On the other hand, the first derivative of the function $\sqrt{x}$ is infinity in the point $x = 0$; therefore, approximation of $\sqrt{x}$ in the neighborhood of $x = 0$ by polynomials is considerably more difficult.

***Exercise 3.14.*** Using 2, 3, and 4 Gaussian quadrature points, evaluate each of the following integrals:

$$I_1 = \int_0^{\pi} \sin x \, dx, \qquad I_2 = \int_0^1 \sqrt{x}\, dx, \qquad I_3 = \int_{0.1}^{1.1} \sqrt{x}\, dx$$

and compute the relative errors. (Hint: Use the transformation (3.10a).) (Partial answer: for $n = 3$, $I_2 \cong 0.66917963$, and the relative error is 0.38%.)

***Exercise 3.15.*** An elastic bar is loaded by traction defined as follows:

$$T(x) = \begin{cases} 1 - \dfrac{(x - x_m)^2}{a^2} & \text{for } |x - x_m| \le a \\ 0 & \text{for } |x - x_m| > a \end{cases}$$

where $x_m$ and $a$ are input data. Assuming that $x_k + a \le x_m \le x_{k+1} - a$, compute the load vector term $r_2^{(k)}$ by Gaussian quadrature. Use the smallest number of Gauss points such that $r_2^{(k)}$ is computed exactly. (Hint: Use the mapping $x = x_m + a\xi$, $-1 \le \xi \le +1$.) Answer: $r_2^{(k)} = 4ax_m/3l_k$.

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## CHAPTER 4

# EXTENSIONS AND THEIR CONVERGENCE RATES IN ONE DIMENSION

In Chapter 3 we saw that the finite element method selects that $u_{FE}$ from the finite element space of admissible functions $\tilde{S}$ which minimizes the energy norm of the error:

$$\|u_{EX} - u_{FE}\|_{E(\Omega)} = \min_{u \in \tilde{S}} \|u_{EX} - u\|_{E(\Omega)}. \tag{4.1}$$

We have noted that this relationship indicates that the error $\|e\|_{E(\Omega)}$ depends on $u_{EX}$ and the space $\tilde{S}$ which is determined by the mesh, the polynomial degrees of elements, and the mapping functions, collectively called *discretization*. Discretization is controlled by the users of finite element computer programs, either directly or through procedures designed to select or modify certain discretization parameters automatically on the basis of data generated in the course of the solution process.

Engineering computations are performed for the purpose of obtaining information concerning the expected response of physical systems to certain imposed conditions, generally called loads. This information is then used in making engineering decisions. Obviously, the computed data must be of such quality that decisions based on them will be substantially the same *as if the exact solution were known*. Therefore, we wish to select the discretization so that $u_{FE}$ is close to $u_{EX}$, in some sense. In general, we wish to determine functionals $\Psi_i(u_{FE})$ $(i = 1, 2, \ldots)$, such as displacements, stresses, reactions, stress intensity factors, etc., so that:

$$\left| \frac{\Psi_i(u_{EX}) - \Psi_i(u_{FE})}{\Psi_i(u_{EX})} \right| \le \tau_i \qquad (i = 1, 2, \ldots) \tag{4.2}$$

where $\tau_i$ represents specific tolerances. The question naturally arises: How can we tell whether $\Psi_i(u_{FE})$ is close to $\Psi_i(u_{EX})$ if we do not know $u_{EX}$? The answer is: by performing *extensions* and certain tests on the finite element solutions. Both the estimation and control of the errors of discretization are based on extensions.

Extensions are systematic changes of discretization so that the number of degrees of freedom is increased at each change. More precisely, a sequence of finite element spaces $S_1, S_2, S_3 \ldots$ with progressively improved approximation properties is created and the corresponding solutions obtained. If extension is based on mesh refinement, then the process is called *h-extension*†. If extension is based on increasing the polynomial degree of elements, then the process is called *p-extension*‡. If extension is based on a combination of *concurrent mesh refinement and increase in the polynomial degree of elements*, then it is called *hp-extension*. Extensions provide information on the basis of which we can draw conclusions concerning the overall quality of the finite element solution $u_{FE}$ and the accuracy of functionals computed from $u_{FE}$. When convergence of the finite element solutions corresponding to spaces $S_1, S_2, \ldots$ is of interest, then we refer to *h-, p-, or hp-convergence*; when aspects of implementation are emphasized, then we refer to *the h-, p-, or hp-versions* of the finite element method.

An important consideration is the *performance of extensions*. Engineering problems can be quite large, and engineering computations generally have multiple goals. In addition, several load cases are usually considered. The finite element discretizations, therefore, must be such that all engineering data computed from the finite element solution should concurrently satisfy conditions (4.2) for all load cases. The available resources, human and machine time and disk storage space, are generally limited. Therefore, it is not feasible to perform extensions in the practical engineering environment unless the extension process is efficient. There are very substantial differences in the performance of the various extension processes. We first consider the performance of extension processes in terms of the relationship between the error measured in energy norm and the number of degrees of freedom $N$.

Because the finite element solution minimizes the strain energy of the error (4.1), the strain energy of the error is a logical measure of the overall quality of $u_{FE}$. It is not the only important measure, however. We will see that smallness of error in energy norm does not necessarily guarantee that the error in all quantities of interest is small. Also, we must bear in mind that the number of degrees of freedom is not an accurate measure of resource requirements. The computer time depends on the sparsity of the matrices, the organization of data, and several factors which depend, in turn, on the particulars of implementation and the characteristics of the computer used. Considerations of human time involved in data preparation and verification of the accuracy of computed data

† $h$ represents the size of elements. h-Extension involves letting $h_{max} \to 0$.
‡ $p$ represents the polynomial degree of elements. p-Extensions involve letting $p_{min} \to \infty$.



are usually of overwhelming importance. It is, however, very difficult to quantify these factors. Thus, while the relationship between the error in energy norm and the number of degrees of freedom is the most readily quantifiable and best understood measure of performance, it is an imperfect measure and must be considered along with other factors when evaluating alternative strategies for the estimation and control of discretization errors.

## 4.1. RATES OF CONVERGENCE IN ENERGY NORM

A well developed, elaborate, theoretical basis exists for the estimation of error in energy norm for the h-, p-, and hp-extension processes. The basic ideas are illustrated for very simple problems in the following. For the other cases, a summary of the relevant theorems is given. A detailed survey, covering one- and two-dimensional cases, is presented in [4.1].

### 4.1.1. h-Convergence, Uniform Mesh Refinement, $p = 1$

Let us consider the following problem: Find $u_{EX} \in \mathring{E}(\Omega)$ such that:

$$\int_0^l (AEu'_{EX}v' + cu_{EX}v)\, dx = \int_0^l fv\, dx \quad \text{for all} \quad v \in \mathring{E}(\Omega) \tag{4.3}$$

and let us first assume that $AE$, $c$, and $f$ are such that $u''_{EX}$ is a bounded, continuous function with $|u''_{EX}| \le C$ in the interval $0 \le x \le l$.

Subdivide the interval $0 \le x \le l$ into $n$ elements of equal length. (Here $n = M(\Delta)$.) The length of each element is then: $h = l/n$. Let $u_n$ be the linear interpolant of $u_{EX}$, that is, $u_n$ is a continuous, piecewise, linear function such that:

$$u_n(jh) = u_{EX}(jh), \qquad j = 0, 1, \ldots, n. \tag{4.4}$$

We denote the error of interpolation on the $k$th element by $\bar{e}_k$:

$$\bar{e}_k(x) \stackrel{\text{def}}{=} u_{EX}(x) - u_n(x), \qquad (k-1)h \le x \le kh, \qquad k = 1, 2, \ldots, n. \tag{4.5}$$

Because $\bar{e}_k(x)$ vanishes at the endpoints of the element, there is a point $\bar{x}_k$ where $|\bar{e}_k|$ is maximal. In this point $\bar{e}'_k = 0$. (See Figure 4.1.) Then:

$$\bar{e}'_k(x) = \int_{\bar{x}_k}^{x} \bar{e}''_k(t)\, dt = \int_{\bar{x}_k}^{x} u''_{EX}(t)\, dt, \qquad (k-1)h \le x \le kh. \tag{4.6}$$

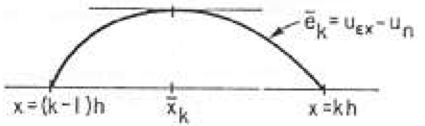


FIGURE 4.1.

Since $|u''_{EX}| \le C$, we have:

$$\max|\tilde{e}'_k(x)| \le hC, \qquad (k-)h \le x \le kh. \tag{4.7}$$

Let us now expand $\tilde{e}_k$ into a Taylor series about the point $\bar{x}_k$, and let us assume that $\bar{x}_k$ is located such that $kh - \bar{x}_k \le h/2$. We now have:

$$\tilde{e}_k(kh) = 0 = \tilde{e}_k(\bar{x}_k) + (kh - \bar{x}_k)\tilde{e}'_k(\bar{x}_k) + \tfrac{1}{2}(kh - \bar{x}_k)^2\tilde{e}''_k(t) \tag{4.8}$$

where $t$ is a point between $\bar{x}_k$ and $x = kh$. Because $\tilde{e}'_k(\bar{x}_k) = 0$:

$$\max|\tilde{e}_k(\bar{x}_k)| = \frac{1}{2}|kh - \bar{x}_k|^2|\tilde{e}''_k(t)| \le \frac{h^2}{8}C. \tag{4.9}$$

If $\bar{x}_k$ is closer to $(k-1)h$ than to $kh$, then we write the Taylor series expression for $\tilde{e}_k((k-1)h)$ instead of $\tilde{e}_k(kh)$ and obtain the same result. Equation (4.9) is a basic result of interpolation theory. In view of (4.6) and (4.9), the strain energy of the error of $\tilde{e}$ is:

$$\mathscr{U}(\tilde{e}) = \frac{1}{2}\int_0^l (AE(\tilde{e}')^2 + c\tilde{e}^2)\,dx = \frac{1}{2}\sum_{k=1}^{n}\int_{(k-1)h}^{kh} (AE(\tilde{e}'_k)^2 + c\tilde{e}_k^2)\,dx$$
$$\le \frac{1}{2}nh\left(K_1(Ch)^2 + K_2\left(\frac{h^2}{8}C\right)^2\right) \tag{4.10a}$$

where $K_1$ and $K_2$ are constants chosen so that $AE(x) \le K_1$ and $c(x) \le K_2$ in the interval $0 \le x \le l$. Clearly, $K_1$, $K_2$ are independent of $h$. Since $nh = l$, there is a constant $K$ such that:

$$\mathscr{U}(\tilde{e}) \le \tfrac{1}{2}KlC^2h^2. \tag{4.10b}$$

Finally, because the energy norm of the error of the finite element solution $e \overset{\text{def}}{=} u_{EX} - u_{FE}$ is less than or equal to the energy norm of the error of $\tilde{e}$ (see (4.1)), we have:

$$\|e\|_{E(\Omega)} = \sqrt{\mathscr{U}(e)} \le kCh \tag{4.11a}$$



where $C$ depends on $AE$, $c$, $f$ but is independent of $h$. This estimate is a typical *a priori* estimate. *A priori* estimates are based on some general information about the exact solution. For example, in this case, we were concerned with solutions which satisfy $|u''_{EX}| \le C$. In actual problems $C$ is not known, or known very inaccurately. *A priori* estimates indicate how fast the error changes as the discretization is changed. To obtain error estimates for specific problems, additional information is necessary. The source of additional information is the finite element solution. Estimates which employ such information are called *a posteriori* estimates. We will demonstrate that it is possible to obtain very accurate *a posteriori* estimates.

In order to allow comparison between discretization strategies, we will write *a priori* estimates in terms of the number of degrees of freedom. In the one dimensional case for $p = 1$ $N \approx l/h$, therefore we can write (4.11*a*) in the form:

$$\|e\|_{E(\Omega)} \le \frac{k}{N}. \tag{4.11b}$$

The constant $k$ in (4.11*b*) is not the same as the constant $k$ in (4.11*a*). Expressions of this type should be interpreted to mean that there is a constant such that the inequality holds.

We will consider problems for which the exact solution is of the form:

$$u_{EX} = \begin{cases} -u_s\dfrac{x}{l}\left(1 - \dfrac{a}{l}\right)^{\lambda} & x \le a \\ u_s\left(\dfrac{x}{l} - \dfrac{a}{l}\right)^{\lambda} - u_s\dfrac{x}{l}\left(1 - \dfrac{a}{l}\right)^{\lambda} & x > a, \end{cases} \tag{4.12}$$

where $u_s$ is a constant and $0 \le a < l$. The solution is said to be *smooth* if $\lambda$ is large. If $\lambda \ge 2$, then the foregoing analysis is valid for this class of problems. In our formulation of the finite element method, called the *displacement formulation*, we require only that $\mathscr{U}(u_{EX}) < \infty$. It is left to the reader to show that this condition is satisfied when $\lambda > 1/2$.

***Exercise 4.1.*** Show that if $u_{EX}$ is of the form (4.12) and $\lambda > 1/2$, then $\mathscr{U}(u_{EX})$ is finite.

In many practical problems the solutions are not smooth. Consequently, (4.12) with $1/2 < \lambda < 2$ is representative. For any $\lambda > 1/2$, the relationship between the error in energy norm and the number of degrees of freedom is:

$$\|e\|_{E(\Omega)} \le \frac{k}{N^{\beta}} \tag{4.13a}$$

where $k$ and $\beta$ are constants, $k$ is independent of $h$ but is dependent on the polynomial degree of elements $p$, and:

$$\beta = \min\left(p, \lambda - \frac{1}{2}\right). \tag{4.13b}$$

Detailed proof of (4.13*a*,*b*) can be found in [4.2]. This estimate is valid under much less restrictive assumptions. We have restricted our statement to the class of problems (4.12) in order to avoid having to give a precise definition for the smoothness parameter $\lambda$ for the general case. In its general form this estimate shows that any function in $E(\Omega)$ can be approximated arbitrarily closely in energy norm by using sufficiently fine mesh.

The form of expression (4.13*a*) is typical of h- and p-extensions. These estimates are "sharp" for large $N$ values in the sense that the less than or equal sign ($\leq$) can be replaced by "approximately equal" ($\approx$) when $N$ is large. Taking the logarithm of both sides we have:

$$\log\|e\|_{E(\Omega)} \approx \log k - \beta \log N. \tag{4.14}$$

If we plot $\log\|e\|_{E(\Omega)}$ versus $\log N$, then, for large $N$, we see a downward sloping straight line with the slope $-\beta$. The absolute value of the slope, $\beta$, is called the *asymptotic rate of convergence in energy norm* or, simply, the rate of convergence. When the estimate is of the form (4.13*a*,*b*) then the rate of convergence is said to be *algebraic*. The rate of convergence is a measure of how difficult it is to control the error in energy norm. When $\beta$ is large, then the error decreases rapidly as $N$ is increased. When $\beta$ is small, then the error decreases slowly. Of course, the error also depends on $k$. There are methods for estimating $k$ from the results of extensions. This will be discussed in Section 4.2.

***Example 4.1.*** Let $AE = 1$, $c = 1$, $l = 1$ and define $f$ so that $u_{EX}$ is of the form (4.12) with $a = 0$ and $\lambda = 0.65$. Solve this problem using a sequence of uniform meshes and uniform p-distribution of $p = 1$. Plot the relative error in energy norm:

$$(e_r)_E \overset{\text{def}}{=} \frac{\|u_{EX} - u_{FE}\|_{E(\Omega)}}{\|u_{EX}\|_{E(\Omega)}}$$

against $N$ on log–log scale. Repeat the computations for $\lambda = 1.2$ and $\lambda = 2.0$.

*Solution:* The results are shown in Figure 4.2. Observe that the asymptotic convergence rates predicted by (4.13*b*) are realized.

### 4.1.2. h-Convergence, Optimal or Nearly Optimal Mesh Refinement

Equations (4.13*a*,*b*) indicate that when $u_{EX}$ is smooth, i.e., $\lambda$ is large, then the rate of convergence is controlled by the polynomial degree of elements.



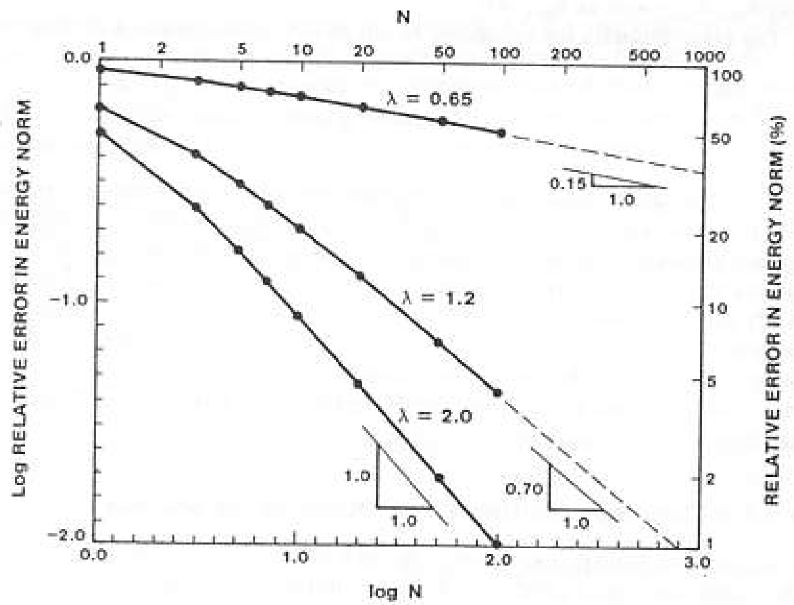


FIGURE 4.2. h-Convergence; uniform mesh refinement, $p = 1$

When $\lambda - 1/2 < p$, i.e., $u_{EX}$ is not smooth, then we can still realize the same rate of convergence if we use properly refined sequences of meshes. However, proper mesh refinement depends on $u_{EX}$, which is not known. Therefore, optimal meshes can be generated only by *adaptive procedures*. Adaptive mesh refinement involves the computation of a finite element solution using an initial, coarse mesh. The relative error contribution (in energy norm) of each element is then estimated. This estimate is based on the idea that the finite element solution $u_{FE}$ can be viewed as the exact solution to a problem which differs from the original problem by the loading function only. The loading corresponding to $u_{FE}$ is called *apparent loading*. For example, if $p = 1$, then there is generally a jump in the axial force term $AEu'_{FE}$ between neighboring elements. These jumps can be interpreted as a series of concentrated forces applied at the nodal points. The jumps also provide information about the second derivative of $u_{EX}$, which, as we have seen, governs the error of the finite element solution. By comparing the apparent loading with the original loading, we can determine which elements contribute most to the total error of approximation. We then refine the mesh, for example, by halving those elements where the estimated strain energy due to the discrepancy between the original loading and the apparent loading is the greatest, and repeat the computations. The resulting sequence of meshes can have the property that the size of the largest element ($h_{max}$) divided by the smallest element ($h_{min}$) is bounded as $h_{min} \to 0$. Such sequences of meshes are called *quasiuniform* meshes*. Quasiuniform meshes are created by the adaptive process when $u_{EX}$ is smooth. When $u_{EX}$ is not smooth (e.g., $1/2 < \lambda < 3/2$), then the adaptive process creates sequences of meshes such that $h_{max}/h_{min} \to \infty$ as $h_{min} \to 0$.

The error estimate for sequences of adaptively refined meshes is then:

$$\|e\|_{E(\Omega)} \le \frac{k}{N^p}. \tag{4.15}$$

That is, the rate of convergence is independent of the smoothness parameter $\lambda$ of the exact solution. Of course, adaptive procedures increase the volume of computations, but the benefit gained by the faster rate of convergence is much greater than the increase in computations. For example, if $u_{EX}$ is of the form (4.12) and $\lambda$ is close to 0.5 (say $\lambda = 0.65$), then it is impractical to reduce the relative error in energy norm to under 1% by uniform mesh refinement; whereas the problem is tractable by adaptive methods. In general, the less smooth the exact solution, the greater the benefits gained through the use of adaptive methods. Detailed analysis is available in [4.2].

### 4.1.3. p-Convergence, Uniform or Quasiuniform Meshes

If $u_{EX}$ is smooth (e.g., $u_{EX} = \sin(\pi x/l)$, then the rate of p-convergence is faster than algebraic, that is, the error in energy norm plotted against the number of degrees of freedom on log–log scale is not a straight line but a downward curving line. For smooth solutions p-convergence is very rapid.

If $u_{EX}$ is not smooth, then we need to distinguish between two cases: (a) Let us assume that the solution is of the form of (4.12), $x = a$ is a nodal point, and $\lambda$ is not an integer. In this case we have:

$$\|e\|_{E(\Omega)} \le \frac{k}{N^{2\lambda-1}}. \tag{4.16a}$$

That is, the rate of p-convergence is exactly twice the rate of h-convergence when h-extensions are based on uniform mesh refinement, provided that $p \ge \lambda - 1/2$ in the h-extension process (see (4.13*a,b*)), and faster otherwise. (b) If $x = a$ is not a nodal point and $\lambda$ is not an integer, then we have:

$$\|e\|_{E(\Omega)} \le \frac{k}{N^{\lambda-1/2}}. \tag{4.16b}$$

That is, the rate of p-convergence is the same as the rate of h-convergence when h-extensions are based on uniform mesh refinement, provided that

* Compare with geometric meshes, defined in section 4.1.4.



$p \ge \lambda - 1/2$ in the h-extension process, and faster otherwise. From this we see that the rate of p-convergence cannot be slower than the rate of h-convergence when uniform mesh refinement is used. Of these two cases, case (a) is representative of an important class of practical problems, whereas problems analogous to case (b) occur relatively rarely in practical problems that involve elliptic partial differential equations.

***Example 4.2.*** As in Example 4.1, let $AE = 1$, $c = 1$, $l = 1$ and define $f$ so that $u_{EX}$ is of the form (4.12) with $a = 0$ and $\lambda = 0.65$. Let $\Delta$ be a mesh of two elements of equal length and compute the finite element solutions for $p = 1, 2, \ldots, 16$. Plot the relative error in energy norm vs. $N$ on log–log scale. Repeat the computations for $\lambda = 1.2$.

*Solution:* The results are shown in Figure 4.3. Note that the asymptotic convergence rates are exactly twice those of the corresponding values in Figure 4.2. In the case $\lambda = 2$ the exact solution is a polynomial of degree 2; hence, for $p \ge 2$, the exact solution is obtained.

### 4.1.4. hp-Convergence

We have seen that when sequences of optimal or nearly optimal meshes are used, then the rate of h-convergence is independent of the smoothness

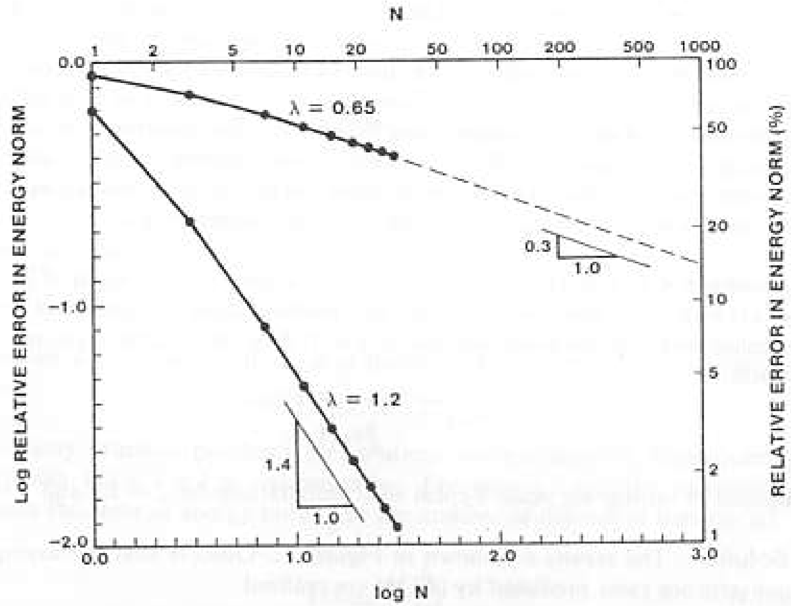


FIGURE 4.3. p-Convergence; uniform mesh, $M(\Delta) = 2$

parameter $\lambda$ and is equal to the polynomial degree of elements. Clearly, the fastest rate of convergence is obtained when *both* the meshes and p-distributions are optimal. When the solution is of the type (4.12) and $\lambda$ is not an integer, then the optimal meshes are graded so that the sizes of elements decrease in *geometric progression* toward to point $x = a$ (which is often called a *point of singularity*). For example, when $a = 0$ then the nodal points are defined as follows:

$$x_{j+1} = \begin{cases} 0 & \text{for } j = 0 \\ lq^{M(\Delta)-j} & \text{for } j = 1, 2, \ldots, M(\Delta) \end{cases} \tag{4.17}$$

where $0 < q < 1$ is the common factor of the geometric progression. Such *geometric meshes* have the property that the ratio $h_{max}/h_{min}$ is unbounded as $h_{min} \to 0$. Interestingly, the asymptotically optimal value of $q$ is independent of the smoothness parameter $\lambda$ and is given by: $q = (\sqrt{2} - 1)^2 \approx 0.17$. In general, overrefinement is preferable to underrefinement and often the values $q = 0.15$ and $q = 0.10$ are used in practice. Because the mesh is independent of $\lambda$, mesh design in hp-extensions is much simpler than in the case of adaptive h-extensions.

Unlike the common factor $q$, optimal p-distributions depend on $\lambda$. The smallest element is assigned $p = 1$ and the largest element is assigned $p = (2\lambda - 1)(M(\Delta) - 1)$ rounded to the nearest integer. The optimal p-distribution varies linearly between these values. If the p-distribution is restricted to be uniform, then optimal results are obtained for $p = p_{max}$. (For proof see [4.2].) In this way, over that part of the domain where the solution is very smooth, high p-values are used to take advantage of the fact that p-extensions are very efficient in the case of smooth solutions. Strong mesh grading is used in the vicinity of singular points to take advantage of the fact that with proper mesh grading the rate of convergence is independent of the smoothness parameter. Because the mesh grading is independent of $\lambda$ and the performance of hp-extensions is not sensitive to the p-distribution, hp-extensions are very *robust*.

In the case of hp extensions, the rate of convergence is *exponential*:

$$\|e\|_{E(\Omega)} \leq \frac{k}{\exp(\beta N^{\gamma})} \tag{4.18}$$

where $k$, $\beta$, and $\gamma$ are positive constants. From (4.18) we have:

$$\log\|e\|_{E(\Omega)} \leq \log k - \beta N^{\gamma} \log e \tag{4.19}$$

where e is the base of natural logarithm. Thus, if we plot $\log\|e\|_{E(\Omega)}$ vs. $N^{\gamma}$, then we see a downward sloping straight line for large $N$*. It has been shown in [4.2] that under certain conditions which are normally met in practice $\gamma = 1/2$.

* Compare (4.19) with (4.14).



***Example 4.3.*** Solve the problem of Example 4.1 for the case of $\lambda = 0.65$ using a sequence of meshes graded in geometric progression toward $x = 0$ with $q = 0.15$ (see (4.17)) and a sequence of p-distributions such that $p$ ranges from 1 to $M(\Delta)$ with the smallest element assigned $p = 1$ and the largest $p = M(\Delta)$. Plot the logarithm of the relative error in energy norm against $N^{1/2}$.

*Solution:* The results are shown in Figure 4.4. In designing the mesh we made use of the fact that we know the location of the singular points. In the case of elliptic boundary value problems, the locations of singular points are known *a priori*, and it is feasible to grade the mesh in geometric progression around singular points.

### 4.1.5. p-Convergence, Geometric Meshes

If we use a fixed geometric mesh and increase the polynomial degree of elements uniformly, then, typically, convergence is very rapid at first, because at low p-values the error is coming from that part of the domain where the solution is smooth (and the elements are large). For low p-values the rate of convergence is as if the exact solution were smooth. As the polynomial degree of elements is increased, the error caused by the singularity begins to dominate and the

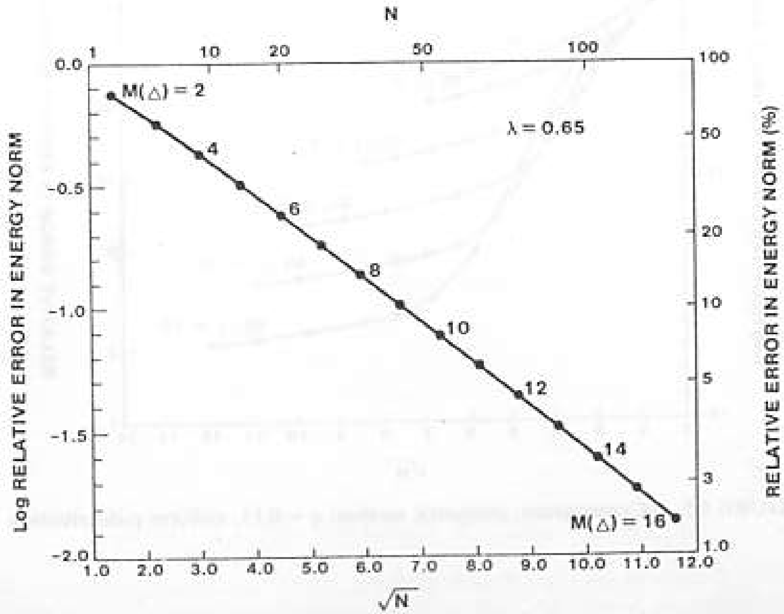


FIGURE 4.4. hp-Convergence; geometric mesh: $q = 0.15$; graded p-distribution

convergence rate slows to the convergence rate of p-extensions. (See (4.16*a*,*b*).) Of course, we would like to select the mesh so that the desired level of precision is reached before the rate of convergence slows. By increasing $M(\Delta)$ we can extend the steep part of the convergence path. This is demonstrated by the following example.

***Example 4.4.*** Once again, let $AE = 1$, $c = 1$, $l = 1$ and define $f$ so that $u_{EX}$ is of the form (4.12) with $a = 0$ and $\lambda = 0.65$. Create six meshes of 3, 4, 6, 8, 10, 12 elements and grade each mesh in geometric progression with $q = 0.15$. Compute the finite element solutions corresponding to $p = 1, 2, 3, 4, 5, 6, 8, 10, 12$. Plot the logarithm of the relative error in energy norm vs. $N^{1/2}$.

*Solution:* The results are shown in Figure 4.5. Note that the envelope of the family of curves shown in Figure 4.5 is very nearly a straight line, which is characteristic of hp-convergence. This envelope represents the optimal combination of $M(\Delta)$ and $p$ for uniform p-distributions. In practice the slowing of the convergence rate can be detected and the number of elements increased to keep the convergence path close to the optimal one.

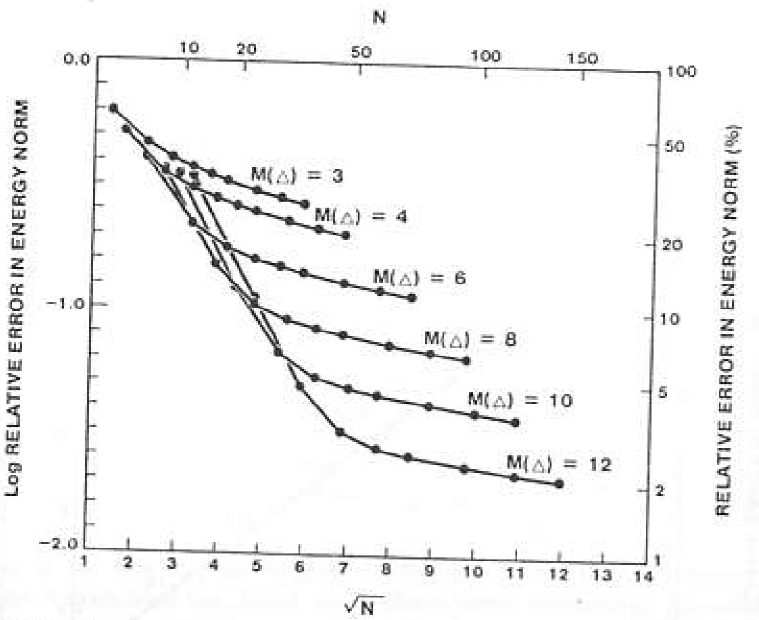


FIGURE 4.5. p-Convergence; geometric meshes: $q = 0.15$; uniform p-distribution



***Exercise 4.2.*** The exact solution of the problem in Exercise 3.10 is:

$$u_{EX} = \frac{\alpha\mathcal{T}}{\sqrt{\dfrac{c}{AE}}\cosh\left(\sqrt{\dfrac{c}{AE}}\,l\right) + \dfrac{k}{AE}\sinh\left(\sqrt{\dfrac{c}{AE}}\,l\right)}\sinh\left(\sqrt{\frac{c}{AE}}\,x\right).$$

What is the rate of convergence in energy norm if:
uniform mesh refinement and $p = 2$ are used?
uniform mesh refinement and $p = 3$ are used?
one finite element and $p$-extension are used?

## 4.2. A POSTERIORI ESTIMATION OF ERROR IN ENERGY NORM

We have seen in Section 4.1 that in the case of h- and p-extensions the *a priori* estimate is of the form:

$$\|u_{EX} - u_{FE}\|_{E(\Omega)} \le \frac{k}{N^{\beta}} \tag{4.20}$$

where $k$ and $\beta$ are positive constants, $N$ is the number of degrees of freedom. We have noted that these estimators are "sharp" for large $N$ values. Hence, the "less than or equal" ($\le$) can be replaced by "approximately equal" ($\approx$) in (4.20) when $N$ is large. We now outline a procedure for obtaining an *a posteriori* estimate for $\|u_{EX} - u_{FE}\|_{E(\Omega)}$ which is based on (4.20) and utilizes data generated by p-extensions.

Define the error $e \overset{\text{def}}{=} u_{EX} - u_{FE}$. Since $u_{EX} \in \tilde{E}(\Omega)$ and $u_{FE} \in \tilde{E}(\Omega)$, $e$ lies in the same space as the function $w$ introduced in Section 2.4. Therefore, (2.48) holds for $e$:

$$\|e\|^2_{E(\Omega)} \equiv \tfrac{1}{2}\mathcal{B}(e, e) = \Pi(u_{FE}) - \Pi(u_{EX}). \tag{4.21}$$

Using (4.21), we can write (4.20) in the following form:

$$\|u_{EX} - u_{FE}\|^2_{E(\Omega)} = \Pi(u_{FE}) - \Pi(u_{EX}) \approx \frac{k^2}{N^{2\beta}}. \tag{4.22}$$

We have three unknowns: $\Pi(u_{EX})$, $k$, and $\beta$. If we have three values of $\Pi(u_{FE})$ and $N$ corresponding to three different values of $p$, then we have three equations for computing the unknowns. We denote these three values by $\Pi_p$, $\Pi_{p-1}$, $\Pi_{p-2}$;

$N_p, N_{p-1}, N_{p-2}$; and $\Pi(u_{EX})$ by $\Pi$. Then, from (4.22) we have:

$$\frac{\log \dfrac{\Pi - \Pi_p}{\Pi - \Pi_{p-1}}}{\log \dfrac{\Pi - \Pi_{p-1}}{\Pi - \Pi_{p-2}}} \approx \frac{\log \dfrac{N_{p-1}}{N_p}}{\log \dfrac{N_{p-2}}{N_{p-1}}}. \tag{4.23}$$

Denoting the right hand side of (4.23) by $Q$, we have:

$$\frac{\Pi - \Pi_p}{\Pi - \Pi_{p-1}} \approx \left(\frac{\Pi - \Pi_{p-1}}{\Pi - \Pi_{p-2}}\right)^Q. \tag{4.24}$$

To obtain an estimate of the exact potential energy $\Pi$, we need to solve (4.24). Computational experience has shown this estimate to be reliable and generally accurate, with the accuracy of the estimate increasing with the accuracy of $\Pi_p$. Specific examples will be presented in connection with two-dimensional problems.

This method of error estimation is based on the assumption that $\Pi_p$ converges monotonically. For this to occur, it is necessary that the sequence of finite element spaces $S_{p-2}, S_{p-1}, S_p$ have the property $S_{p-2} \subset S_{p-1} \subset S_p$. In p-extensions the finite element spaces have this property; however, in h-extensions the refined mesh is generally not imbedded in the coarser ones. Hence, monotonicity is generally not guaranteed. The problem with mesh-imbedment in h-extensions is that it is computationally expensive.

The exponent $\beta$ has a good practical meaning. If error in energy norm is plotted against the number of degrees of freedom, as in Figure 4.2, for example, then $\beta$ is the approximate slope of this error curve. In the case of p-extensions we would like to design the mesh so that $\beta$ is increasing when the desired accuracy is reached. This point is discussed in Section 10.2.4.

In the case of hp-extensions the *a priori* estimate is of the form (4.18). We may set $\gamma = 1/2$ and compute estimates for $k$, $\beta$, and $\Pi(u_{EX})$, analogously to the case described in this section.

***Exercise 4.3.*** Assume that strain energy values have been computed for a sequence of finite element spaces $S_{p-2} \subset S_{p-1} \subset S_p$. Prove the following:

1. If loading is by imposed tractions or temperature, that is, $\tilde{E}(\Omega) = \mathring{E}(\Omega)$ and $\mathscr{F}(v) \neq 0$, the computed strain energy values *increase* monotonically.
2. If loading is by imposed displacements, that is, $\tilde{E}(\Omega) \neq \mathring{E}(\Omega)$ and $\mathscr{F}(v) = 0$, the computed strain energy values *decrease* monotonically.
3. If loading is by imposed displacements *and* tractions or temperature, that is, $\tilde{E}(\Omega) \neq \mathring{E}(\Omega)$ and $\mathscr{F}(v) \neq 0$, the computed strain energy values do not, in general, converge monotonically. However, the energy norm of the error does.



The foregoing discussion was confined to the error measured in energy norm. It is possible to derive estimates for the error measured in other norms also. For some estimates for the h-version we refer to [4.3]. For error estimates in eigenvalue computations we refer to [4.4].

## 4.3. REFERENCES

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