and 7 for three-dimensional analyses.

The same kind of advantage arises on occasion in other problems but in general the optimum element may have to be determined from case to case.

In Sec. 2.6 of Chapter 2 we have shown that the order of error in the approximation to the unknown function is $O(h^{p+1})$, where h is the element 'size' and p is the degree of the complete polynomial present in the expansion. Clearly, as the element shape functions increase in degree so will the order of error increase, and convergence to the exact solution becomes more rapid. While this says nothing about the magnitude of error at a particular subdivision, it is clear that we should seek element shape functions with the highest complete polynomial for a given number of degrees of freedom.

8.2 Standard and hierarchical concepts

The essence of the finite element method already stated in Chapters 2 and 3 is in approximating the unknown (displacement) by an expansion given in Eqs (2.1) and (3.3). For a scalar variable u this can be written as

$$u \approx \hat{u} = \sum_{i=1}^{n} N_i a_i = \mathbf{N}\mathbf{a} \tag{8.1}$$

where n is the total number of functions used and a_i are the unknown parameters to be determined.

We have explicitly chosen to identify such variables with the values of the unknown function at element nodes, thus making

$$u_i = a_i \tag{8.2}$$

The shape functions so defined will be referred to as 'standard' ones and are the basis of most finite element programs. If polynomial expansions are used and the element satisfies Criterion 1 of Chapter 2 (which specifies that rigid body displacements cause no strain), it is clear that a constant value of a_i specified at all nodes must result in a constant value of $\hat{u}$:

$$\hat{u} = \left(\sum_{i=1}^{n} N_i\right) u_0 = u_0 \tag{8.3}$$

when $a_i = u_0$. It follows that

$$\sum_{i=1}^{n} N_i = 1 \tag{8.4}$$

at all points of the domain. This important property is known as a *partition of unity*[1] which we will make extensive use of in Chapter 16. The first part of this chapter will deal with such *standard shape functions*.

A serious drawback exists, however, with 'standard' functions, since when element refinement is made totally new shape functions have to be generated and hence all calculations repeated. It would be of advantage to avoid this difficulty by considering the expression (8.1) as a *series* in which the shape function N_i does not depend on the number of nodes in the mesh n. This indeed is achieved with *hierarchic shape functions* to which the second part of this chapter is devoted.

The hierarchic concept is well illustrated by the one-dimensional (elastic bar) problem of Fig. 8.1. Here for simplicity elastic properties are taken as constant ($D = E$) and the body force b is assumed to vary in such a manner as to produce the exact solution shown on the figure (with zero displacements at both ends).

Two meshes are shown and a linear interpolation between nodal points assumed. For both standard and hierarchic forms the coarse mesh gives

$$K_{11}^c a_1^c = f_1 \tag{8.5}$$

For a fine mesh two additional nodes are added and with the standard shape function the equations requiring solution are

$$\begin{bmatrix} K_{11}^F & K_{12}^F & 0 \\ K_{21}^F & K_{22}^F & K_{23}^F \\ 0 & K_{32}^F & K_{33}^F \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{8.6}$$

In this form the zero matrices have been automatically inserted due to element interconnection which is here obvious, and we note that as no coefficients are the same, the new equations have to be resolved. [Equation (2.13) shows how these coefficients are calculated and the reader is encouraged to work these out in detail.]

With the 'hierarchic' form using the shape functions shown, a similar form of equation arises and an identical approximation is achieved (being simply given by a series of straight segments). The *final* solution is identical but the meaning of the parameters a_i^* is now different, as shown in Fig. 8.1.

Quite generally,

$$K_{11}^F = K_{11}^c \tag{8.7}$$

as an identical shape function is used for the first variable. Further, in this particular case the off-diagonal coefficients are zero and the final equations become, for the fine mesh,

$$\begin{bmatrix} K_{11}^c & 0 & 0 \\ 0 & K_{22}^F & 0 \\ 0 & 0 & K_{33}^F \end{bmatrix} \begin{Bmatrix} a_1^* \\ a_2^* \\ a_3^* \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{8.8}$$

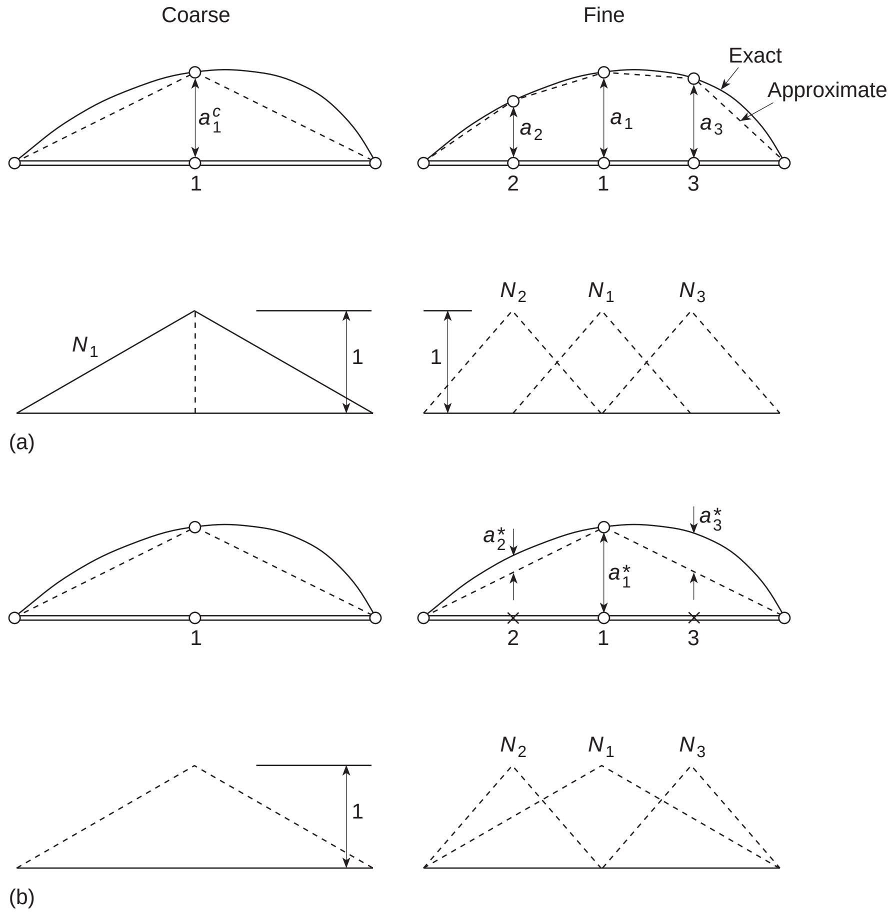

Fig. 8.1 A one-dimensional problem of stretching of a uniform elastic bar by prescribed body forces. (a) 'Standard approximation. (b) Hierarchic approximation.

The 'diagonality' feature is only true in the one-dimensional problem, but in general it will be found that the matrices obtained using hierarchic shape functions are more nearly diagonal and hence imply better conditioning than those with standard shape functions.

Although the variables are now not subject to the obvious interpretation (as local displacement values), they can be easily transformed to those if desired. Though it is not usual to use hierarchic forms in linearly interpolated elements their derivation in polynomial form is simple and very advantageous.

The reader should note that with hierarchic forms it is convenient to consider the finer mesh as still using the same, coarse, elements but now adding additional refining functions.

Hierarchic forms provide a link with other approximate (orthogonal) series solutions. Many problems solved in classical literature by trigonometric, Fourier series, expansion are indeed particular examples of this approach.

In the following sections of this chapter we shall consider the development of shape functions for high order elements with many boundary and internal degree of freedoms. This development will generally be made on simple geometric forms and the reader may well question the wisdom of using increased accuracy for such simple shaped domains, having already observed the advantage of generalized finite element methods in fitting arbitrary domain shapes. This concern is well founded, but in the next chapter we shall show a general method to map high order elements into quite complex shapes.

Part 1 'Standard' shape functions

Two-dimensional elements

8.3 Rectangular elements – some preliminary considerations

Conceptually (especially if the reader is conditioned by education to thinking in the cartesian coordinate system) the simplest element form of a two-dimensional kind is that of a rectangle with sides parallel to the x and y axes. Consider, for instance, the rectangle shown in Fig. 8.2 with nodal points numbered 1 to 8, located as shown, and at which the values of an unknown function u (here representing, for instance, one of the components of displacement) form the element parameters. How can suitable C_0 continuous shape functions for this element be determined?

Let us first assume that u is expressed in polynomial form in x and y. To ensure interelement continuity of u along the top and bottom sides the variation must be linear. Two points at which the function is common between elements lying above or below exist, and as two values uniquely determine a linear function, its identity all along these sides is ensured with that given by adjacent elements. Use of this fact was already made in specifying linear expansions for a triangle.

Similarly, if a cubic variation along the vertical sides is assumed, continuity will be preserved there as four values determine a unique cubic polynomial. Conditions for satisfying the first criterion are now obtained.

To ensure the existence of constant values of the first derivative it is necessary that all the linear polynomial terms of the expansion be retained.

Finally, as eight points are to determine uniquely the variation of the function only eight coefficients of the expansion can be retained and thus we could write

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 y^2 + \alpha_6 xy^2 + \alpha_7 y^3 + \alpha_8 xy^3 \qquad (8.9)$$

The choice can in general be made unique by retaining the lowest possible expansion terms, though in this case apparently no such choice arises.† The reader will easily verify that all the requirements have now been satisfied.

† Retention of a higher order term of expansion, ignoring one of lower order, will usually lead to a poorer approximation though still retaining convergence,[2] providing the linear terms are always included.

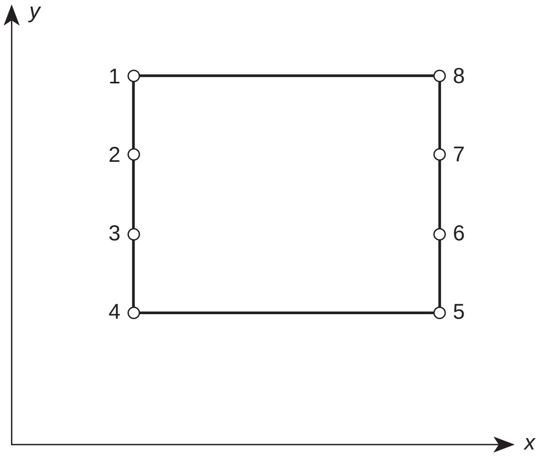

Fig. 8.2 A rectangular element.

Substituting coordinates of the various nodes a set of simultaneous equations will be obtained. This can be written in exactly the same manner as was done for a triangle in Eq. (4.4) as

$$\begin{Bmatrix} u_1 \\ \vdots \\ u_8 \end{Bmatrix} = \begin{bmatrix} 1, & x_1, & y_1, & x_1 y_1, & y_1^2, & x_1 y_1^2, & y_1^3, & x_1 y_1^3 \\ . & . & . & . & . & . & . & . \\ 1, & x_8, & y_8, & . & . & . & . & x_8 y_8^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_8 \end{Bmatrix} \tag{8.10}$$

or simply as

$$\mathbf{u}^e = \mathbf{C}\boldsymbol{\alpha} \tag{8.11}$$

Formally,

$$\boldsymbol{\alpha} = \mathbf{C}^{-1}\mathbf{u}^e \tag{8.12}$$

and we could write Eq. (8.9) as

$$u = \mathbf{P}\boldsymbol{\alpha} = \mathbf{P}\mathbf{C}^{-1}\mathbf{u}^e \tag{8.13}$$

in which

$$\mathbf{P} = [1, x, y, xy, y^2, xy^2, y^3, xy^3] \tag{8.14}$$

Thus the shape functions for the element defined by

$$u = \mathbf{N}\mathbf{u}^e = [N_1, N_2, \ldots, N_8]\mathbf{u}^e \tag{8.15}$$

can be found as

$$\mathbf{N} = \mathbf{P}\mathbf{C}^{-1} \tag{8.16}$$

This process has, however, some considerable disadvantages. Occasionally an inverse of **C** may not exist[2,3] and *always* considerable algebraic difficulty is experienced in obtaining an expression for the inverse in general terms suitable for all element geometries. It is therefore worthwhile to consider whether shape functions $N_i(x, y)$ can be written down directly. Before doing this some general properties of these functions have to be mentioned.

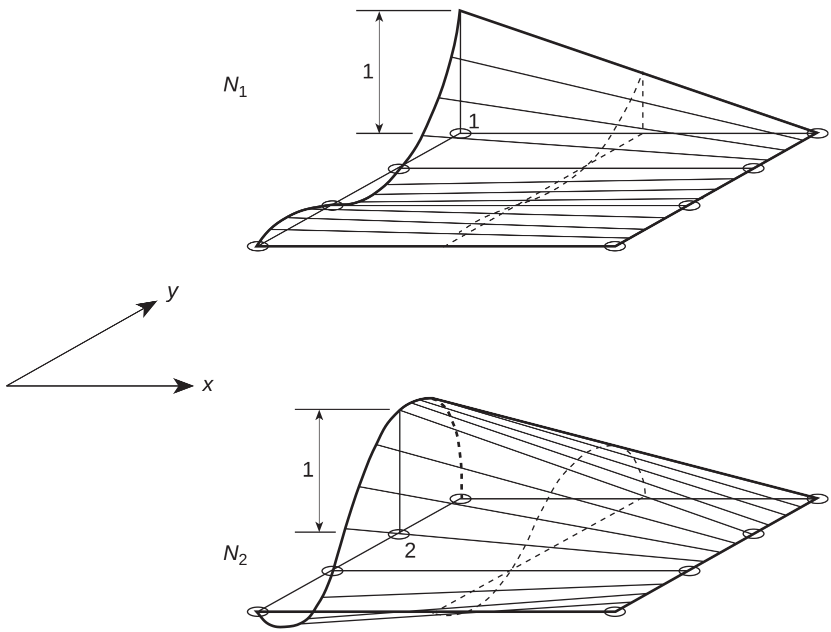

Fig. 8.3 Shape functions for elements of Fig. 8.2.

Inspection of the defining relation, Eq. (8.15), reveals immediately some important characteristics. Firstly, as this expression is valid for all components of $\mathbf{u}^e$,

$$N_i(x_j, y_j) = \delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

where δ_{ij} is known as the Kronecker delta. Further, the basic type of variation along boundaries defined for continuity purposes (e.g., linear in x and cubic in y in the above example) must be retained. The typical form of the shape functions for the elements considered is illustrated isometrically for two typical nodes in Fig. 8.3. It is clear that these could have been written down directly as a product of a suitable linear function in x with a cubic function in y. The easy solution of this example is not always as obvious but given sufficient ingenuity, a direct derivation of shape functions is always preferable.

It will be convenient to use normalized coordinates in our further investigation. Such normalized coordinates are shown in Fig. 8.4 and are chosen so that their values are ± 1 on the faces of the rectangle:

$$\begin{aligned} \xi &= \frac{x - x_c}{a} \qquad \mathrm{d}\xi = \frac{\mathrm{d}x}{a} \\ \eta &= \frac{y - y_c}{b} \qquad \mathrm{d}\eta = \frac{\mathrm{d}y}{b} \end{aligned} \tag{8.17}$$

Once the shape functions are known in the normalized coordinates, translation into actual coordinates or transformation of the various expressions occurring, for instance, in the stiffness derivation is trivial.

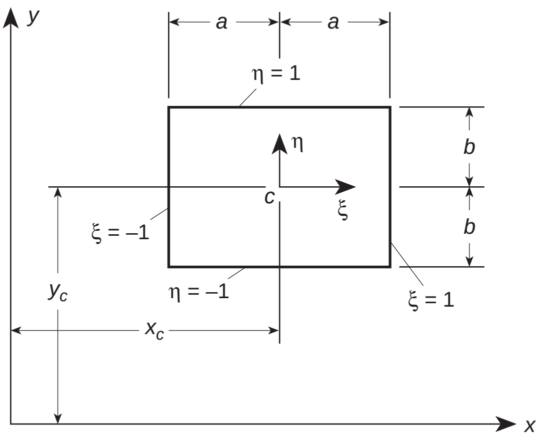

Fig. 8.4 Normalized coordinates for a rectangle.

8.4 Completeness of polynomials

The shape function derived in the previous section was of a rather special form [see Eq. (8.9)]. Only a linear variation with the coordinate x was permitted, while in y a full cubic was available. The complete polynomial contained in it was thus of order 1. In general use, a convergence order corresponding to a linear variation would occur despite an increase of the total number of variables. Only in situations where the linear variation in x corresponded closely to the exact solution would a higher order of convergence occur, and for this reason elements with such 'preferential' directions should be restricted to special use, e.g., in narrow beams or strips. In general, we shall seek element expansions which possess the highest order of a complete polynomial for a minimum of degrees of freedom. In this context it is useful to recall the Pascal triangle (Fig. 8.5) from which the number of terms

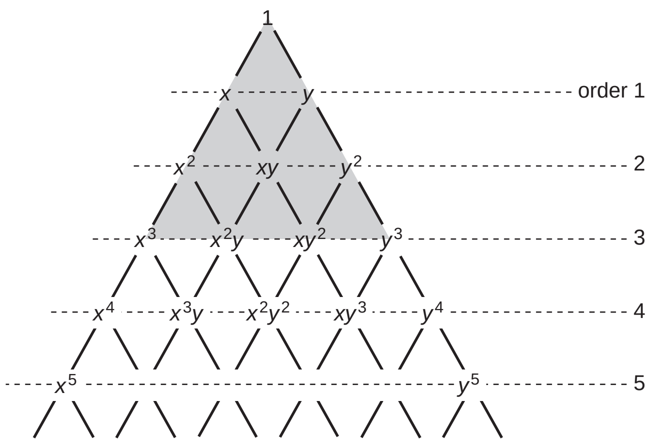

Fig. 8.5 The Pascal triangle. (Cubic expansion shaded – 10 terms).

occurring in a polynomial in two variables x, y can be readily ascertained. For instance, first-order polynomials require three terms, second-order require six terms, third-order require ten terms, etc.

8.5 Rectangular elements – Lagrange family[4–6]

An easy and systematic method of generating shape functions of any order can be achieved by simple products of appropriate polynomials in the two coordinates. Consider the element shown in Fig. 8.6 in which a series of nodes, external and internal, is placed on a regular grid. It is required to determine a shape function for the point indicated by the heavy circle. Clearly the product of a fifth-order polynomial in ξ which has a value of unity at points of the second column of nodes and zero elsewhere and that of a fourth-order polynomial in η having unity on the coordinate corresponding to the top row of nodes and zero elsewhere satisfies all the interelement continuity conditions and gives unity at the nodal point concerned.

Polynomials in one coordinate having this property are known as Lagrange polynomials and can be written down directly as

$$l_k^n(\xi) = \frac{(\xi-\xi_0)(\xi-\xi_1)\cdots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\cdots(\xi-\xi_n)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\cdots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\cdots(\xi_k-\xi_n)} \tag{8.18}$$

giving unity at ξ_k and passing through n points.

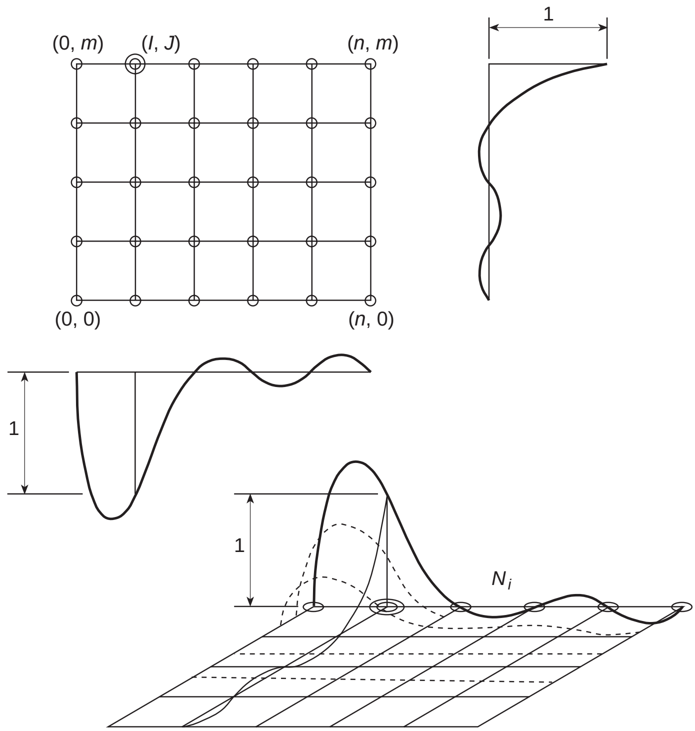

Fig. 8.6 A typical shape function for a Lagrangian element ($n = 5$, $m = 4$, $I = 1$, $J = 4$).

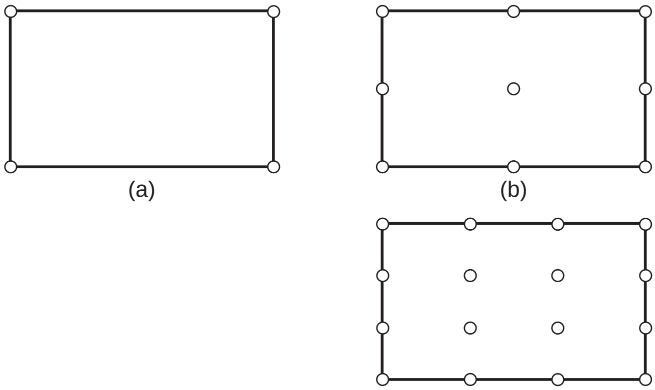

Fig. 8.7 Three elements of the Lagrange family: (*a*) linear, (*b*) quadratic, and (c) cubic.

Thus in two dimensions, if we label the node by its column and row number, I, J, we have

$$N_i \equiv N_{IJ} = l_I^n(\xi) l_J^m(\eta) \tag{8.19}$$

where n and m stand for the number of subdivisions in each direction.

Figure 8.7 shows a few members of this unlimited family where $m = n$.

Indeed, if we examine the polynomial terms present in a situation where $n = m$ we observe in Fig. 8.8, based on the Pascal triangle, that a large number of polynomial terms is present above those needed for a complete expansion.[7] However, when mapping of shape functions is considered (Chapter 9) some advantages occur for this family.

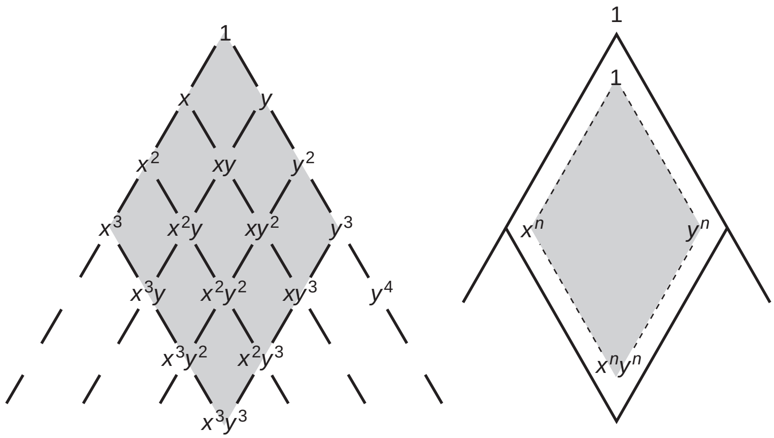

Fig. 8.8 Terms generated by a lagrangian expansion of order 3×3 (or $n \times n$). Complete polynomials of order 3 (or n).

8.6 Rectangular elements – 'serendipity' family[4,5]

It is usually more efficient to make the functions dependent on nodal values placed on the element boundary. Consider, for instance, the first three elements of Fig. 8.9. In each a progressively increasing and equal number of nodes is placed on the element boundary. The variation of the function on the edges to ensure continuity is linear, parabolic, and cubic in increasing element order.

To achieve the shape function for the first element it is obvious that a product of linear lagrangian polynomials of the form

$$\tfrac{1}{4}(\xi + 1)(\eta + 1) \tag{8.20}$$

gives unity at the top right corners where $\xi = \eta = 1$ and zero at all the other corners. Further, a linear variation of the shape function of all sides exists and hence continuity is satisfied. Indeed this element is identical to the lagrangian one with $n = 1$.

Introducing new variables

$$\xi_0 = \xi\xi_i \qquad \eta_0 = \eta\eta_i \tag{8.21}$$

in which ξ_i, η_i are the normalized coordinates at node i, the form

$$N_i = \tfrac{1}{4}(1 + \xi_0)(1 + \eta_0) \tag{8.22}$$

allows all shape functions to be written down in one expression.

As a linear combination of these shape functions yields any arbitrary linear variation of u, the second convergence criterion is satisfied.

The reader can verify that the following functions satisfy all the necessary criteria for quadratic and cubic members of the family.

'Quadratic' element

Corner nodes:

$$N_i = \tfrac{1}{4}(1 + \xi_0)(1 + \eta_0)(\xi_0 + \eta_0 - 1) \tag{8.23}$$

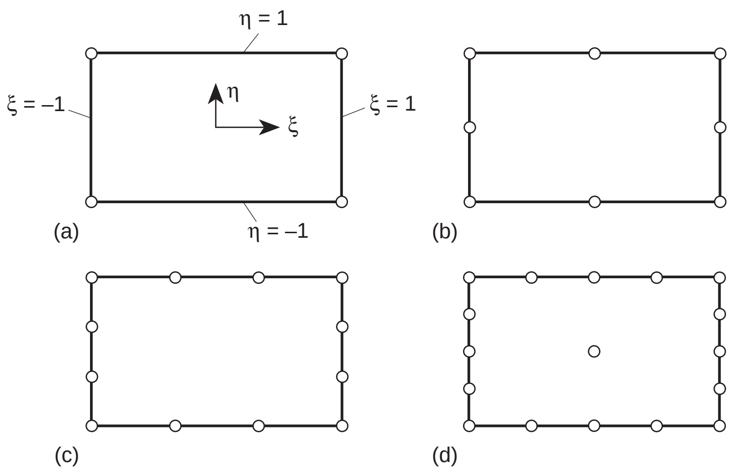

Fig. 8.9 Rectangles of boundary node (serendipity) family: (*a*) linear, (*b*) quadratic, (*c*) cubic, (*d*) quartic.

Mid-side nodes:

$$\xi_i = 0 \qquad N_i = \tfrac{1}{2}(1-\xi^2)(1+\eta_0)$$
$$\eta_i = 0 \qquad N_i = \tfrac{1}{2}(1+\xi_0)(1-\eta^2)$$

'Cubic' element

Corner nodes:

$$N_i = \tfrac{1}{32}(1+\xi_0)(1+\eta_0)[-10+9(\xi^2+\eta^2)] \tag{8.24}$$

Mid-side nodes:

$$\xi_i = \pm 1 \qquad \text{and} \qquad \eta_i = \pm\tfrac{1}{3}$$
$$N_i = \tfrac{9}{32}(1+\xi_0)(1-\eta^2)(1+9\eta_0)$$

with the remaining mid-side node expression obtained by changing variables.

In the next, quartic, member[8] of this family a central node is added so that all terms of a complete fourth-order expansion will be available. This central node adds a shape function $(1-\xi^2)(1-\eta^2)$ which is zero on all outer boundaries.

The above functions were originally derived by inspection, and progression to yet higher members is difficult and requires some ingenuity. It was therefore appropriate

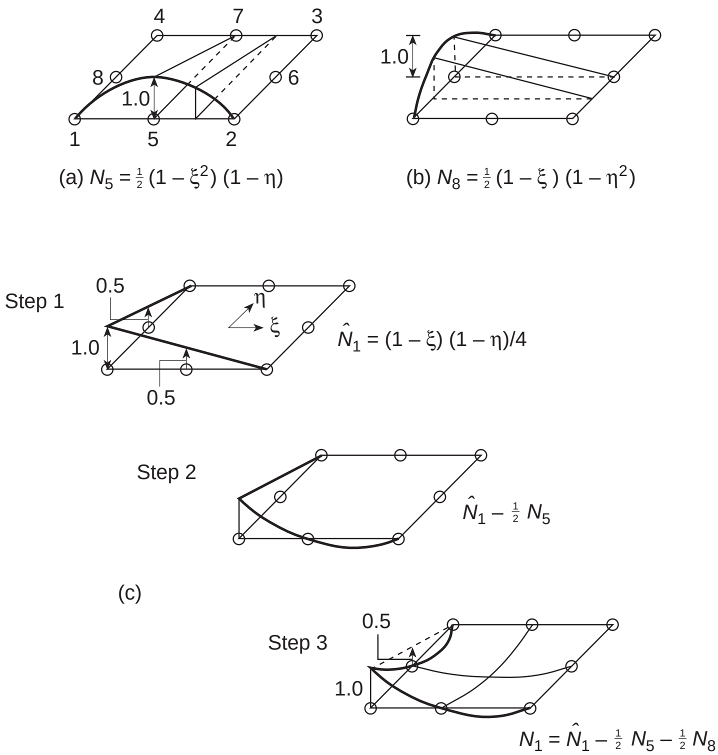

Fig. 8.10 Systematic generation of 'serendipity' shape functions.

to name this family 'serendipity' after the famous princes of Serendip noted for their chance discoveries (Horace Walpole, 1754).

However, a quite systematic way of generating the 'serendipity' shape functions can be devised, which becomes apparent from Fig. 8.10 where the generation of a quadratic shape function is presented.[7,9]

As a starting point we observe that for *mid-side* nodes a lagrangian interpolation of a quadratic $\times$ linear type suffices to determine N_i at nodes 5 to 8. N_5 and N_8 are shown at Fig. 8.10(*a*) and (*b*). For a *corner* node, such as Fig. 8.10(*c*), we start with a bilinear lagrangian family $\hat{N}_1$ and note immediately that while $\hat{N}_1 = 1$ at node 1, it is not zero at nodes 5 or 8 (step 1). Successive subtraction of $\frac{1}{2}N_5$ (step 2) and $\frac{1}{2}N_8$ (step 3) ensures that a zero value is obtained at these nodes. The reader can verify that the expressions obtained coincide with those of Eq. (8.23).

Indeed, it should now be obvious that for all higher order elements the *mid-side* and *corner shape* functions can be generated by an identical process. For the former a simple multiplication of *m*th-order and first-order lagrangian interpolations suffices. For the latter a combination of bilinear corner functions, together with appropriate fractions of mid-side shape functions to ensure zero at appropriate nodes, is necessary.

Similarly, it is quite easy to generate shape functions for elements with different numbers of nodes along each side by a systematic algorithm. This may be very

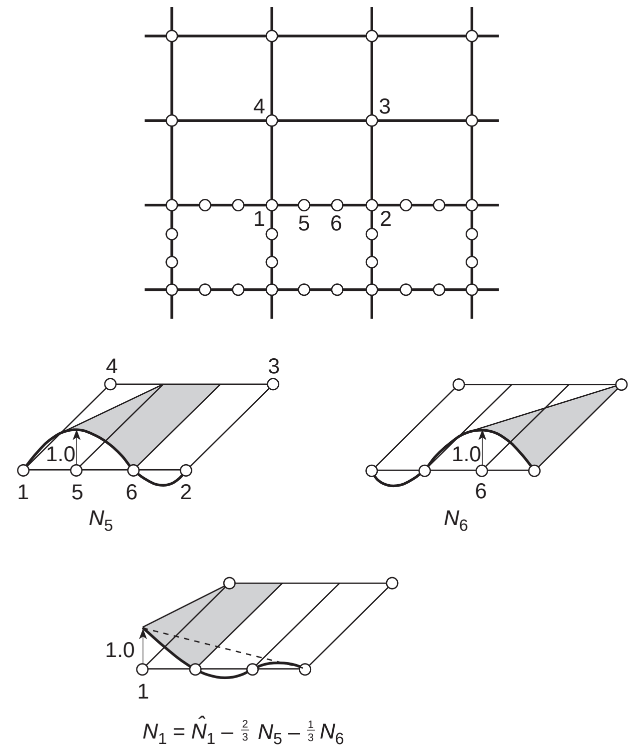

Fig. 8.11 Shape functions for a transition 'serendipity' element, cubic/linear.

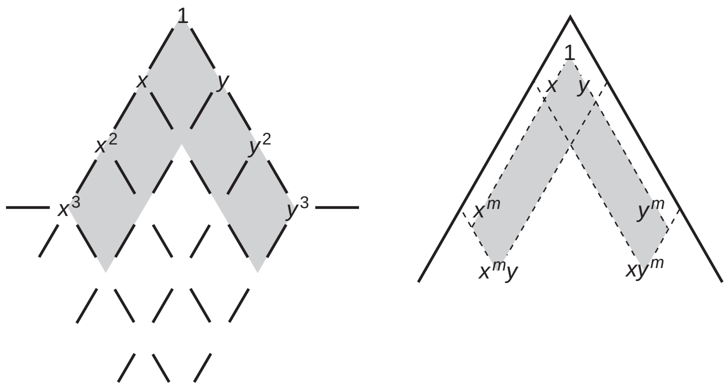

Fig. 8.12 Terms generated by edge shape functions in serendipity-type elements (3×3 and $m \times m$).

desirable if a transition between elements of different order is to be achieved, enabling a different order of accuracy in separate sections of a large problem to be studied. Figure 8.11 illustrates the necessary shape functions for a cubic/linear transition. Use of such special elements was first introduced in reference 9, but the simpler formulation used here is that of reference 7.

With the mode of generating shape functions for this class of elements available it is immediately obvious that fewer degrees of freedom are now necessary for a given complete polynomial expansion. Figure 8.12 shows this for a cubic element where only two surplus terms arise (as compared with six surplus terms in a lagrangian of the same degree).

It is immediately evident, however, that the functions generated by nodes placed only along the edges will not generate complete polynomials beyond cubic order. For higher order ones it is necessary to supplement the expansion by internal nodes (as was done in the quartic element of Fig. 8.9) or by the use of 'nodeless' variables which contain appropriate polynomial terms.

8.7 Elimination of internal variables before assembly – substructures

Internal nodes or nodeless internal parameters yield in the usual way the element properties (Chapter 2)

$$\frac{\partial \Pi^e}{\partial \mathbf{a}^e} = \mathbf{K}^e \mathbf{a}^e + \mathbf{f}^e \tag{8.25}$$

As $\mathbf{a}^e$ can be subdivided into parts which are common with other elements, $\bar{\mathbf{a}}^e$, and others which occur in the particular element only, $\bar{\bar{\mathbf{a}}}^e$, we can immediately write

$$\frac{\partial \Pi}{\partial \bar{\bar{\mathbf{a}}}^e} = \frac{\partial \Pi^e}{\partial \bar{\bar{\mathbf{a}}}^e} = \mathbf{0}$$

and eliminate $\bar{\bar{\mathbf{a}}}^e$ from further consideration. Writing Eq. (8.25) in a partitioned form we have

$$\frac{\partial \Pi^e}{\partial \mathbf{a}^e} = \begin{Bmatrix} \dfrac{\partial \Pi^e}{\partial \bar{\mathbf{a}}^e} \\ \dfrac{\partial \Pi^e}{\partial \bar{\bar{\mathbf{a}}}^e} \end{Bmatrix} = \begin{bmatrix} \bar{\mathbf{K}}^e & \hat{\mathbf{K}}^e \\ \hat{\mathbf{K}}^{e\mathrm{T}} & \bar{\bar{\mathbf{K}}}^e \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{a}}^e \\ \bar{\bar{\mathbf{a}}}^e \end{Bmatrix} + \begin{Bmatrix} \bar{\mathbf{f}}^e \\ \bar{\bar{\mathbf{f}}}^e \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \Pi^e}{\partial \bar{\mathbf{a}}^e} \\ \mathbf{0} \end{Bmatrix} \tag{8.26}$$

From the second set of equations given above we can write

$$\bar{\bar{\mathbf{a}}}^e = -(\bar{\bar{\mathbf{K}}}^e)^{-1}(\hat{\mathbf{K}}^{e\mathrm{T}}\bar{\mathbf{a}}^e + \bar{\bar{\mathbf{f}}}^e) \tag{8.27}$$

which on substitution yields

$$\frac{\partial \Pi^e}{\partial \bar{\mathbf{a}}^e} = \mathbf{K}^{*e}\bar{\mathbf{a}}^e + \mathbf{f}^{*e} \tag{8.28}$$

in which

$$\begin{aligned} \mathbf{K}^{*e} &= \bar{\mathbf{K}}^e - \hat{\mathbf{K}}^e(\bar{\bar{\mathbf{K}}}^e)^{-1}\hat{\mathbf{K}}^{e\mathrm{T}} \\ \mathbf{f}^{*e} &= \bar{\mathbf{f}}^e - \hat{\mathbf{K}}^e(\bar{\bar{\mathbf{K}}}^e)^{-1}\bar{\bar{\mathbf{f}}}^e \end{aligned} \tag{8.29}$$

Assembly of the total region then follows, by considering only the element boundary variables, thus giving a considerable saving in the equation-solving effort at the expense of a few additional manipulations carried out at the element stage.

Perhaps a structural interpretation of this elimination is desirable. What in fact is involved is the separation of a part of the structure from its surroundings and determination of its solution separately for any prescribed displacements at the interconnecting boundaries. $\mathbf{K}^{*e}$ is now simply the overall stiffness of the separated structure and $\mathbf{f}^{*e}$ the equivalent set of nodal forces.

If the triangulation of Fig. 8.13 is interpreted as an assembly of pin-jointed bars the reader will recognize immediately the well-known device of 'substructures' used frequently in structural engineering.

Such a substructure is in fact simply a complex element from which the internal degrees of freedom have been eliminated.

Immediately a new possibility for devising more elaborate, and presumably more accurate, elements is presented.

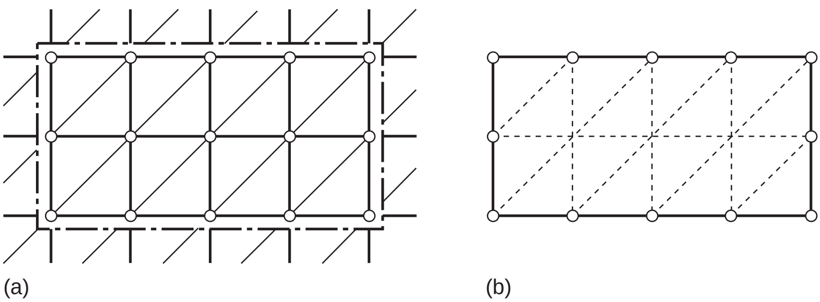

Fig. 8.13 Substructure of a complex element.

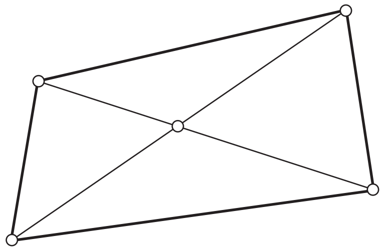

Fig. 8.14 A quadrilateral made up of four simple triangles.

Figure 8.13(*a*) can be interpreted as a continuum field subdivided into triangular elements. The substructure results in fact in one complex element shown in Fig. 8.13(*b*) with a number of boundary nodes.

The only difference from elements derived in previous sections is the fact that the unknown u is now not approximated internally by one set of smooth shape functions but by a series of piecewise approximations. This presumably results in a slightly poorer approximation but an economic advantage may arise if the total computation time for such an assembly is saved.

Substructuring is an important device in complex problems, particularly where a repetition of complicated components arises.

In simple, small-scale finite element analysis, much improved use of simple triangular elements was found by the use of simple subassemblies of the triangles (or indeed tetrahedra). For instance, a quadrilateral based on four triangles from which the central node is eliminated was found to give an economic advantage over direct use of simple triangles (Fig. 8.14). This and other subassemblies based on triangles are discussed in detail by Doherty *et al.*[10]

8.8 Triangular element family

The advantage of an arbitrary triangular shape in approximating to any boundary shape has been amply demonstrated in earlier chapters. Its apparent superiority here over rectangular shapes needs no further discussion. The question of generating more elaborate higher order elements needs to be further developed.

Consider a series of triangles generated on a pattern indicated in Fig. 8.15. The number of nodes in each member of the family is now such that a complete polynomial expansion, of the order needed for interelement compatibility, is ensured. This follows by comparison with the Pascal triangle of Fig. 8.5 in which we see the number of nodes coincides exactly with the number of polynomial terms required. This particular feature puts the triangle family in a special, privileged position, in which the inverse of the $\mathbf{C}$ matrices of Eq. (8.11) will always exist.[3] However, once again a direct generation of shape functions will be preferred – and indeed will be shown to be particularly easy.

Before proceeding further it is useful to define a special set of normalized coordinates for a triangle.

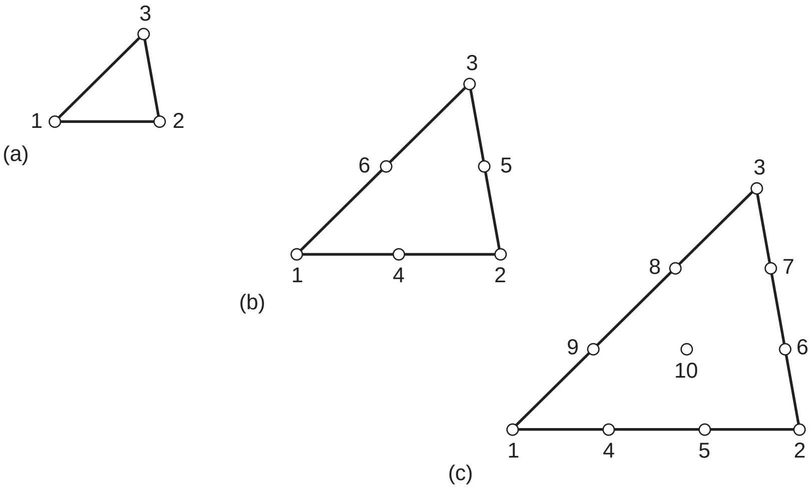

Fig. 8.15 Triangular element family: (*a*) linear, (*b*) quadratic, and (*c*) cubic.

8.8.1 Area coordinates

While cartesian directions parallel to the sides of a rectangle were a natural choice for that shape, in the triangle these are not convenient.

A new set of coordinates, L_1, L_2, and L_3 for a triangle 1, 2, 3 (Fig. 8.16), is defined by the following linear relation between these and the cartesian system:

$$\begin{aligned} x &= L_1 x_1 + L_2 x_2 + L_3 x_3 \\ y &= L_1 y_1 + L_2 y_2 + L_3 y_3 \\ 1 &= L_1 + L_2 + L_3 \end{aligned} \tag{8.30}$$

To every set, L_1, L_2, L_3 (which are not independent, but are related by the third equation), there corresponds a unique set of cartesian coordinates. At point 1, $L_1 = 1$ and $L_2 = L_3 = 0$, etc. A linear relation between the new and cartesian coordinates implies

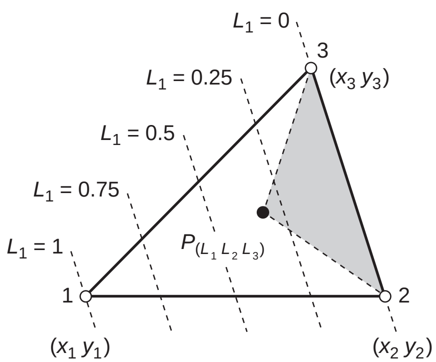

Fig. 8.16 Area coordinates.

that contours of L_1 are equally placed straight lines parallel to side 2–3 on which $L_1 = 0$, etc.

Indeed it is easy to see that an alternative definition of the coordinate L_1 of a point P is by a ratio of the area of the shaded triangle to that of the total triangle:

$$L_1 = \frac{\text{area } P23}{\text{area } 123} \tag{8.31}$$

Hence the name *area coordinates*.

Solving Eq. (8.30) gives

$$\begin{aligned} L_1 &= \frac{a_1 + b_1 x + c_1 y}{2\Delta} \\ L_2 &= \frac{a_2 + b_2 x + c_2 y}{2\Delta} \\ L_3 &= \frac{a_3 + b_3 x + c_3 y}{2\Delta} \end{aligned} \tag{8.32}$$

in which

$$\Delta = \tfrac{1}{2}\det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{area } 123 \tag{8.33}$$

and

$$a_1 = x_2 y_3 - x_3 y_2 \qquad b_1 = y_2 - y_3 \qquad c_1 = x_3 - x_2$$

etc., with cyclic rotation of indices 1, 2, and 3.

The identity of expressions with those derived in Chapter 4 [Eqs (4.5b) and (4.5c)] is worth noting.

8.8.2 Shape functions

For the first element of the series [Fig. 8.15(*a*)], the shape functions are simply the area coordinates. Thus

$$N_1 = L_1 \qquad N_2 = L_2 \qquad N_3 = L_3 \tag{8.34}$$

This is obvious as each individually gives unity at one node, zero at others, and varies linearly everywhere.

To derive shape functions for other elements a simple recurrence relation can be derived.[3] However, it is very simple to write an arbitrary triangle of order M in a manner similar to that used for the lagrangian element of Sec. 8.5.

Denoting a typical node i by three numbers I, J, and K corresponding to the position of coordinates L_{1i}, L_{2i}, and L_{3i} we can write the shape function in terms of three lagrangian interpolations as [see Eq. (8.18)]

$$N_i = l_I^I(L_1)\, l_J^J(L_2)\, l_K^K(L_3) \tag{8.35}$$

In the above l_I^I, etc., are given by expression (8.18), with L_1 taking the place of ξ, etc.

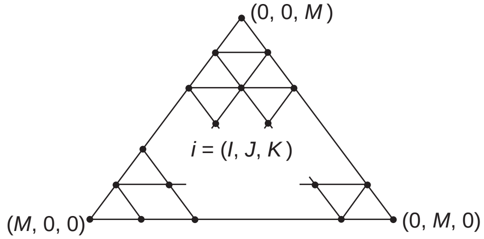

Fig. 8.17 A general triangular element.

It is easy to verify that the above expression gives

$$N_i = 1 \qquad \text{at} \quad L_1 = L_{1I}, \quad L_2 = L_{2J}, \quad L_3 = L_{3K}$$

and zero at all other nodes.

The highest term occurring in the expansion is

$$L_1^I L_2^J L_3^K$$

and as

$$I + J + K \equiv M$$

for all points the polynomial is also of order M.

Expression (8.35) is valid for quite arbitrary distributions of nodes of the pattern given in Fig. 8.17 and simplifies if the spacing of the nodal lines is equal (i.e., $1/m$). The formula was first obtained by Argyris *et al.*[11] and formalized in a different manner by others.[7,12]

The reader can verify the shape functions for the second- and third-order elements as given below and indeed derive ones of any higher order easily.

Quadratic triangle [Fig. 8.15(*b*)]

Corner nodes:

$$N_1 = (2L_1 - 1)L_1, \qquad \text{etc.}$$

Mid-side nodes:

$$N_4 = 4L_1L_2, \qquad \text{etc.}$$

Cubic triangle [Fig. 8.15(*c*)]

Corner nodes:

$$N_1 = \tfrac{1}{2}(3L_1 - 1)(3L_1 - 2)L_1, \qquad \text{etc.} \tag{8.36}$$

Mid-side nodes:

$$N_4 = \tfrac{9}{2}L_1L_2(3L_1 - 1), \qquad \text{etc.} \tag{8.37}$$

and for the internal node:

$$N_{10} = 27L_1L_2L_3$$

The last shape again is a 'bubble' function giving zero contribution along boundaries – and this will be found to be useful in many other contexts (see the mixed forms in Chapter 12).

The quadratic triangle was first derived by Veubeke[13] and used later in the context of plane stress analysis by Argyris.[14]

When element matrices have to be evaluated it will follow that we are faced with integration of quantities defined in terms of area coordinates over the triangular region. It is useful to note in this context the following exact integration expression:

$$\iint_{\Delta} L_1^a L_2^b L_3^c \, \mathrm{d}x \, \mathrm{d}y = \frac{a!\, b!\, c!}{(a+b+c+2)!} 2\Delta \tag{8.38}$$

One-dimensional elements

8.9 Line elements

So far in this book the continuum was considered generally in two or three dimensions. 'One-dimensional' members, being of a kind for which exact solutions are generally available, were treated only as trivial examples in Chapter 2 and in Sec. 8.2. In many practical two- or three-dimensional problems such elements do in fact appear in conjunction with the more usual continuum elements – and a unified treatment is desirable. In the context of elastic analysis these elements may represent lines of reinforcement (plane and three-dimensional problems) or sheets of thin lining material in axisymmetric bodies. In the context of field problems of the type discussed in Chapter 7 lines of drains in a porous medium of lesser conductivity can be envisaged.

Once the shape of such a function as displacement is chosen for an element of this kind, its properties can be determined, noting, however, that derived quantities such as strain, etc., have to be considered only in one dimension.

Figure 8.18 shows such an element sandwiched between two adjacent quadratic-type elements. Clearly for continuity of the function a quadratic variation of the

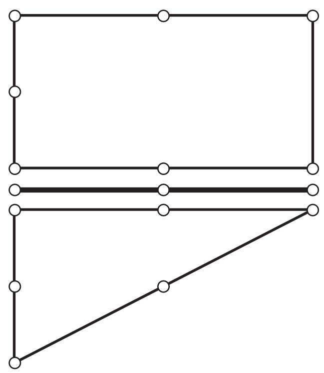

Fig. 8.18 A line element sandwiched between two-dimensional elements.

unknown with the one variable ξ is all that is required. Thus the shape functions are given directly by the Lagrange polynomial as defined in Eq. (8.18).

Three-dimensional elements

8.10 Rectangular prisms – Lagrange family

In a precisely analogous way to that given in previous sections equivalent elements of three-dimensional type can be described.

Now, for interelement continuity the simple rules given previously have to be modified. What is necessary to achieve is that along a whole face of an element the nodal values define a unique variation of the unknown function. With incomplete polynomials, this can be ensured only by inspection.

Shape function for such elements, illustrated in Fig. 8.19, will be generated by a direct product of three Lagrange polynomials. Extending the notation of Eq. (8.19) we now have

$$N_i \equiv N_{IJK} = l_I^n l_J^m l_K^p \tag{8.39}$$

for n, m, and p subdivisions along each side.

This element again is suggested by Zienkiewicz *et al.*[5] and elaborated upon by Argyris *et al.*[6] All the remarks about internal nodes and the properties of the formulation with mappings (to be described in the next chapter) are applicable here.

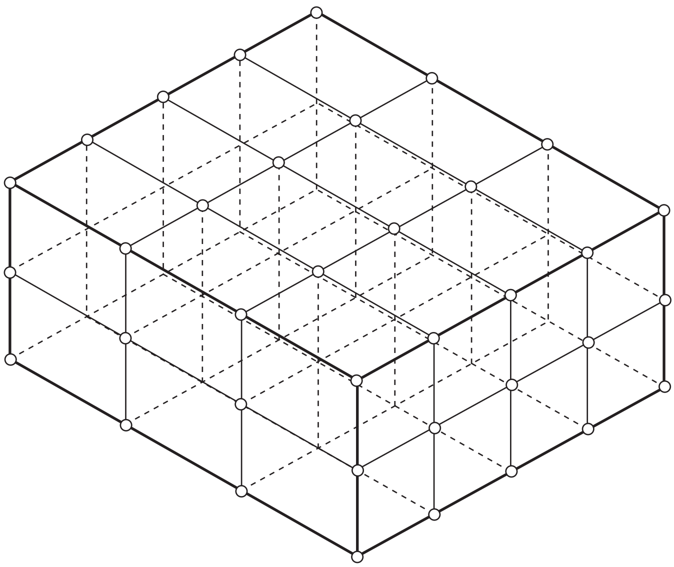

Fig. 8.19 Right prism of Lagrange family.

8.11 Rectangular prisms – 'serendipity' family[4,9,15]

The family of elements shown in Fig. 8.20 is precisely equivalent to that of Fig. 8.9. Using now three normalized coordinates and otherwise following the terminology of Sec. 8.6 we have the following shape functions:

'Linear' element (8 nodes)

$$N_i = \tfrac{1}{8}(1+\xi_0)(1+\eta_0)(1+\zeta_0) \qquad (8.40)$$

which is identical with the linear lagrangian element.

'Quadratic' element (20 nodes)

Corner nodes:

$$N_i = \tfrac{1}{8}(1+\xi_0)(1+\eta_0)(1+\zeta_0)(\xi_0+\eta_0+\zeta_0-2) \qquad (8.41)$$

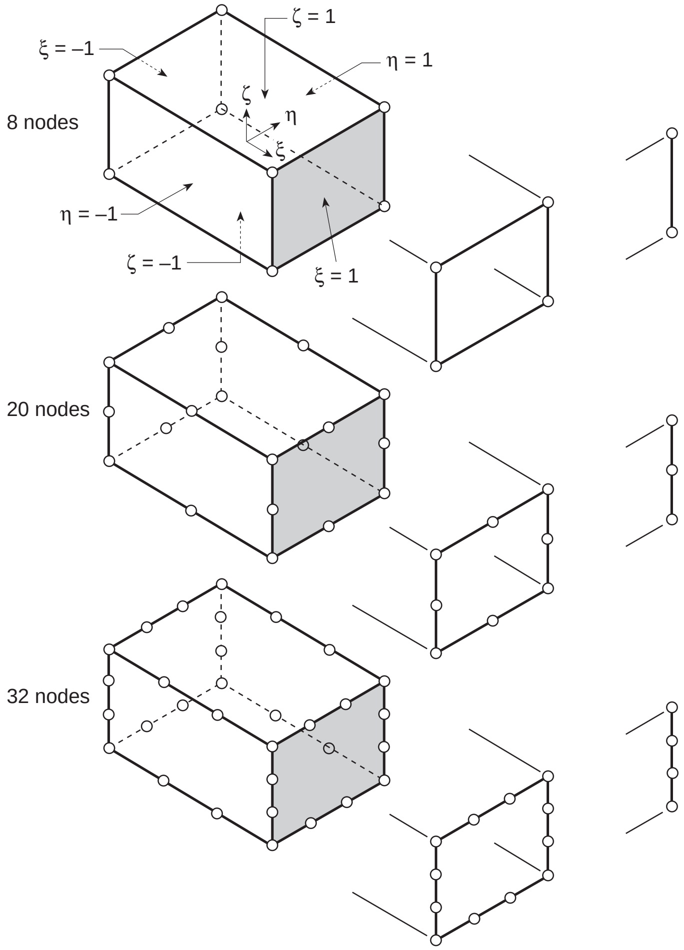

Fig. 8.20 Right prisms of boundary node (serendipity) family with corresponding sheet and line elements.

Typical mid-side node:

$$\xi_i = 0 \qquad \eta_i = \pm 1 \qquad \zeta_i = \pm 1$$
$$N_i = \tfrac{1}{4}(1 - \xi^2)(1 + \eta_0)(1 + \zeta_0)$$

'Cubic' elements (32 nodes)

Corner node:

$$N_i = \tfrac{1}{64}(1 + \xi_0)(1 + \eta_0)(1 + \zeta_0)[9(\xi^2 + \eta^2 + \zeta^2) - 19] \tag{8.42}$$

Typical mid-side node:

$$\xi_i = \pm \tfrac{1}{3} \qquad \eta_i = \pm 1 \qquad \zeta_i = \pm 1$$
$$N_i = \tfrac{9}{64}(1 - \xi^2)(1 + 9\xi_0)(1 + \eta_0)(1 + \zeta_0)$$

When $\zeta = 1 = \zeta_0$ the above expressions reduce to those of Eqs (8.22)–(8.24). Indeed such elements of three-dimensional type can be joined in a compatible manner to sheet or line elements of the appropriate type as shown in Fig. 8.20.

Once again the procedure for generating the shape functions follows that described in Figs 8.10 and 8.11 and once again elements with varying degrees of freedom along the edges can be derived following the same steps.

The equivalent of a Pascal triangle is now a tetrahedron and again we can observe the small number of surplus degrees of freedom – a situation of even greater magnitude than in two-dimensional analysis.

8.12 Tetrahedral elements

The tetrahedral family shown in Fig. 8.21 not surprisingly exhibits properties similar to those of the triangle family.

Firstly, once again complete polynomials in three coordinates are achieved at each stage. Secondly, as faces are divided in a manner identical with that of the previous triangles, the same order of polynomial in two coordinates in the plane of the face is achieved and element compatibility ensured. No surplus terms in the polynomial occur.

8.12.1 Volume coordinates

Once again special coordinates are introduced defined by (Fig. 8.22):

$$\begin{aligned} x &= L_1 x_1 + L_2 x_2 + L_3 x_3 + L_4 x_4 \\ y &= L_1 y_1 + L_2 y_2 + L_3 y_3 + L_4 y_4 \\ z &= L_1 z_1 + L_2 z_2 + L_3 z_3 + L_4 z_4 \\ 1 &= L_1 + L_2 + L_3 + L_4 \end{aligned} \tag{8.43}$$

Solving Eq. (8.43) gives

$$L_1 = \frac{a_1 + b_1 x + c_1 y + d_1 z}{6V} \qquad \text{etc.}$$

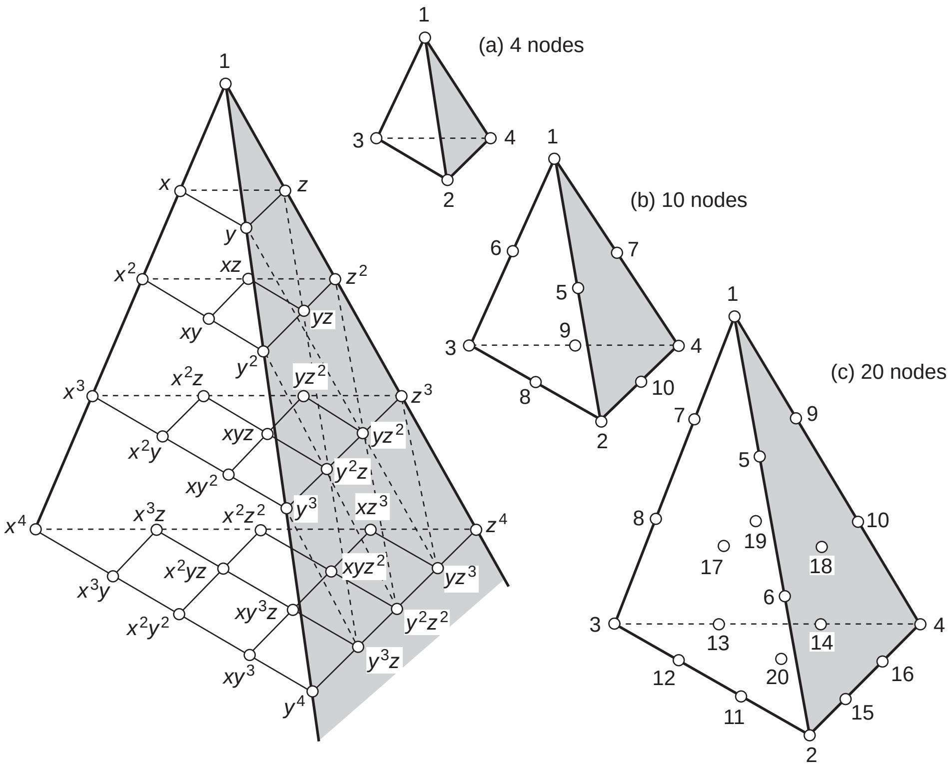

Fig. 8.21 The tetrahedron family: (*a*) linear, (*b*) quadratic, and (*c*) cubic.

where the constants can be identified from Chapter 6, Eq. (6.5). Again the physical nature of the coordinates can be identified as the ratio of volumes of tetrahedra based on an internal point P in the total volume, e.g., as shown in Fig. 8.22:

$$L_1 = \frac{\text{volume } P234}{\text{volume } 1234}, \qquad \text{etc.} \tag{8.44}$$

8.12.2 Shape function

As the volume coordinates vary linearly with the cartesian ones from unity at one node to zero at the opposite face then shape functions for the linear element [Fig. 8.21(a)] are simply

$$N_1 = L_1 \qquad N_2 = L_2, \qquad \text{etc.} \tag{8.45}$$

Formulae for shape functions of higher order tetrahedra are derived in precisely the same manner as for the triangles by establishing appropriate Lagrange-type formulae similar to Eq. (8.35). Leaving this to the reader as a suitable exercise we quote the following:

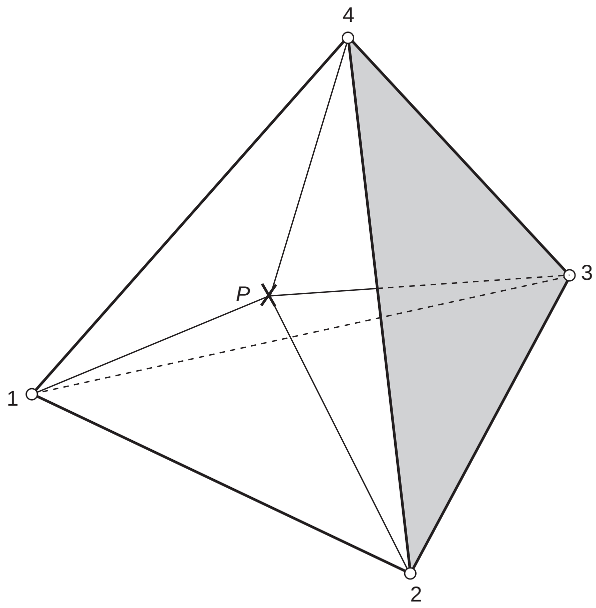

Fig. 8.22 Volume coordinates.

'Quadratic' tetrahedron [Fig. 8.21(b)]

For corner nodes:

$$N_1 = (2L_1 - 1)L_1, \qquad \text{etc.} \tag{8.46}$$

For mid-edge nodes:

$$N_5 = 4L_1L_2, \qquad \text{etc.}$$

'Cubic' tetrahedron

Corner nodes:

$$N_1 = \tfrac{1}{2}(3L_1 - 1)(3L_1 - 2)L_1, \qquad \text{etc.} \tag{8.47}$$

Mid-edge nodes:

$$N_5 = \tfrac{9}{2}L_1L_2(3L_1 - 1), \qquad \text{etc.}$$

Mid-face nodes:

$$N_{17} = 27L_1L_2L_3, \qquad \text{etc.}$$

A useful integration formula may again be here quoted:

$$\iiint_{\text{vol}} L_1^a L_2^b L_3^c L_4^d \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \frac{a!\, b!\, c!\, d!}{(a+b+c+d+3)!} 6V \tag{8.48}$$

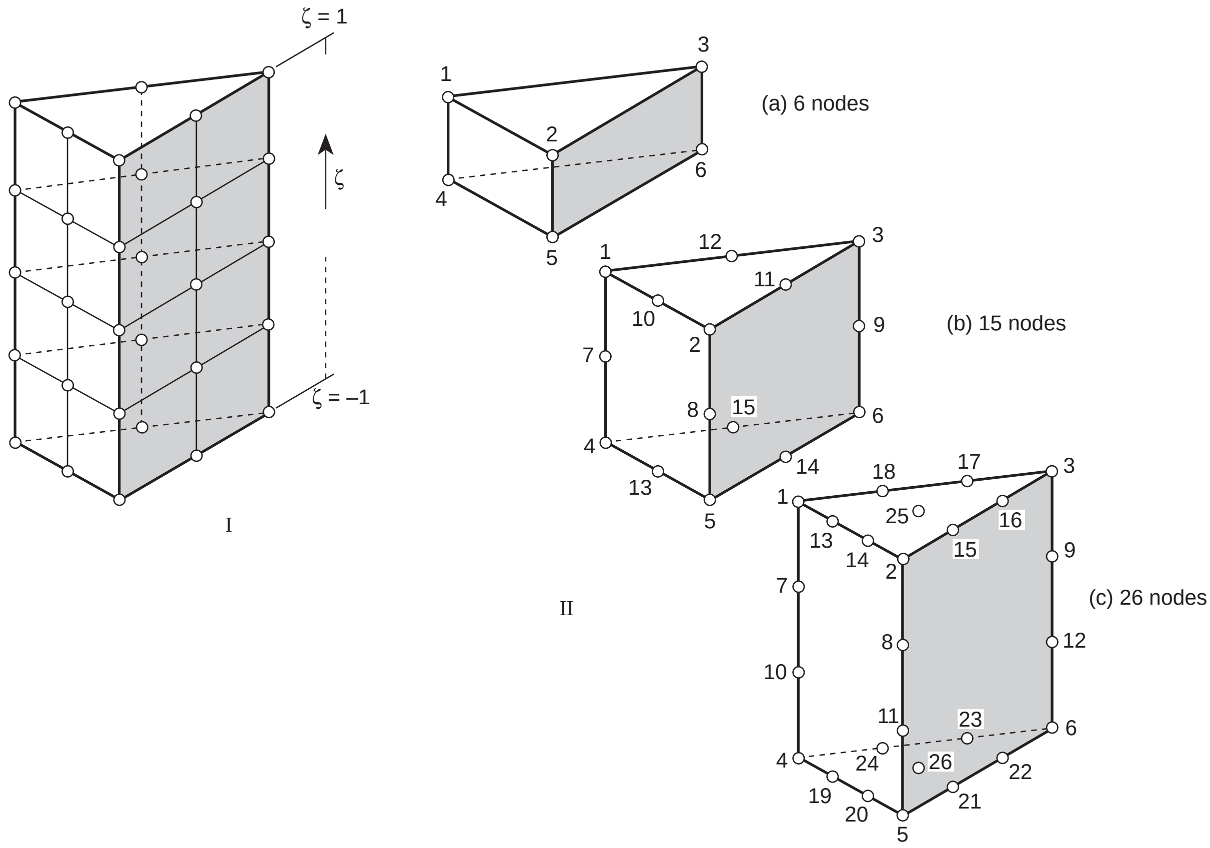

Fig. 8.23 Triangular prism elements (serendipity) family: (*a*) linear, (*b*) quadratic, and (*c*) cubic.

8.13 Other simple three-dimensional elements

The possibilities of simple shapes in three dimensions are greater, for obvious reasons, than in two dimensions. A quite useful series of elements can, for instance, be based on triangular prisms (Fig. 8.23). Here again variants of the product, Lagrange, approach or of the 'serendipity' type can be distinguished. The first element of both families is identical and indeed the shape functions for it are so obvious as not to need quoting.

For the 'quadratic' element illustrated in Fig. 8.23(*b*) the shape functions are

Corner nodes $L_1 = \zeta_1 = 1$:

$$N_1 = \tfrac{1}{2}L_1(2L_1 - 1)(1+\zeta) - \tfrac{1}{2}L_1(1-\zeta^2) \tag{8.49}$$

Mid-edge of triangles:

$$N_{10} = 2L_1L_2(1+\zeta), \qquad \text{etc.} \tag{8.50}$$

Mid-edge of rectangle:

$$N_7 = L_1(1-\zeta^2), \qquad \text{etc.}$$

Such elements are not purely esoteric but have a practical application as 'fillers' in conjunction with 20-noded serendipity elements.

Part 2 Hierarchical shape functions

8.14 Hierarchic polynomials in one dimension

The general ideas of hiearchic approximation were introduced in Sect. 8.2 in the context of simple, linear, elements. The idea of generating higher order hierarchic forms is again simple. We shall start from a one-dimensional expansion as this has been shown to provide a basis for the generation of two- and three-dimensional forms in previous sections.

To generate a polynomial of order p along an element side we do not need to introduce nodes but can instead use parameters without an obvious physical meaning. As shown in Fig. 8.24, we could use here a linear expansion specified by 'standard' functions N_0 and N_1 and add to this a series of polynomials always designed so as to have zero values at the ends of the range (i.e. points 0 and 1).

Thus for a quadratic approximation, we would write over the typical one-dimensional element, for instance,

$$\hat{u} = u_0N_0 + u_1N_1 + a_2N_2 \tag{8.51}$$

where

$$N_0 = -\frac{\xi - 1}{2} \qquad N_1 = \frac{\xi + 1}{2} \qquad N_2 = -(\xi - 1)(\xi + 1) \tag{8.52}$$

using in the above the normalized x-coordinate [viz. Eq. (8.17)].

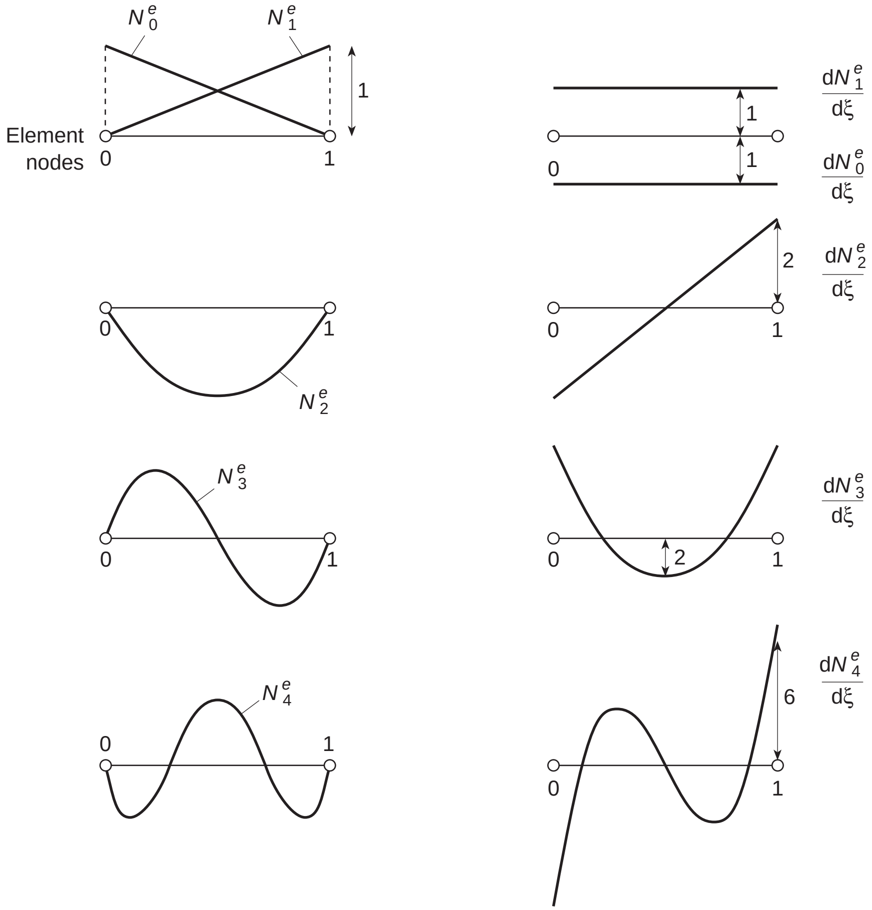

Fig. 8.24 Hierarchical element shape functions of nearly orthogonal form and their derivatives.

We note that the parameter a_2 does in fact have a meaning in this case as it is the magnitude of the departure from linearity of the approximation $\hat{u}$ at the element centre, since N_2 has been chosen here to have the value of unity at that point.

In a similar manner, for a cubic element we simply have to add $a_3 N_3$ to the quadratic expansion of Eq. (8.51), where N_3 is any cubic of the form

$$N_3^e = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 \tag{8.53}$$

and which has zero values at $\xi = \pm 1$ (i.e., at nodes 0 and 1). Again an infinity of choices exists, and we could select a cubic of a simple form which has a zero value at the centre of the element and for which $dN_3/d\xi = 1$ at the same point. Immediately we can write

$$N_3^e = \xi(1 - \xi^2) \tag{8.54}$$

as the cubic function with the desired properties. Now the parameter a_3 denotes the departure of the slope at the centre of the element from that of the first approximation.

We note that we could proceed in a similar manner and define the fourth-order hierarchical element shape function as

$$N_4^e = \xi^2(1-\xi^2) \tag{8.55}$$

but a physical identification of the parameter associated with this now becomes more difficult (even though it is not strictly necessary).

As we have already noted, the above set is not unique and many other possibilities exist. An alternative convenient form for the hierarchical functions is defined by

$$N_p^e(\xi) = \begin{cases} \dfrac{1}{p!}(\xi^p - 1) & p \text{ even} \\ \dfrac{1}{p!}(\xi^p - \xi) & p \text{ odd} \end{cases} \tag{8.56}$$

where p ($\geqslant 2$) is the degree of the introduced polynomial.[16] This yields the set of shape functions:

$$\begin{aligned} N_2^e &= \tfrac{1}{2}(\xi^2 - 1) \qquad & N_3^e &= \tfrac{1}{6}(\xi^3 - \xi) \\ N_4^e &= \tfrac{1}{24}(\xi^4 - 1) \qquad & N_5^e &= \tfrac{1}{120}(\xi^5 - \xi) \qquad \text{etc.} \end{aligned} \tag{8.57}$$

We observe that all derivatives of N_p^e of second or higher order have the value zero at $\xi = 0$, apart from $\mathrm{d}^p N_p^e/\mathrm{d}\xi^p$, which equals unity at that point, and hence, when shape functions of the form given by Eq. (8.57) are used, we can identify the parameters in the approximation as

$$a_p^e = \left.\frac{\mathrm{d}^p \hat{u}}{\mathrm{d}\xi^p}\right|_{\xi=0} \qquad p \geqslant 2 \tag{8.58}$$

This identification gives a general physical significance but is by no means necessary.

In two- and three-dimensional elements a simple identification of the hierarchic parameters on interfaces will automatically ensure C_0 continuity of the approximation.

As mentioned previously, an optimal form of hierarchical function is one that results in a diagonal equation system. This can on occasion be achieved, or at least approximated, quite closely.

In the elasticity problems which we have discussed in the preceding chapters the element matrix $\mathbf{K}^e$ possesses terms of the form [using Eq. (8.17)]

$$K_{lm}^e = \int_{\Omega^e} k \frac{\mathrm{d}N_l^e}{\mathrm{d}x}\frac{\mathrm{d}N_m^e}{\mathrm{d}x}\,\mathrm{d}x = \frac{1}{a}\int_{-1}^{1} k \frac{\mathrm{d}N_l^e}{\mathrm{d}\xi}\frac{\mathrm{d}N_m^e}{\mathrm{d}\xi}\,\mathrm{d}\xi \tag{8.59}$$

If shape function sets containing the appropriate polynomials can be found for which such integrals are zero for $l \neq m$, then orthogonality is achieved and the coupling between successive solutions disappears.

One set of polynomial functions which is known to possess this orthogonality property over the range $-1 \leqslant \xi \leqslant 1$ is the set of Legendre polynomials $P_p(\xi)$, and the shape functions could be defined in terms of integrals of these polynomials.[9] Here we define the Legendre polynomial of degree p by

$$P_p(\xi) = \frac{1}{(p-1)!}\frac{1}{2^{p-1}}\frac{\mathrm{d}^p}{\mathrm{d}\xi^p}[(\xi^2-1)^p] \tag{8.60}$$

and integrate these polynomials to define

$$N_{p+1}^e = \int P_p(\xi)\,\mathrm{d}\xi = \frac{1}{(p-1)!\,2^{p-1}} \frac{\mathrm{d}^{p-1}}{\mathrm{d}\xi^{p-1}}[(\xi^2-1)^p] \tag{8.61}$$

Evaluation for each p in turn gives

$$N_2^e = \xi^2 - 1 \qquad N_3^e = 2(\xi^3 - \xi) \qquad \text{etc.}$$

These differ from the element shape functions given by Eq. (8.57) only by a multiplying constant up to N_3^e, but for $p \geqslant 3$ the differences become significant. The reader can easily verify the orthogonality of the derivatives of these functions, which is useful in computation. A plot of these functions and their derivatives is given in Fig. 8.24.

8.15 Two- and three-dimensional, hierarchic, elements of the 'rectangle' or 'brick' type

In deriving 'standard' finite element approximations we have shown that all shape functions for the Lagrange family could be obtained by a simple multiplication of one-dimensional ones and those for serendipity elements by a combination of such multiplications. The situation is even simpler for hierarchic elements. Here *all* the shape functions can be obtained by a simple multiplication process.

Thus, for instance, in Fig. 8.25 we show the shape functions for a lagrangian nine-noded element and the corresponding hierarchical functions. The latter not only have simpler shapes but are more easily calculated, being simple products of linear and quadratic terms of Eq. (8.56), (8.57), or (8.61). Using the last of these the three functions illustrated are simply

$$\begin{aligned} N_1 &= (1-\xi)(1+\eta)/4 \\ N_2 &= (1-\xi)(1-\eta^2)/2 \\ N_3 &= (1-\xi^2)(1-\eta^2) \end{aligned} \tag{8.62}$$

The distinction between lagrangian and serendipity forms now disappears as for the latter in the present case the last shape function (N_3) is simply omitted.

Indeed, it is now easy to introduce interpolation for elements of the type illustrated in Fig. 8.11 in which a different expansion is used along different sides. This essential characteristic of hierarchical elements is exploited in adaptive refinement (viz. Chapter 15) where new degrees of freedom (or polynomial order increase) is made only when required by the magnitude of the error.

8.16 Triangle and tetrahedron family[16,17]

Once again the concepts of multiplication can be introduced in terms of area (volume) coordinates.

Returning to the triangle of Fig. 8.16 we note that along the side 1–2, L_3 is identically zero, and therefore we have

$$(L_1 + L_2)_{1-2} = 1 \tag{8.63}$$

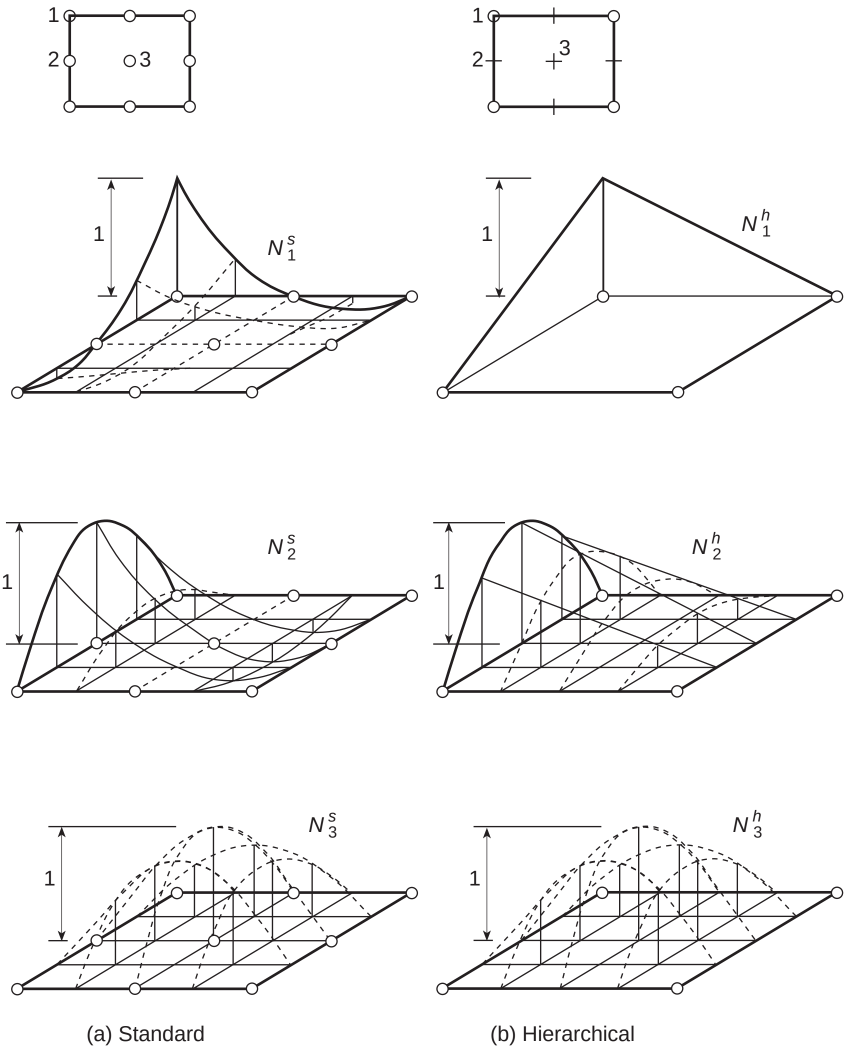

Fig. 8.25 Standard and hierarchic shape functions corresponding to a lagrangian, quadratic element.

If ξ, measured along side 1–2, is the usual non-dimensional local element coordinate of the type we have used in deriving hierarchical functions for one-dimensional elements, we can write

$$L_1|_{1-2} = \tfrac{1}{2}(1-\xi) \qquad L_2|_{1-2} = \tfrac{1}{2}(1+\xi) \tag{8.64}$$

from which it follows that we have

$$\xi = (L_2 - L_1)_{1-2} \tag{8.65}$$

This suggests that we could generate hierarchical shape functions over the triangle by generalizing the one-dimensional shape function forms produced earlier. For example, using the expressions of Eq. (8.56), we associate with the side 1–2 the polynomial of degree p ($\geqslant 2$) defined by

$$N^e_{p(1-2)} = \begin{cases} \dfrac{1}{p!}[(L_2 - L_1)^p - (L_1 + L_2)^p] & p \text{ even} \\ \dfrac{1}{p!}[(L_2 - L_1)^p - (L_2 - L_1)(L_1 + L_2)^{p-1}] & p \text{ odd} \end{cases} \tag{8.66}$$

It follows from Eq. (8.64) that these shape functions are zero at nodes 1 and 2. In addition, it can easily be shown that $N^e_{p(1-2)}$ will be zero all along the sides 3–1 and 3–2 of the triangle, and so C_0 continuity of the approximation $\hat{u}$ is assured.

It should be noted that in this case for $p \geqslant 3$ the number of hierarchical functions arising from the element sides in this manner is insufficient to define a complete polynomial of degree p, and internal hierarchical functions, which are identically zero on the boundaries, need to be introduced; for example, for $p = 3$ the function $L_1 L_2 L_3$ could be used, while for $p = 4$ the three additional functions $L_1^2 L_2 L_3$, $L_1 L_2^2 L_3$, $L_1 L_2 L_3^2$ could be adopted.

In Fig. 8.26 typical hierarchical linear, quadratic, and cubic trial functions for a triangular element are shown. Similar hierarchical shape functions could be generated

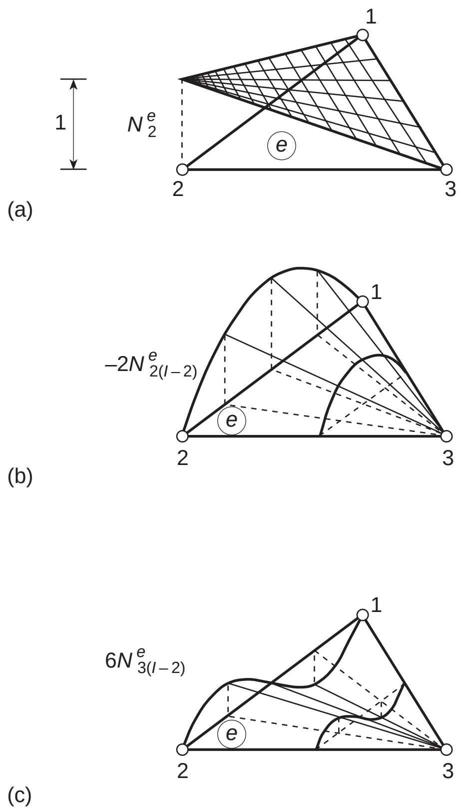

Fig. 8.26 Triangular elements and associated hierarchical shape functions of (*a*) linear, (*b*) quadratic, and (*c*) cubic form.

from the alternative set of one-dimensional shape functions defined in Eq. (8.61). Identical procedures are obvious in the context of tetrahedra.

8.17 Global and local finite element approximation

The very concept of hierarchic approximations (in which the shape functions are not affected by the refinement) means that it is possible to include in the expansion

$$u = \sum_{i=1}^{n} N_i a_i \tag{8.67}$$

functions N which are not local in nature. Such functions may, for instance, be the exact solutions of an analytical problem which in some way resembles the problem dealt with, but do not satisfy some boundary or inhomogeneity conditions. The 'finite element', local, expansions would here be a device for correcting this solution to satisfy the real conditions. This use of the global–local approximation was first suggested by Mote[18] in a problem where the coefficients of this function were fixed. The example involved here is that of a rotating disc with cutouts (Fig. 8.27). The global, known, solution is the analytical one corresponding to a disc without cutout, and finite elements are added locally to modify the solution. Other examples of such 'fixed' solutions may well be those associated with point loads, where the use of the global approximation serves to eliminate the singularity modelled badly by the discretization.

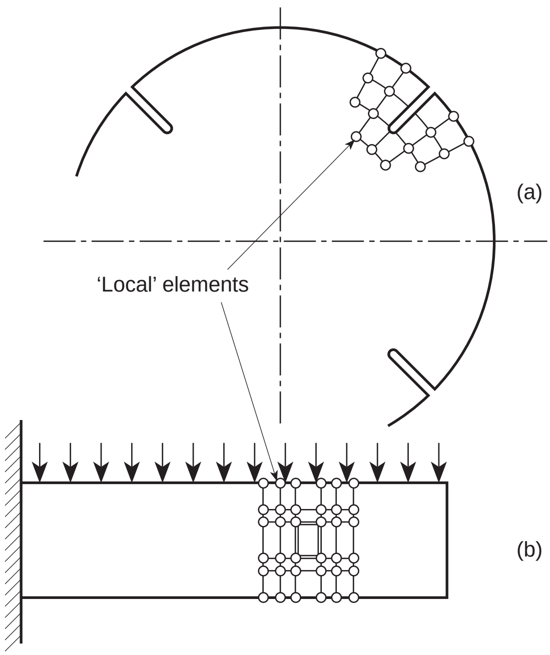

Fig. 8.27 Some possible uses of the local–global approximation: (a) rotating slotted disc, (b) perforated beam.

In some problems the singularity itself is unknown and the appropriate function can be added with an unknown coefficient.

8.18 Improvement of conditioning with hierarchic forms

We have already mentioned that hierarchic element forms give a much improved equation conditioning for steady-state (static) problems due to their form which is more nearly diagonal. In Fig. 8.28 we show the 'condition number' (which is a measure of such diagonality and is defined in standard texts on linear algebra; see Appendix A) for a single cubic element and for an assembly of four cubic elements, using standard and hierarchic forms in their formulation. The improvement of the conditioning is a distinct advantage of such forms and allows the use of iterative solution techniques to be more easily adopted.[19] Unfortunately much of this advantage disappears for transient analysis as the approximation must contain specific modes (see Chapter 17).

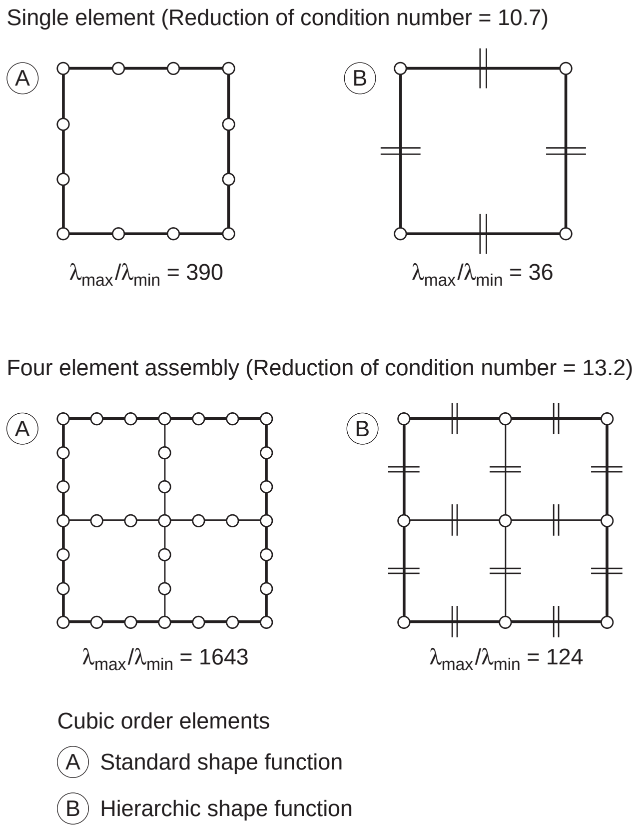

Fig. 8.28 Improvement of condition number (ratio of maximum to minimum eigenvalue of the stiffness matrix) by use of a hierarchic form (elasticity isotropic $\nu = 0.15$).

8.19 Concluding remarks

An unlimited selection of element types has been presented here to the reader – and indeed equally unlimited alternative possibilities exist.[4,9] What of the use of such complex elements in practice? The triangular and tetrahedral elements are limited to situations where the real region is of a suitable shape which can be represented as an assembly of flat facets and all other elements are limited to situations represented by an assembly of right prisms. Such a limitation would be so severe that little practical purpose would have been served by the derivation of such shape functions unless some way could be found of distorting these elements to fit realistic curved boundaries. In fact, methods for doing this are available and will be described in the next chapter.

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