INCREMENTAL ANALYSIS OF ELASTIC-PLASTIC STRUCTURES: Uniaxial Stress- Strain Relations

Uniaxial tensile test

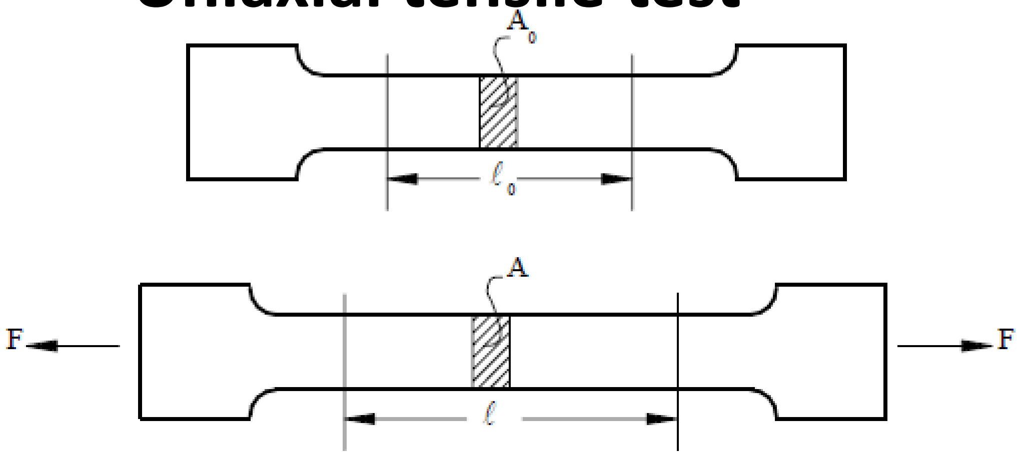

Fig.1.1

$\sigma = F/A_o$ nominal stress ; $t = F/A$ Cauchy stress

$\Delta\ell = \ell - \ell_0$ bar extension ; $\varepsilon = \Delta\ell/\ell_o$ axial strain

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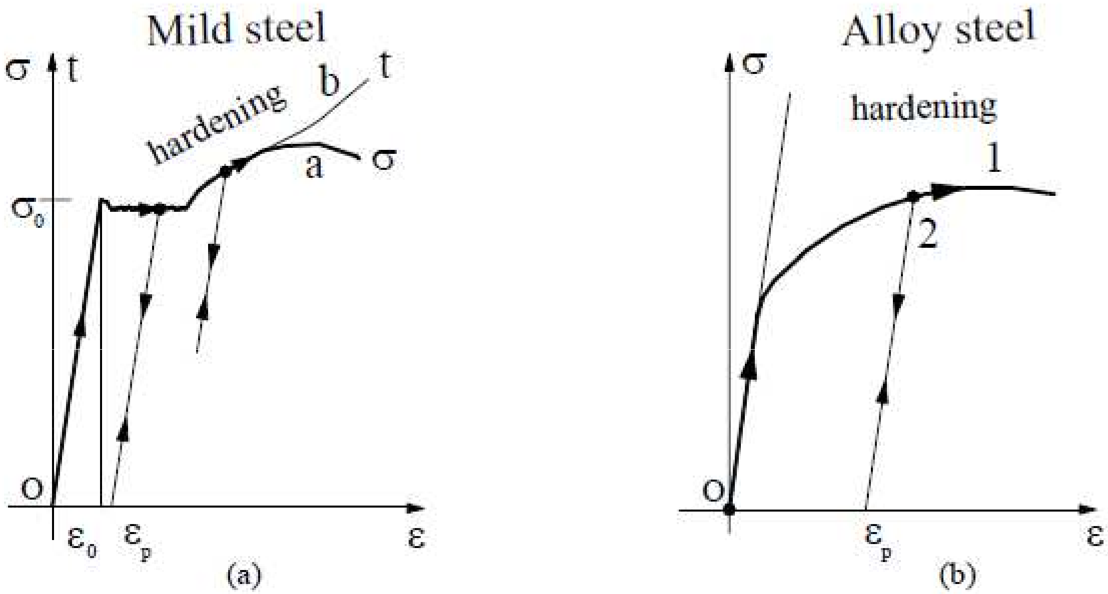

Mild steel, also called plain-carbon steel, is the most common form of steel because its price is relatively low while it provides material properties that are acceptable for many applications. Low carbon steel contains approximately 0.05–0.15% carbon[1] and mild steel contains 0.16–0.29%[1] carbon; making it malleable and ductile, but it cannot be hardened by heat treatment. Mild steel has a relatively low tensile strength, but it is cheap and malleable; surface hardness can be increased through carburizing

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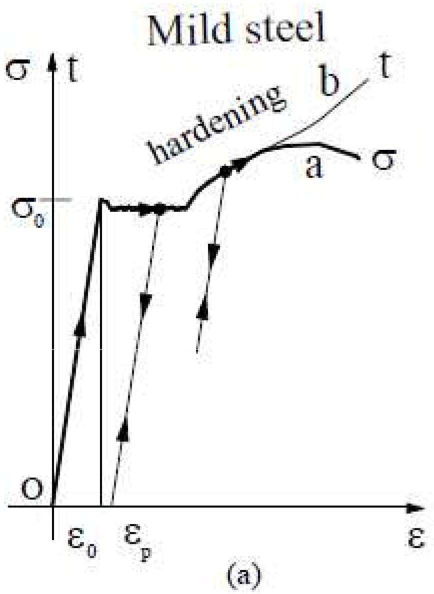
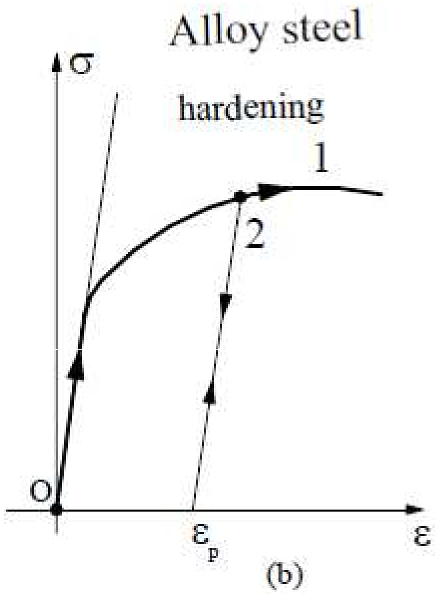
Uniaxial tensile test





Alloy steel is steel that is alloyed with a variety of elements in total amounts between 1.0% and 50% by weight to improve its mechanical properties. Alloy steels are broken down into two groups: low-alloy steels and high-alloy steels. The difference between the two is somewhat arbitrary: Smith and Hashemi define the difference at 4.0%, while Degarmo, et al., define it at 8.0 %.[1][2] Most commonly, the phrase "alloy steel" refers to low-alloy steels

INCREMENTAL ANALYSIS OF ELASTIC-PLASTIC STRUCTURES: Uniaxial tensile test

Remarks:

- σ_o yield stress, plastic stress, yield limit
- $\varepsilon > \varepsilon_o \Rightarrow$ permanent strain $\varepsilon_p \Leftrightarrow$ irreversible deformation
- The slope of the stress-strain diagram is essentially the same as the initial (elastic) slope.
- Hysteretic response- Baushinger effect.

Dissipation in a closed elastic –plastic deformation process:

$$W_d = \underbrace{\oint F d(\Delta\ell) = \oint (A_0\sigma)}_{\text{Pric. Virtual Work+ Homog. Process}} \ell_0 d\varepsilon = A_0 \ell_0 \oint \sigma d\varepsilon = A_0 \ell_0 w_d$$

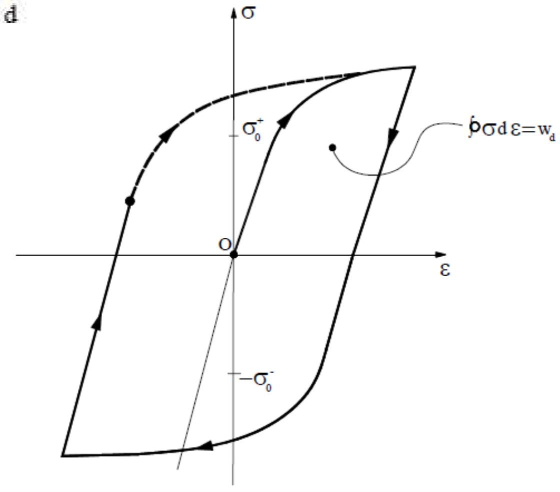


Fig.1.3

INCREMENTAL ANALYSIS OF ELASTIC-PLASTIC STRUCTURES

- Steel $\sigma_o^+ = \sigma_o^-$
- Copper

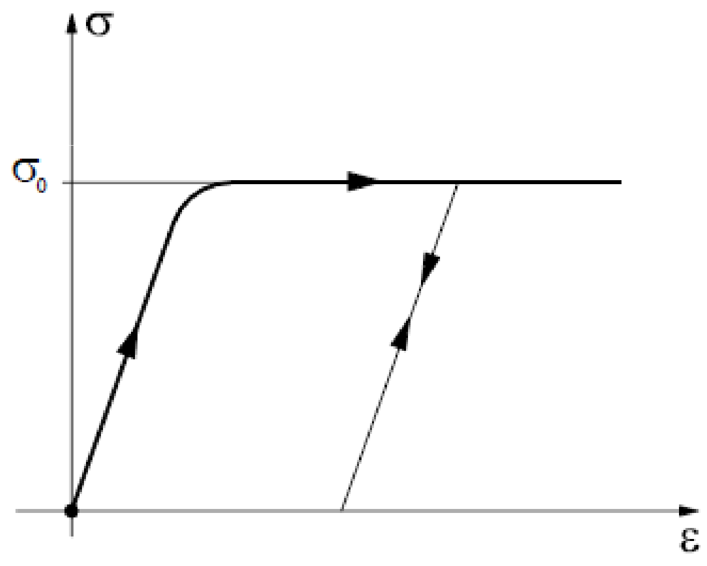


Fig.1.4

INCREMENTAL ANALYSIS OF ELASTIC-PLASTIC STRUCTURES: friction

- Friction

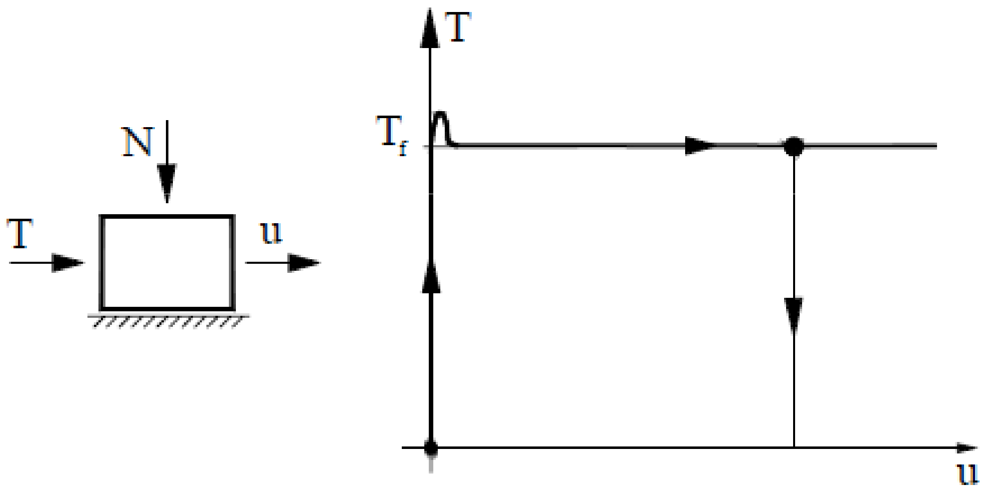


Fig.1.5

Elastic-plastic costitutive models

Basic Hypotheses:

- Plastic strains are attained when the applied stress σ reaches the yield stress σ_o ;
- Additive decomposition of the (total) strain ε as the sum of the elastic (reversible) strain ε_e and the plastic (irreversible) strain ε_p :

$$\varepsilon = \varepsilon_e + \varepsilon_p$$

- The applied stress σ linearly depends on the elastic strain ε_e

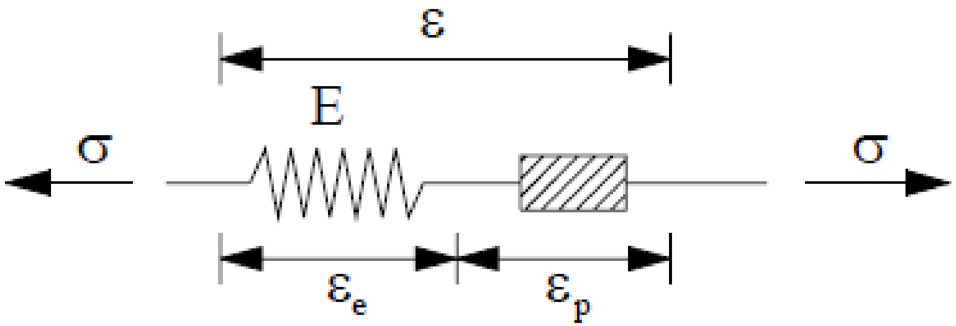
$$\sigma = E\varepsilon_e$$

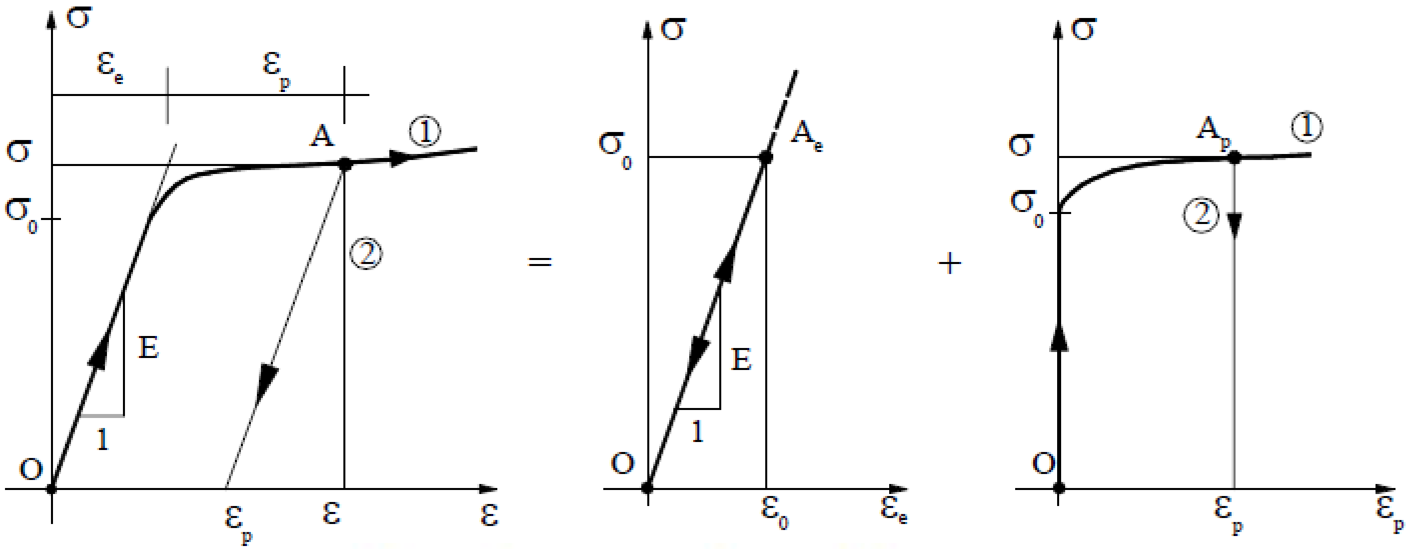
being E the young's modulus.

$$\sigma = E\,(\varepsilon - \varepsilon_p)$$

Elastic-plastic costitutive models

$$\sigma = \mathrm{E}\,(\varepsilon - \varepsilon_p)$$





Rate of dissipation per unit volume $D = \sigma\dot{\varepsilon}_p$

Fig.1.7

Elastic-perfectly plastic material

- The stress cannot exceed the bounds in tension and compression $-\sigma_o^- \le \sigma \le \sigma_o^+$;
- Any value of stress satisfying such inequalities is called plastically admissible;
- For metals $\sigma_o^+ \approx \sigma_o^-$ and the condition of plastic admissibility reads $|\sigma| \le \sigma_o$;
- Plastic yielding can take place only if

$$\phi(\sigma) = |\sigma| - \sigma_o = 0 \quad \underline{\text{yield condition;}}$$

- $\phi(\sigma)$ yielding function;
- Plastic dissipation $w_d = \int_{t_0}^{t_B} \sigma\dot{\varepsilon}dt = \int_0^{\varepsilon_p} \sigma d\varepsilon p$

Elastic-perfectly plastic material

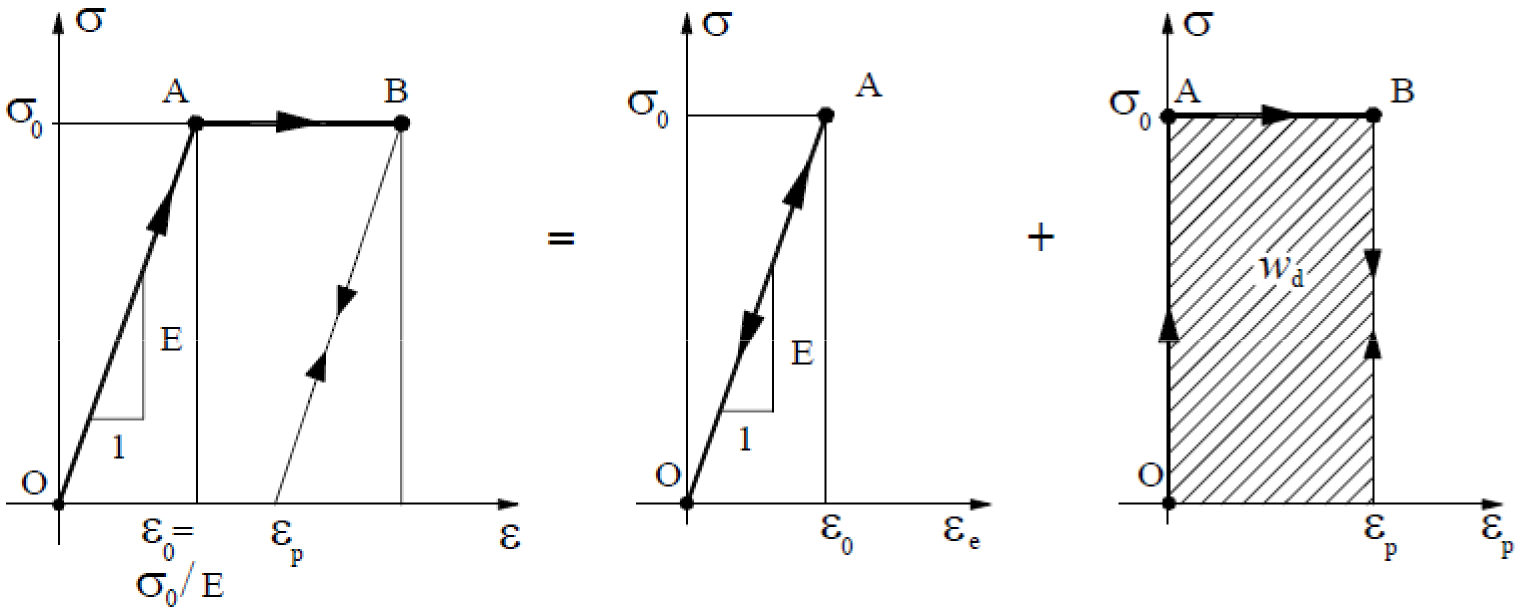


Fig.1.8

Cyclic response

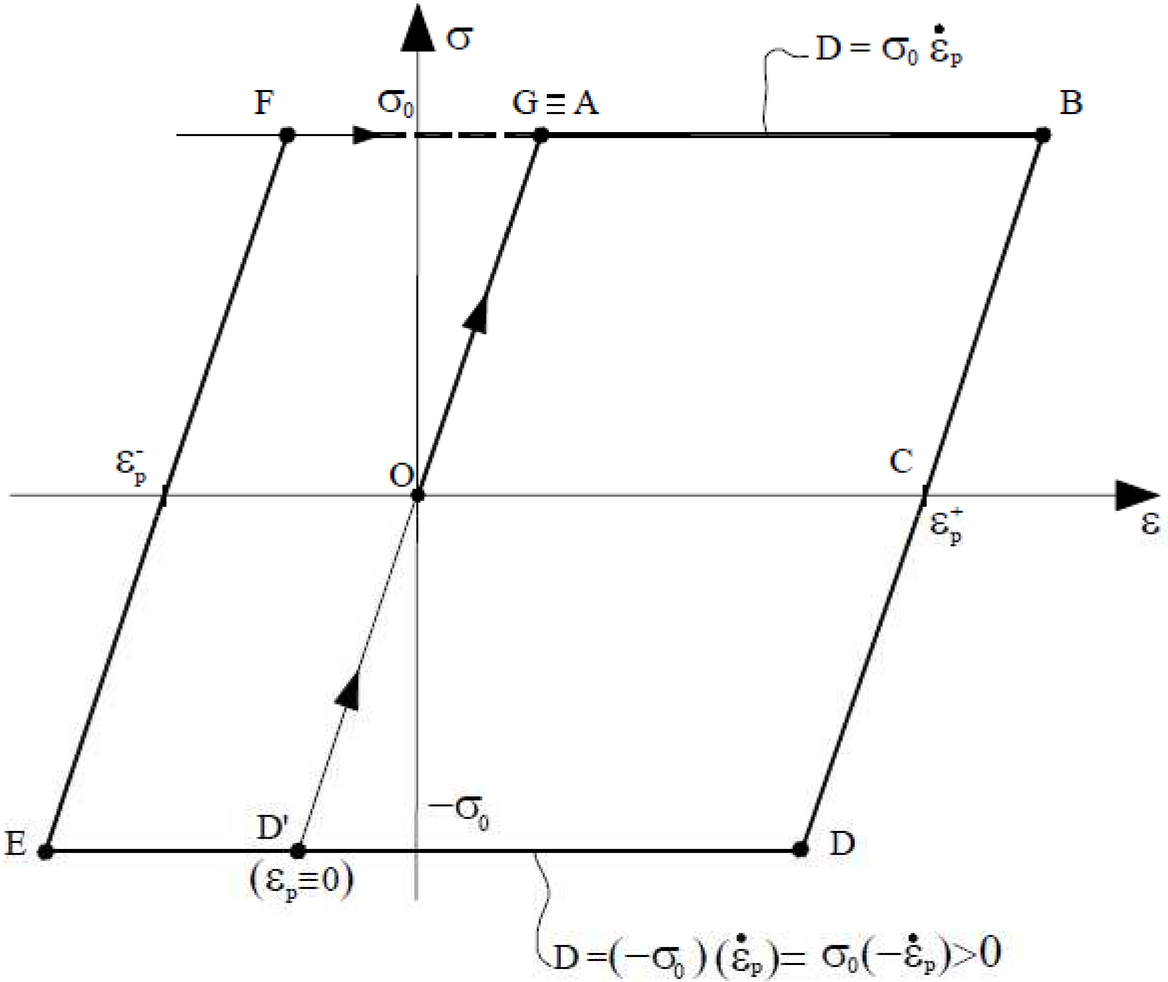


Fig.1.9

In a closed process $A \rightarrow G$ the dissipation per unit volume is $w_d = 2\sigma_0\left(\varepsilon_p^+ - \varepsilon_p^-\right)$.

Rigid perfectly plastic model

Hypothesis: $\varepsilon_e = 0$

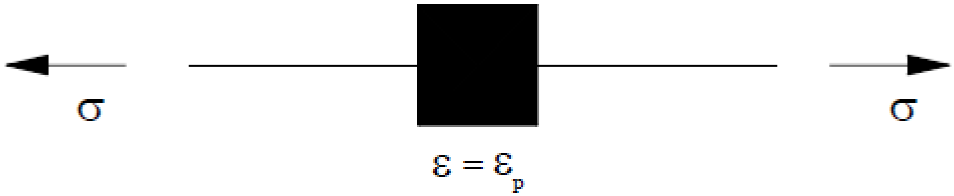


Fig.1.10

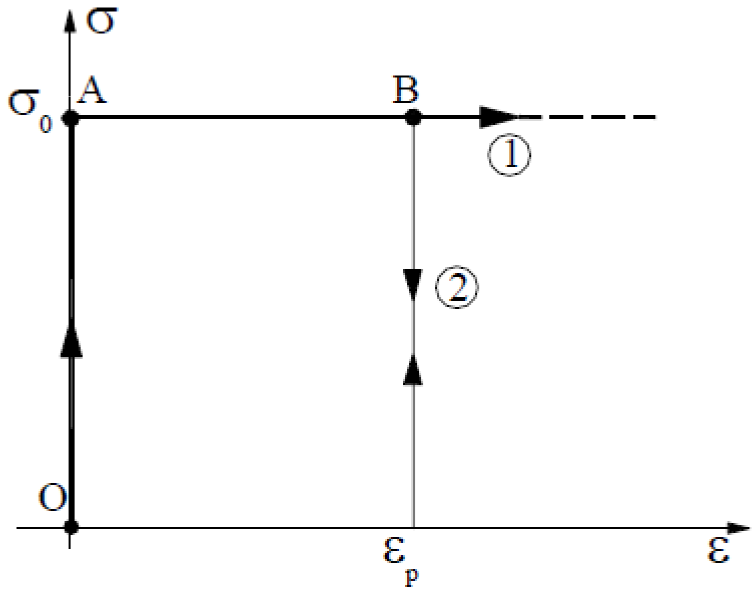
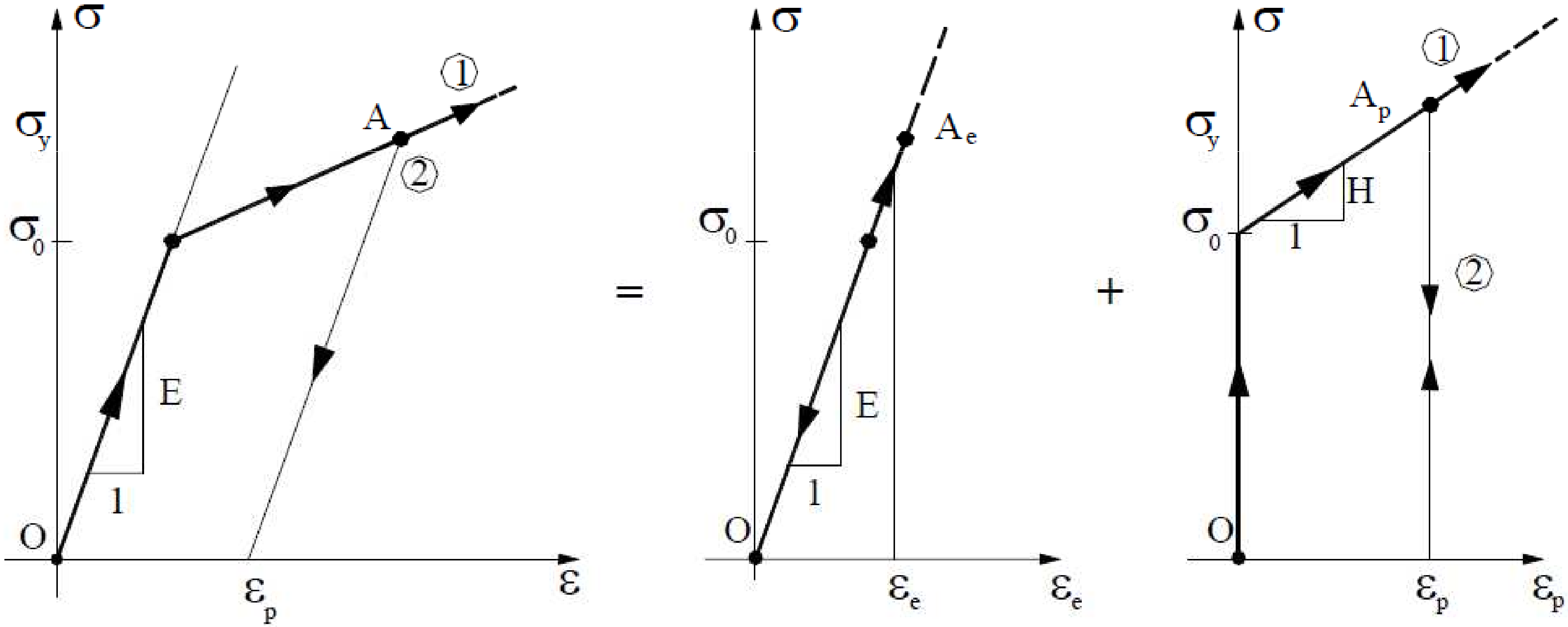


Fig.1.11

Elasto-plastic with linear hardening

The yield stress linearly depends on the permanent strain.



Elasto-plastic with linear hardening

- Linear hardening $\Rightarrow \sigma_y = \sigma_0 + H\varepsilon_p \quad [H] = FL^{-2}$;
- Yield function $\phi(\sigma, \varepsilon_p) = \sigma - \sigma_y = \sigma - \sigma_0 - H\varepsilon_p$ for $\varepsilon_p \geq 0$;
- $H = 0 \Rightarrow$ elastic-perfectly plastic model (EPP);
- $H = 0$, $E \to \infty \Rightarrow$ rigid plastic model;

- Tension-compression yielding: <u>Kinematic hardening</u>

Tension: $\sigma < \sigma_y^+ = \sigma_0 + H\varepsilon_p$

Compression: $\sigma > -\sigma_y^- = -\sigma_0 + H\varepsilon_p$

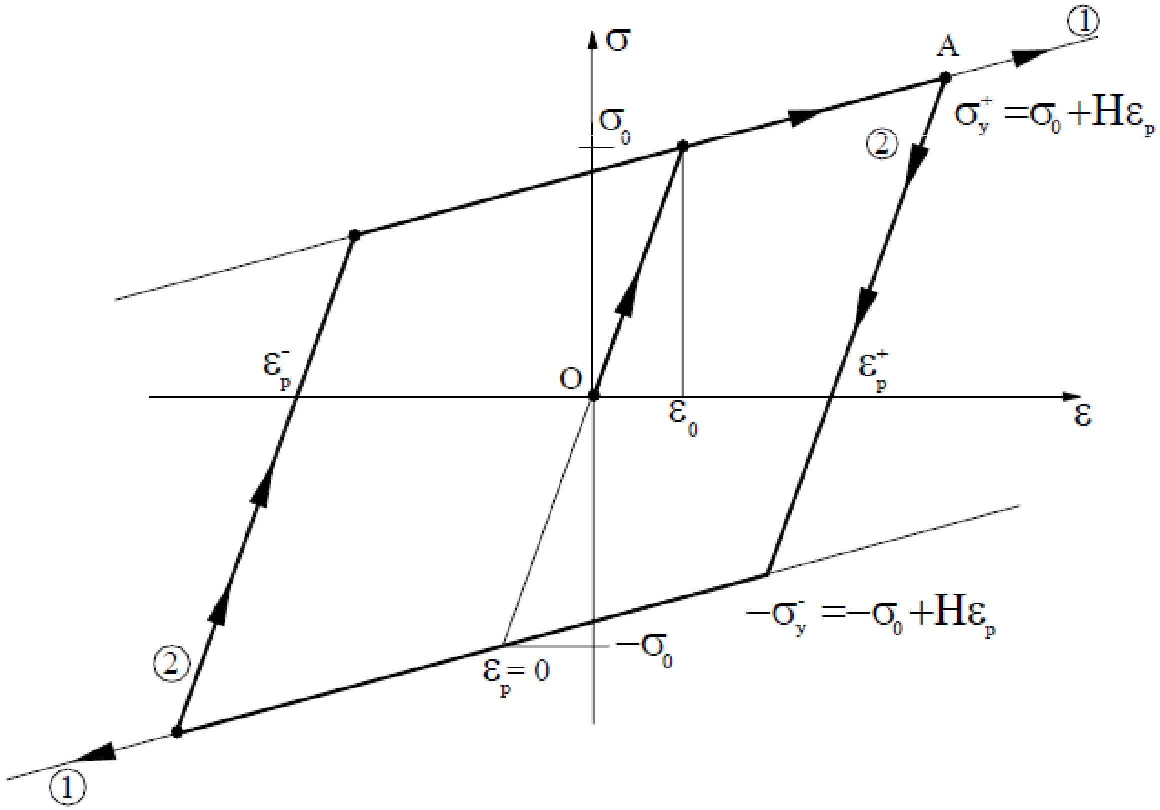
Elasto-plastic with linear hardening

$\Rightarrow$ Plastic admissibility

$$-\sigma + H\varepsilon_p \leq \sigma \leq \sigma_0 + H\varepsilon_p$$

yield function

$$\phi\left(\sigma, \varepsilon_p\right) = \left|\sigma - H\varepsilon_p\right| - \sigma_0 \leq 0$$



Elasto-plastic with linear hardening

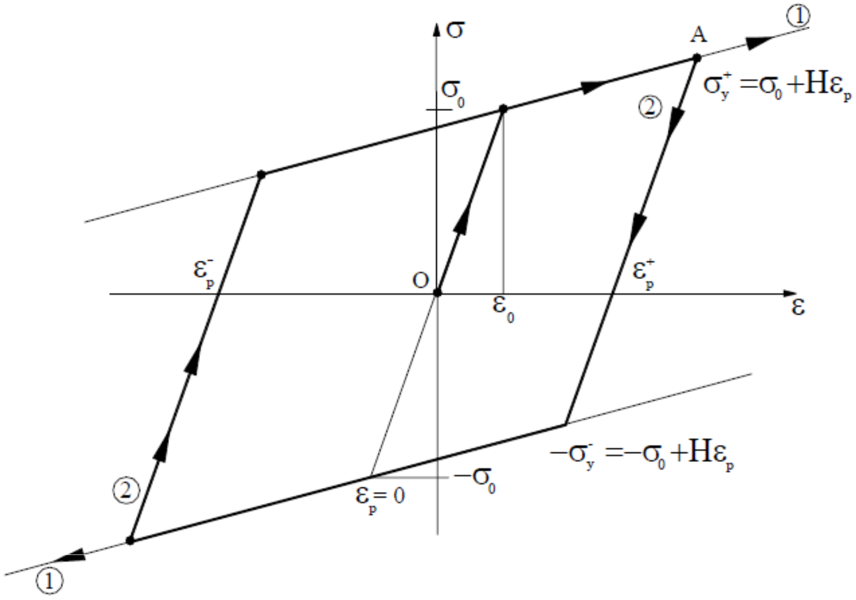


Fig.1.14

- The constitutive equation at an elastic-plastic state (A,B,…) must take into account two different possible evolutions:

$$\left.\begin{array}{l}\text{(1) Elastic-plastic deformation,} \\ \text{(2) Elastic unloading.}\end{array}\right\} \begin{array}{l}\text{Incremental formulation} \\ \text{(non holonomic response)}\end{array}$$

<u>Elastic</u> (Hyperelastic) holonomic-path independent;
<u>Elastic-plastic</u>- non-holonomic- path dependent.

Elasto-plastic with linear hardening

A

A1

$d\varepsilon > 0$

$d\sigma > 0$

$d\sigma = E\left(d\varepsilon - d\varepsilon_p\right)$ Elastic response

$d\sigma = H d\varepsilon_p$ Hardening

$\Rightarrow \boxed{d\varepsilon_p = \dfrac{E}{E+H} d\varepsilon}$

$$\boxed{d\sigma = \frac{EH}{E+H} d\varepsilon}$$

A2

$d\varepsilon < 0$

$d\sigma < 0$

$$\boxed{d\sigma = E d\varepsilon}$$

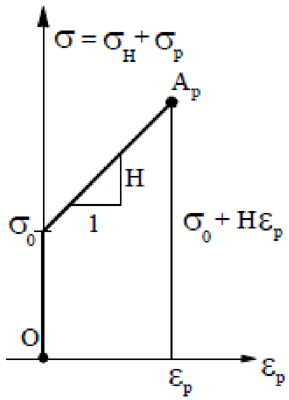
Elastic response $\boxed{d\varepsilon_p = 0}$

Elasto-plastic with linear hardening

$$
\text{B}\begin{cases}
\text{B1} \quad d\sigma = E\left(d\varepsilon - d\varepsilon_p\right) & \text{Elastic response} \\
d\varepsilon < 0 \quad d\sigma = H d\varepsilon_p & \text{Hardening} \\
d\sigma < 0 \quad \boxed{d\sigma = \dfrac{EH}{E+H} d\varepsilon} &
\end{cases}
\quad \Rightarrow \boxed{d\varepsilon_p = \frac{E}{E+H} d\varepsilon}
$$

$$
\begin{cases}
\text{B2} \quad \boxed{d\sigma = E d\varepsilon} & \\
d\varepsilon > 0 & \text{Elastic response} \quad \boxed{d\varepsilon_p = 0} \\
d\sigma > 0 &
\end{cases}
$$

Elasto-plastic with linear hardening



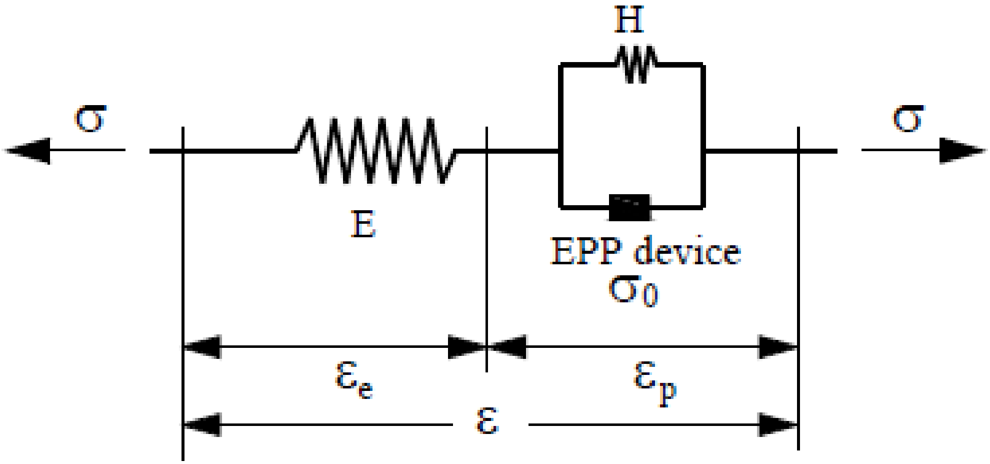
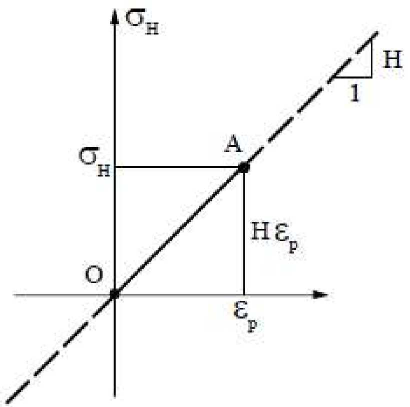
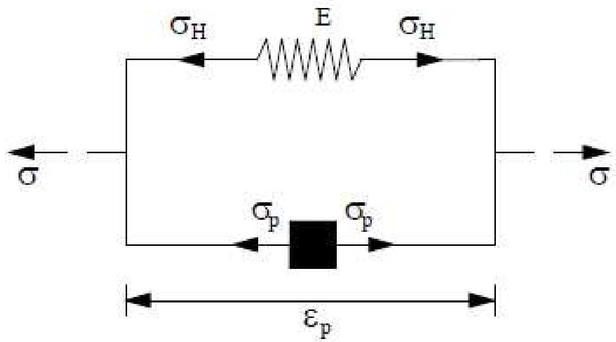
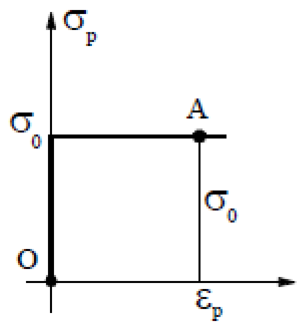


Fig.1.16







$$\sigma = \sigma_H + \sigma_p$$

Elastic-plastic unixial constitutive equations (Incremental form)

Let us consider stress and strain rate, respectively, defined as

$$\dot{\sigma} = \frac{d\sigma}{dt}, \qquad \dot{\varepsilon} = \frac{d\varepsilon}{dt}$$

where t can be considered as an increasing variable in the deformation process $(dt > 0)$, not necessarily coincident with time.

The constitutive equation can be expressed in terms of strain and stress rates as follows:

- additivity of strains $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p$
- elastic response $\dot{\sigma} = E\dot{\varepsilon}_e = E(\dot{\varepsilon} - \dot{\varepsilon}_p)$
- loading-unloading condition

Elastic-plastic unixial constitutive equations (Incremental form)

- If $\phi(\sigma, \varepsilon_p) < 0 \Rightarrow$ elastic response $\dot{\sigma} = E\dot{\varepsilon}$, $\dot{\varepsilon} = \dot{\varepsilon}_e$, $\dot{\varepsilon}_p = 0$
- If $\phi(\sigma, \varepsilon_p) = 0 \Rightarrow$ elastic-plastic state: the response depends on the evolution of the yield function

$$d\phi(\sigma, \varepsilon_p) = \dot{\phi}(\sigma, \varepsilon_p)\,dt = \left(\frac{\partial \phi}{\partial \sigma}\dot{\sigma} + \frac{\partial \phi}{\partial \varepsilon_p}\dot{\varepsilon}_p\right)dt$$

$$\dot{\phi}(\sigma, \varepsilon_p) = \frac{\partial \phi}{\partial \sigma}\dot{\sigma} + \frac{\partial \phi}{\partial \varepsilon_p}\dot{\varepsilon}_p$$

Elastic-plastic unixial constitutive equations (Incremental form)

- If $\dot{\phi}\left(\sigma,\varepsilon_p\right)<0 \Rightarrow$ Elastic unloading $\boxed{\dot{\varepsilon}_p=0\,,\ \dot{\sigma}=E\dot{\varepsilon}}$

  (Note that if $\dfrac{\partial\phi}{\partial\sigma}\dot{\sigma}<0 \Rightarrow \dot{\phi}\left(\sigma,\varepsilon_p<0\right)$)

- If $\dot{\phi}\left(\sigma,\varepsilon_p\right)=0 \Rightarrow$ Elastic –plastic deformation $\dot{\varepsilon}_p\neq 0$

  Flow rule $\dot{\varepsilon}_p=\left(\dfrac{\partial\phi}{\partial\sigma}\right)\dot{\lambda}\,,\ \dot{\lambda}\geq 0$ plastic multiplier; defined the variables $n=\dfrac{\partial\phi}{\partial\sigma}\,, h=\dfrac{\partial\phi}{\partial\varepsilon_p}$

$$\dot{\phi}\left(\sigma,\varepsilon_p\right)=n\dot{\sigma}+hn\dot{\lambda}=n\left[E\dot{\varepsilon}-\left(En-h\right)\dot{\lambda}\right]=0$$

$$\boxed{\Rightarrow\dot{\lambda}=\frac{E}{En-h}\dot{\varepsilon}},\ \boxed{\dot{\varepsilon}_p=\frac{En}{En-h}\dot{\varepsilon}},\ \boxed{\dot{\sigma}=E\left[1-\frac{En}{En-h}\right]\dot{\varepsilon}}.$$

Elastic-plastic unixial constitutive equations (Incremental form)

Example: Elastic perfectly plastic model

$$\dot{\phi}(\sigma) = |\sigma| - \sigma_0 \qquad n = \frac{\partial \phi}{\partial \sigma} = \frac{\sigma}{|\sigma|} \qquad h = \frac{\partial \phi}{\partial \varepsilon_p} = 0$$

If $\phi = \dot{\phi} = \dfrac{\partial \phi}{\partial \sigma}\dot{\sigma} = \dfrac{\sigma\dot{\sigma}}{|\sigma|} = 0$, then

$$\boxed{\dot{\lambda} = \frac{\sigma}{|\sigma|}\dot{\varepsilon} > 0\,,\ \dot{\varepsilon}_p = \dot{\varepsilon}\,,\ \dot{\sigma} = 0}$$

Elastic-plastic uniaxial constitutive equations (Incremental form)

Example: Elastic linear hardening model

$$\phi = \left|\sigma - H\varepsilon_p\right| - \sigma_0\,,\quad n = \frac{\sigma - H\varepsilon_p}{\left|\sigma - H\varepsilon_p\right|}\,,\quad h = -nH$$

$$\boxed{\dot{\lambda} = \frac{\sigma - H\varepsilon_p}{\left|\sigma - H\varepsilon_p\right|}\left[\frac{E}{E+H}\right]\dot{\varepsilon} > 0\,,\quad \dot{\varepsilon}_p = \frac{E}{E+H}\dot{\varepsilon}\,,\quad \dot{\sigma} = E\left[1 - \frac{E}{E+H}\right]\dot{\varepsilon} = \frac{EH}{E+H}\dot{\varepsilon}}$$

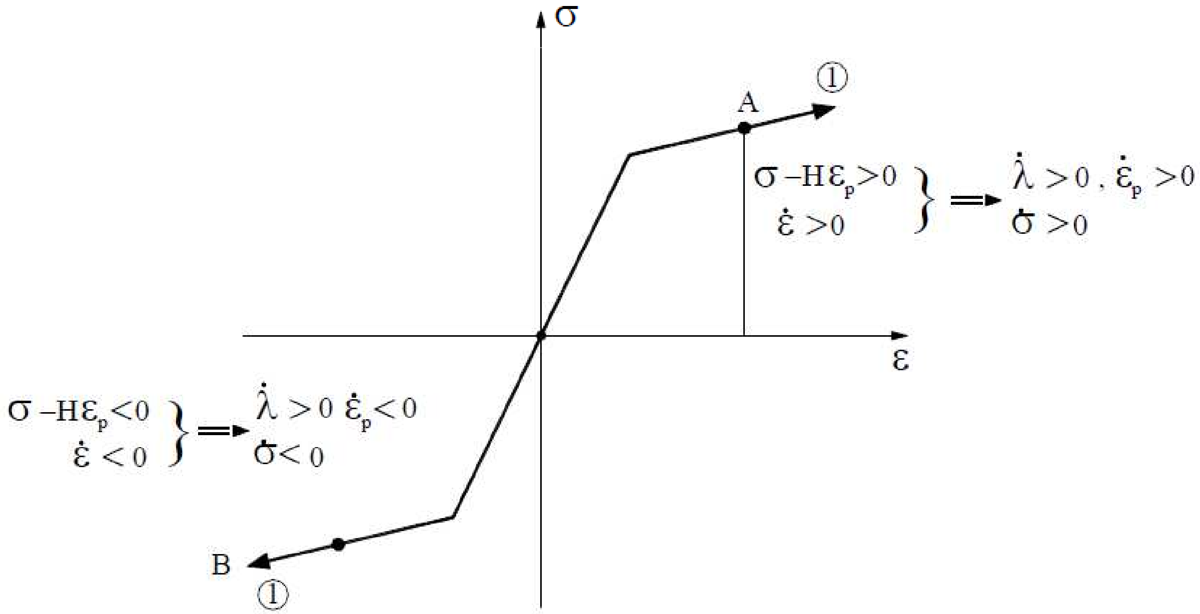


Fig.1.19

Elastic-plastic unixial constitutive equations (Incremental form)

Non linear hardening- Ramberg-Osgood model

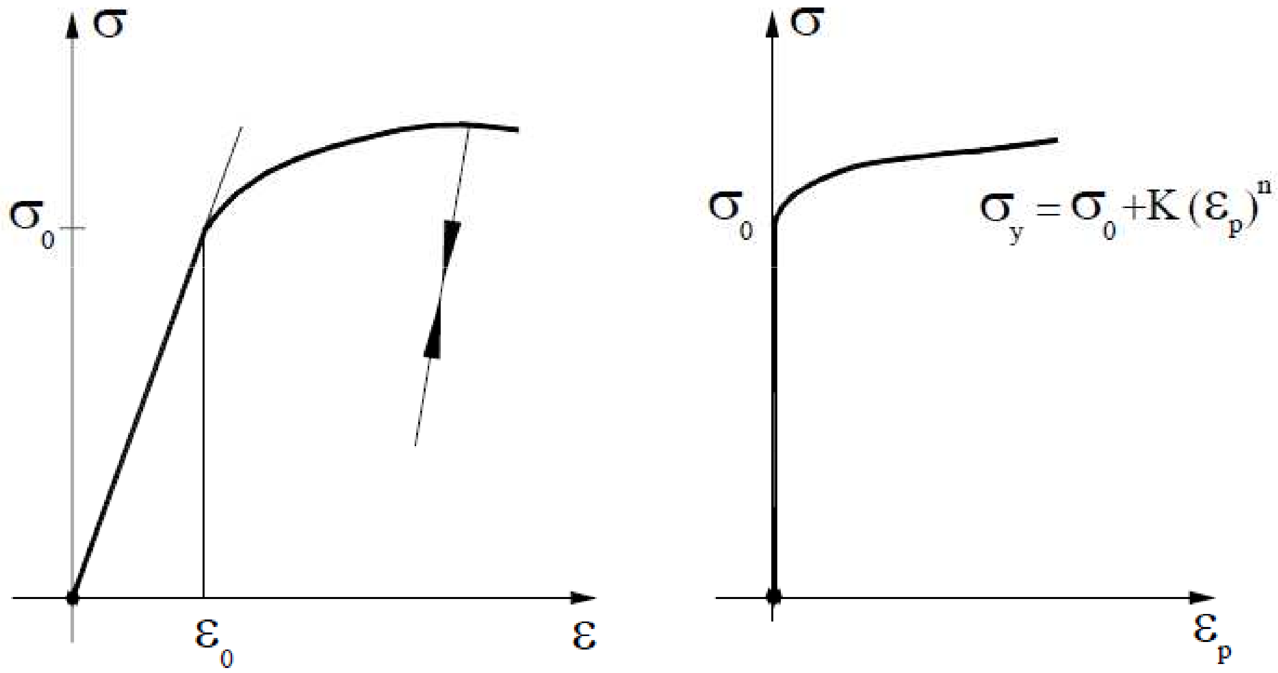
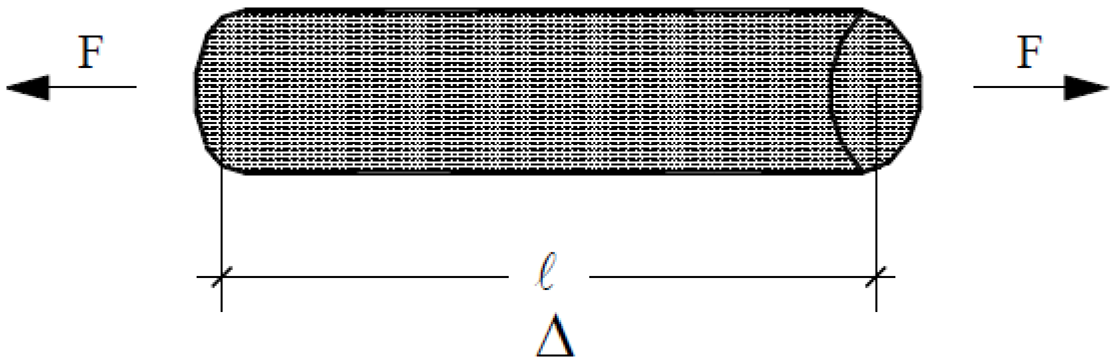


Fig.1.20

Incremental Analysis of EPP Trusses

In each bar of the truss the stress is uniform, so plastic yielding starts simultaneously at all points of a bar.



Incremental Analysis of EPP Trusses

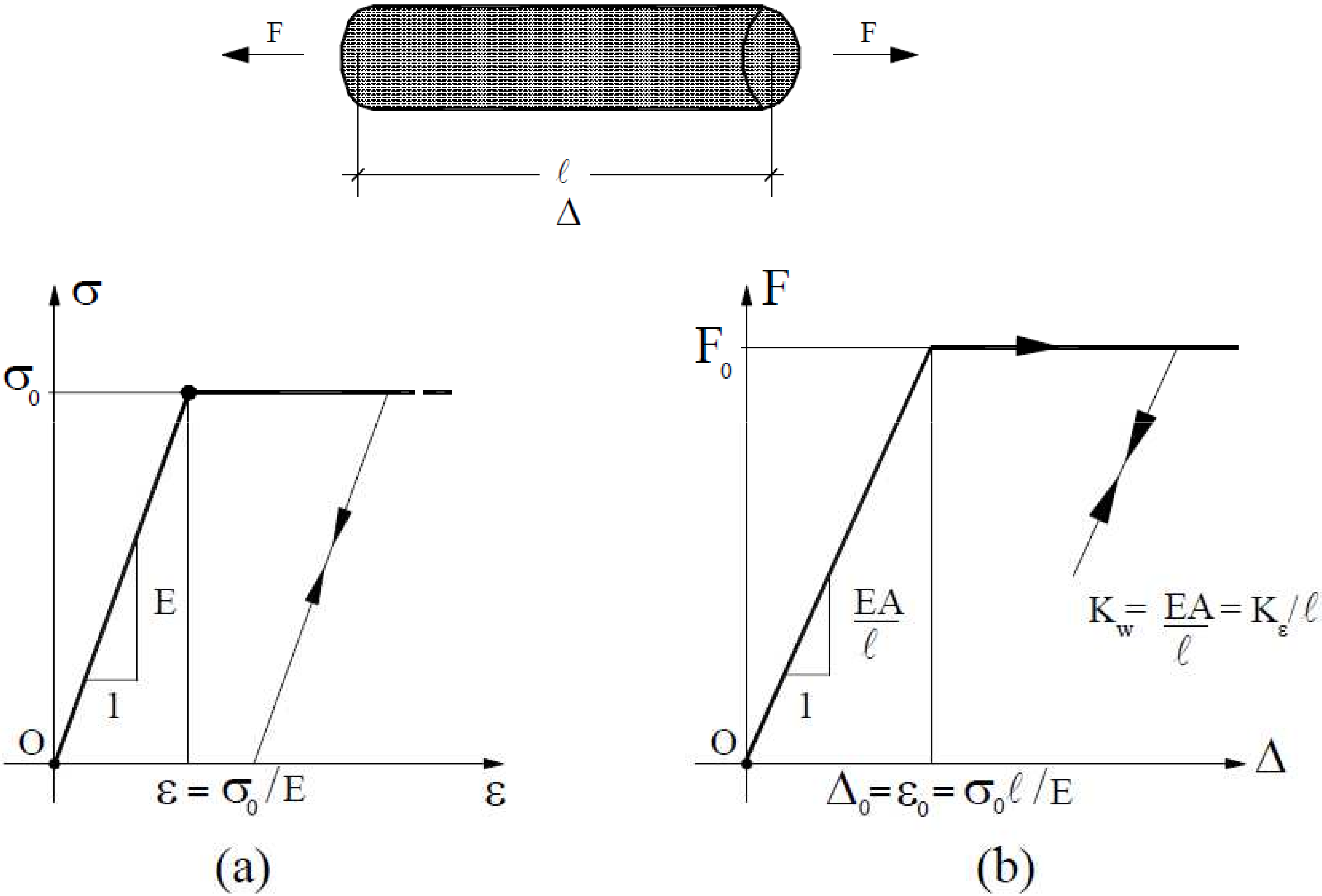
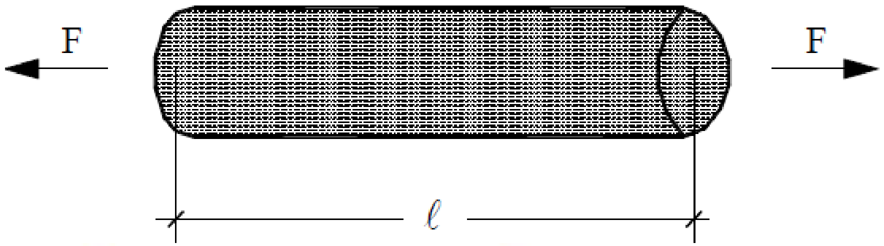


Fig.1.21

Incremental Analysis of EPP Trusses



Once defined the variables: $\sigma = F/A$ tensile stress; $\varepsilon = \Delta/\ell_0$ axial strain, it follows:

- Yielding function $\phi(F) = |F| - F_0 \leq 0$
- Additivity of extensions $\Delta = \Delta_e + \Delta_p$
- Elasticity $F = \dfrac{EA}{\ell}\Delta_e = K_w \Delta_e = K_w(\Delta - \Delta_p)$

❑ <u>Elastic response</u>

If $\phi > 0$ or $\phi = 0$ and $\dot{\phi} < 0 \Rightarrow \boxed{\dot{\Delta}_p = 0,\ \dot{F} = K_W \dot{\Delta}}$

❑ Plastic response

$\phi = \dot{\phi} = \boxed{0 \Rightarrow\ \dot{\Delta}_p = \dot{\Delta},\ \dot{F} = 0}$

$\Rightarrow$ The truss incrementally behaves as if the yielding bar were missing.

Example: statically determinate EPP truss

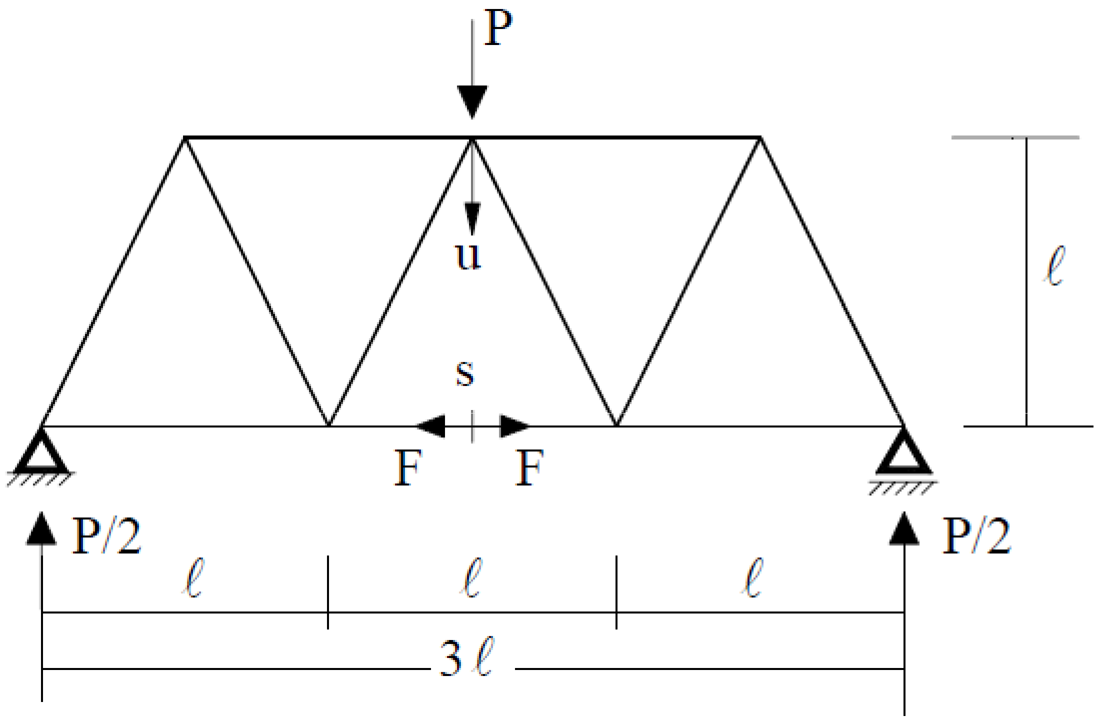


Fig 1.22

In section S the axial force F is obtained by the method of Ritter:

$F\ell = \frac{3}{2}\ell \cdot \frac{P}{2} \Rightarrow F = \frac{3}{4}P$. At yielding $F=F_0 \Rightarrow P_0 = 4/3 F_0$

Example: statically determinate EPP truss

Incrementally the truss turns out to be a mechanism as shown in figure 1.23.

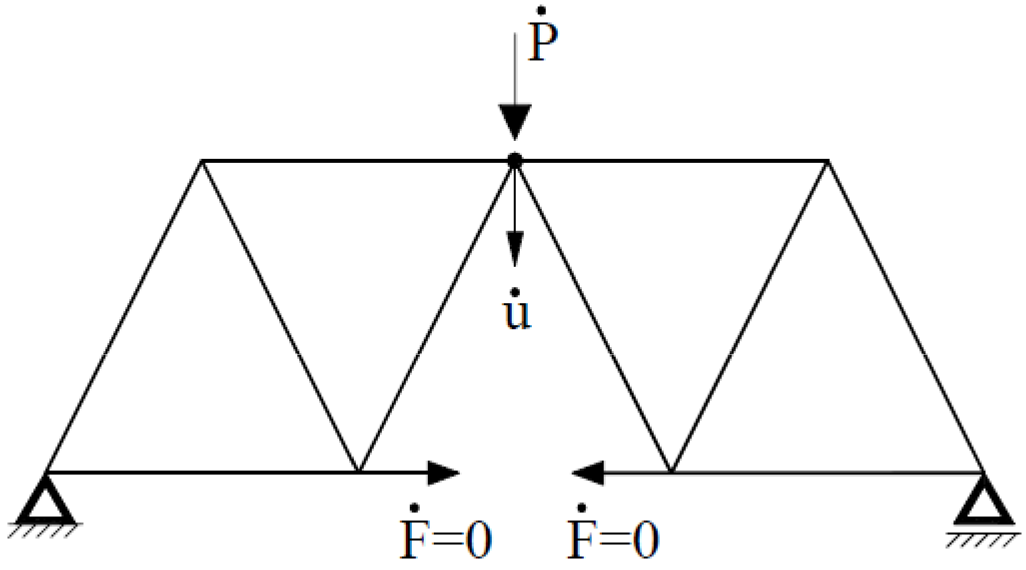
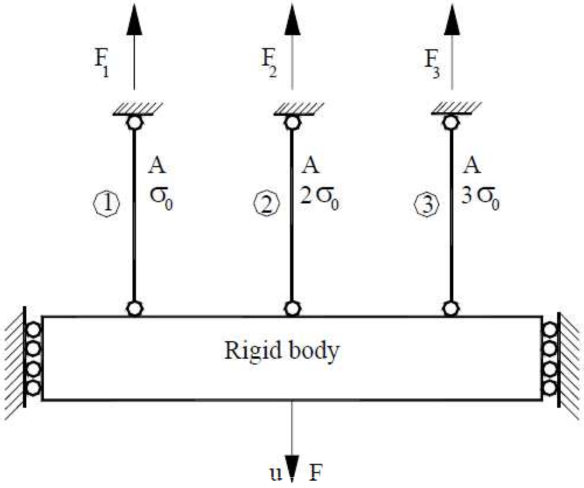


Fig.1.23

The incremental equilibrium of the truss implies $\dot{P}=0$

Example: Statically indeterminate truss - three EPP parallel bars



- <u>Equilibrium</u>

  $$F_1 + F_2 + F_3 = F$$

  $$\dot{F}_1 + \dot{F}_2 + \dot{F}_3 = \dot{F}$$

- <u>Compatibility</u>

  $$\Delta_1 = \Delta_2 = \Delta_3 = u$$

- <u>Constitutive properties</u>

  $$E_1 = E_2 = E_3 = E \Rightarrow K_{w_1} = K_{w_2} = K_{w_3} = K$$

  $$\sigma_{0_1} = \sigma_0 \,,\, \sigma_{0_2} = 2\sigma_0 \,,\, \sigma_{0_3} = 3\sigma_0$$

Example: Statically indeterminate truss - three EPP parallel bars

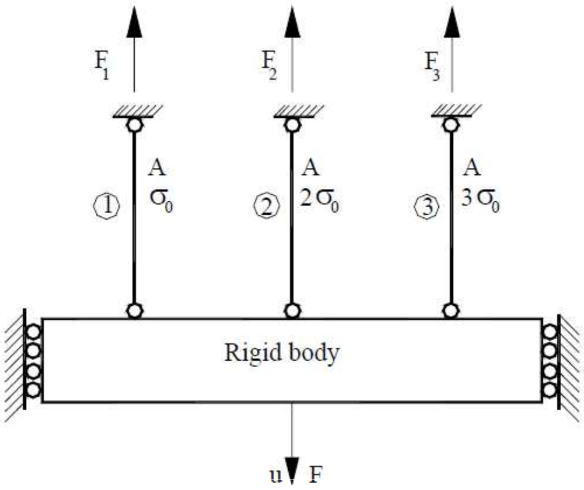


Fig.1.24

Degree of redundancy: $i = 2$

$$\phi_1 = |F_1| - F_0, \quad \phi_2 = |F_2| - 2F_0, \; \phi_3 = |F_3| - 3F_0$$

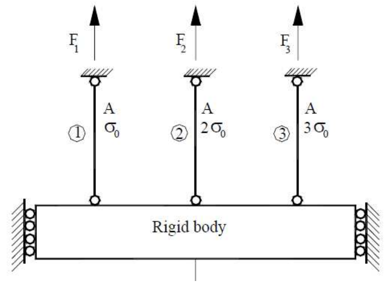
$$F_0 = A\sigma_0, \qquad \Delta_0 = \sigma_0 \frac{\ell}{E}$$

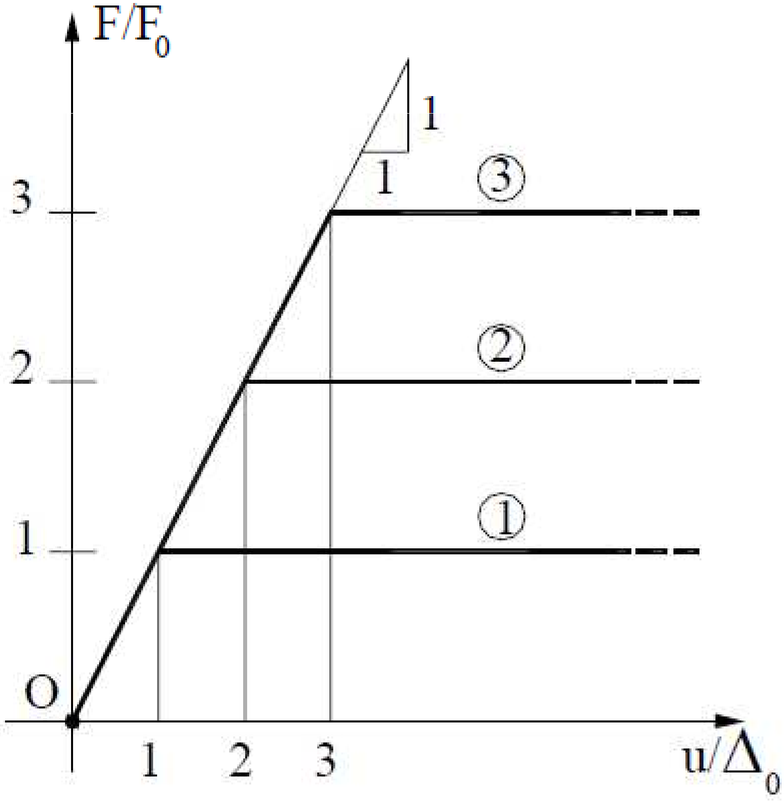
Example: Statically indeterminate truss - three EPP parallel bars

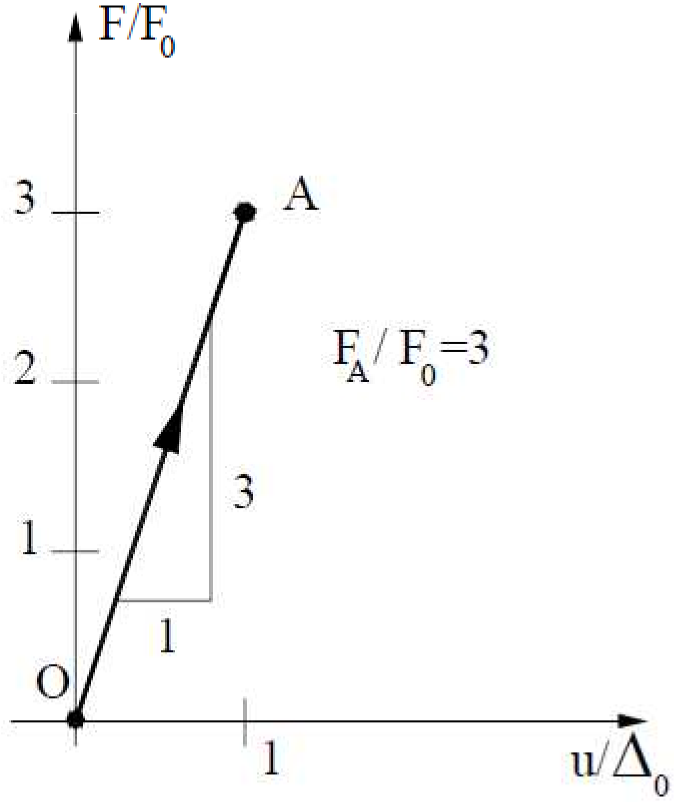
Degree of redundancy: $i = 2$

$$\phi_1 = |F_1| - F_0, \quad \phi_2 = |F_2| - 2F_0, \; \phi_3 = |F_3| - 3F_0$$

$$F_0 = A\sigma_0, \qquad \Delta_0 = \sigma_0 \frac{\ell}{E}$$

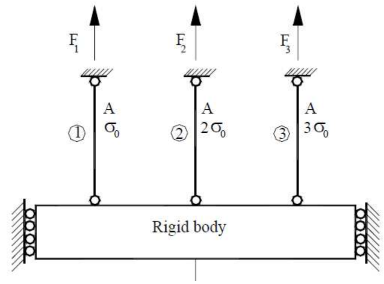


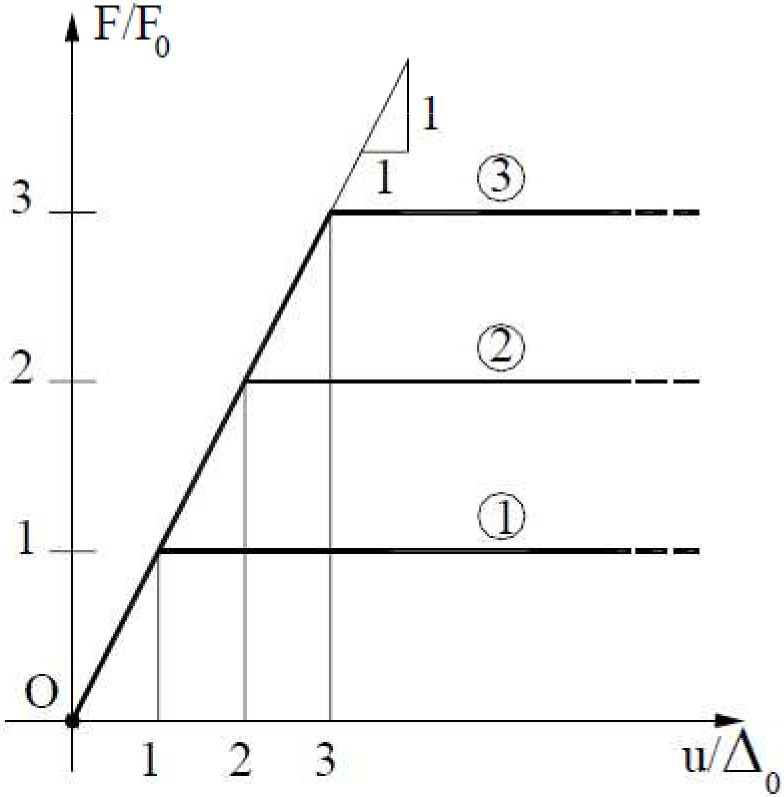


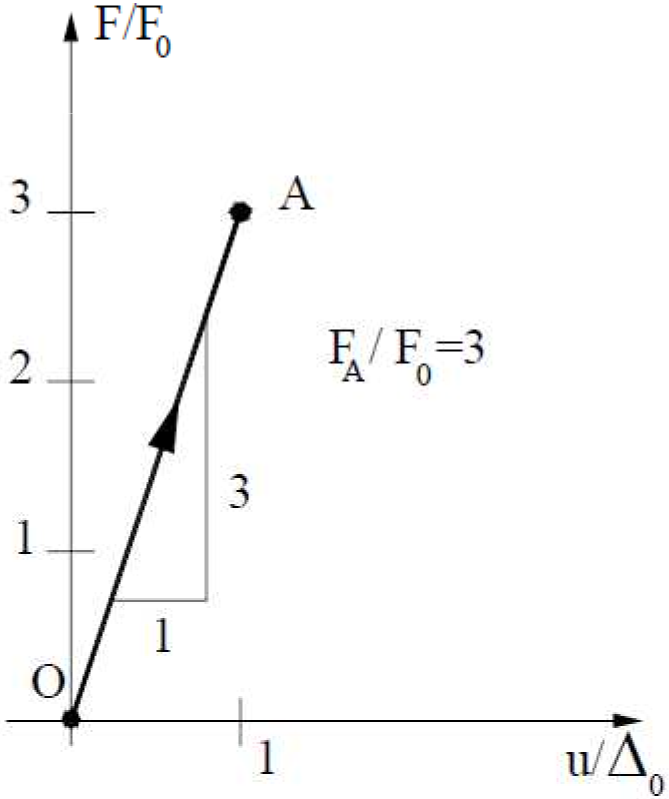


Example: Statically indeterminate truss - three EPP parallel bars

$$\dot{F} = \dot{F}_1 + \dot{F}_2 + \dot{F}_3 = 3\dot{F}_1 = \frac{3EA}{\ell}\dot{u} \Rightarrow \boxed{\frac{\dot{F}}{F_0} = 3\frac{\dot{u}}{\Delta_0}}$$







Example: Statically indeterminate truss - three EPP parallel bars

Step.2 - Yielding bar 1 $\quad \Delta_0 \le u \le 2\Delta_0$

$$\frac{\dot F_1}{F_0} = 0\,,\; \frac{\dot F_2}{F_0} = \frac{\dot u}{\Delta_0}\,,\; \frac{\dot F_3}{F_0} = \frac{\dot u}{\Delta_0} \Rightarrow \frac{\dot F}{F_0} = 2\frac{\dot u}{\Delta_0}\,,\; \dot F_B = 5F_0\,.$$

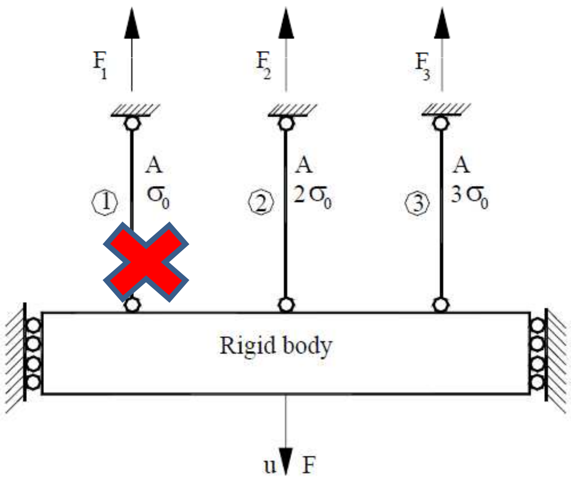


Fig.1.24

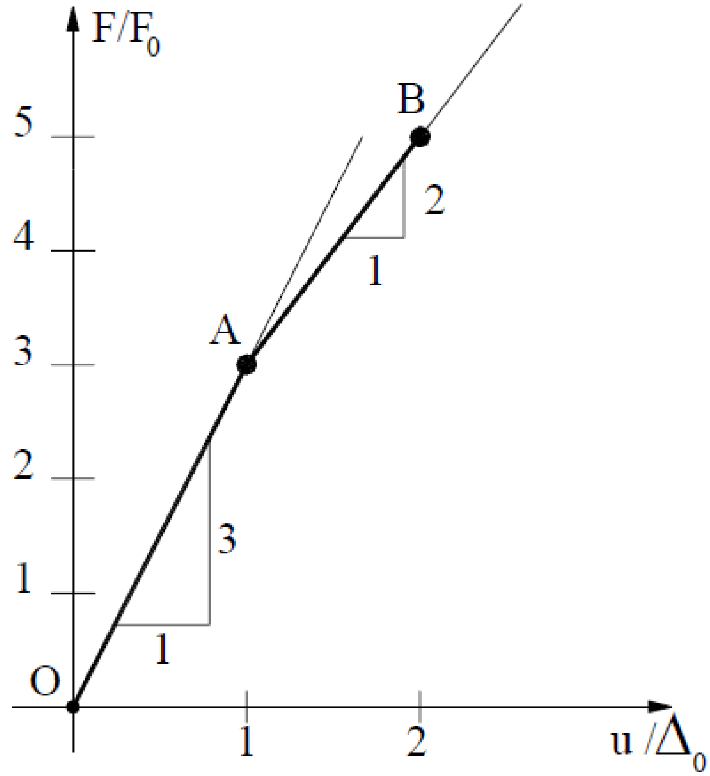


Fig.1.26

Example: Statically indeterminate truss - three EPP parallel bars

Step.3 Yielding bars (1)+(2) $2\Delta_0 \le u \le 3\Delta_0$

$$\frac{\dot F_1}{F_0}=\frac{\dot F_2}{F_0}=0;\ \frac{\dot F_3}{F_0}=\frac{\dot u}{\Delta_0}\Rightarrow\frac{\dot F_1}{F_0}=\frac{\dot F_2}{F_0}=0,\ \frac{\dot F_3}{F_0}=\frac{\dot u}{\Delta_0}$$

$$\Rightarrow\boxed{\frac{\dot F}{F_0}=\frac{\dot u}{\Delta_0}},\ F_c=6F_0\ .$$

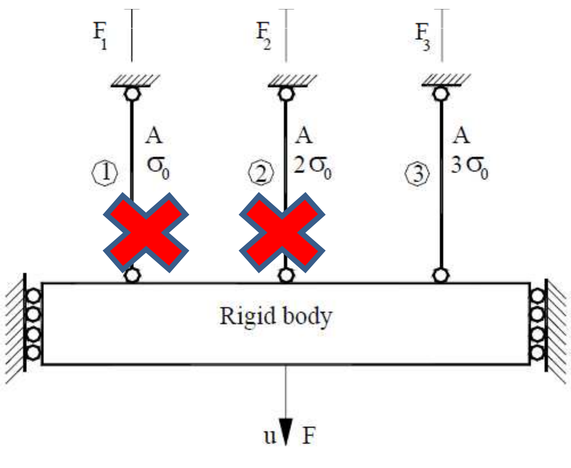
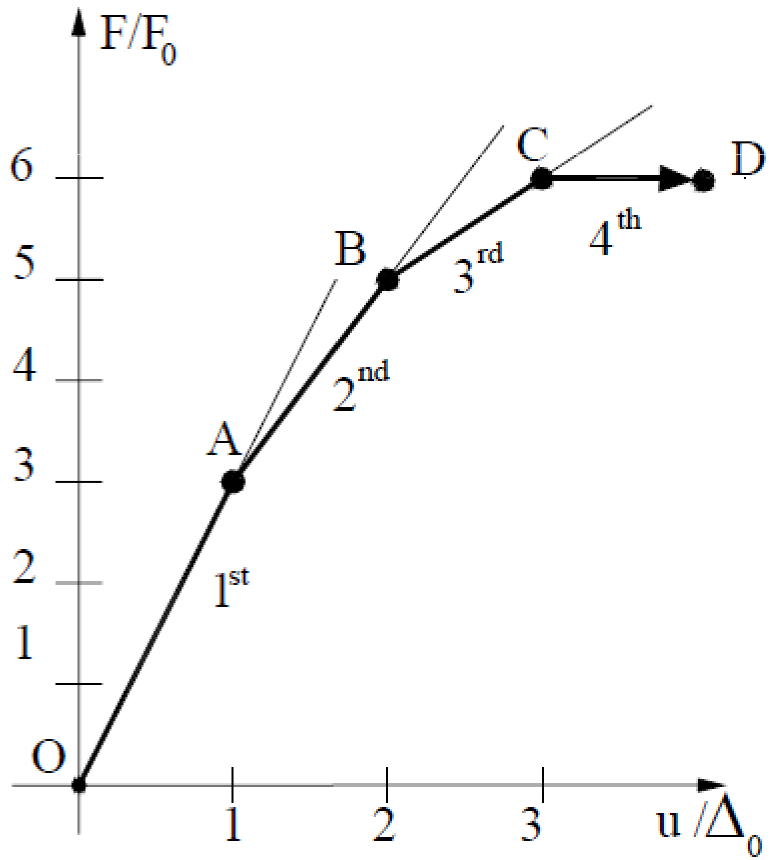


Fig.1.24



Example: Statically indeterminate truss - three EPP parallel bars

Step 4- Yielding bars 1 + 2 + 3 $\quad u \geq 3\Delta_0$

$$\dot{F}_1 = \dot{F}_2 = \dot{F}_3 = 0 \Rightarrow \frac{\dot{F}}{F_0} = 0 \Rightarrow \dot{u} > 0 \Rightarrow \underline{\text{Collapse}}$$

(1) The truss turns into a mechanism $\forall \dot{u} > 0$

(2) Increments of the applied load $\dot{F} > 0$ are not statically admissible.

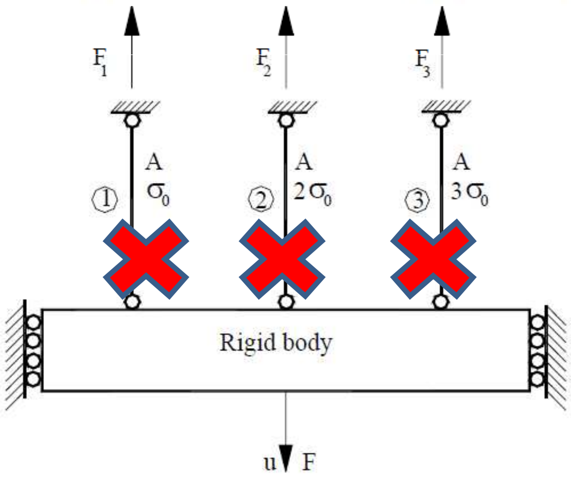
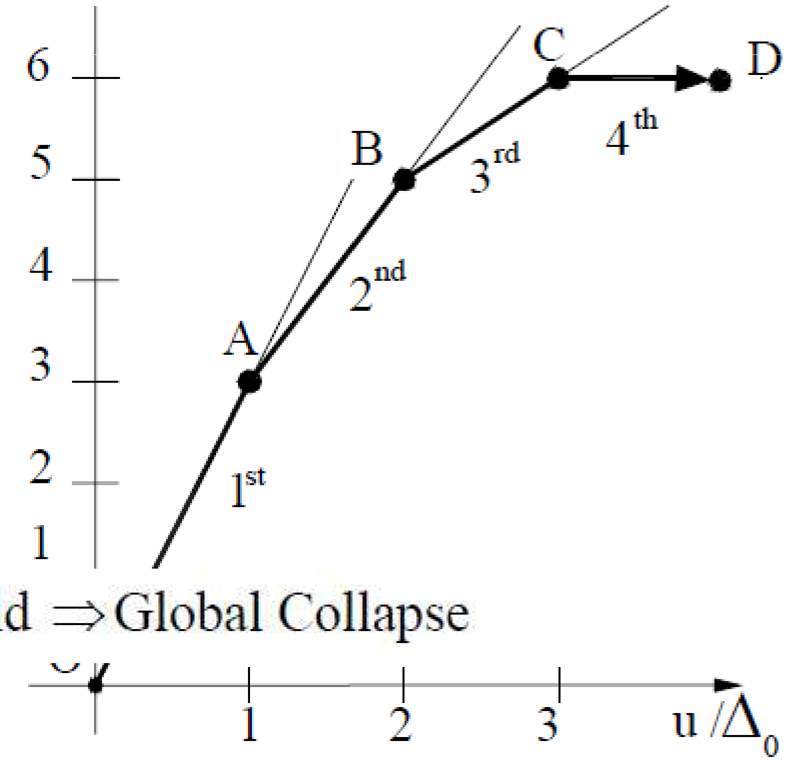


Fig.1.24



Collapse takes place when $(i+1=3)$ bars yield $\Rightarrow$ Global Collapse

Example: Statically indeterminate truss - three EPP parallel bars

Step 5-Unloading Bar Response $u < u_D \,,\ \dot{u} < 0$

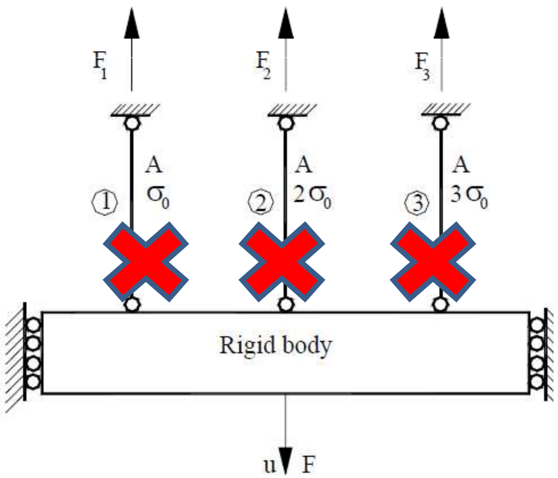
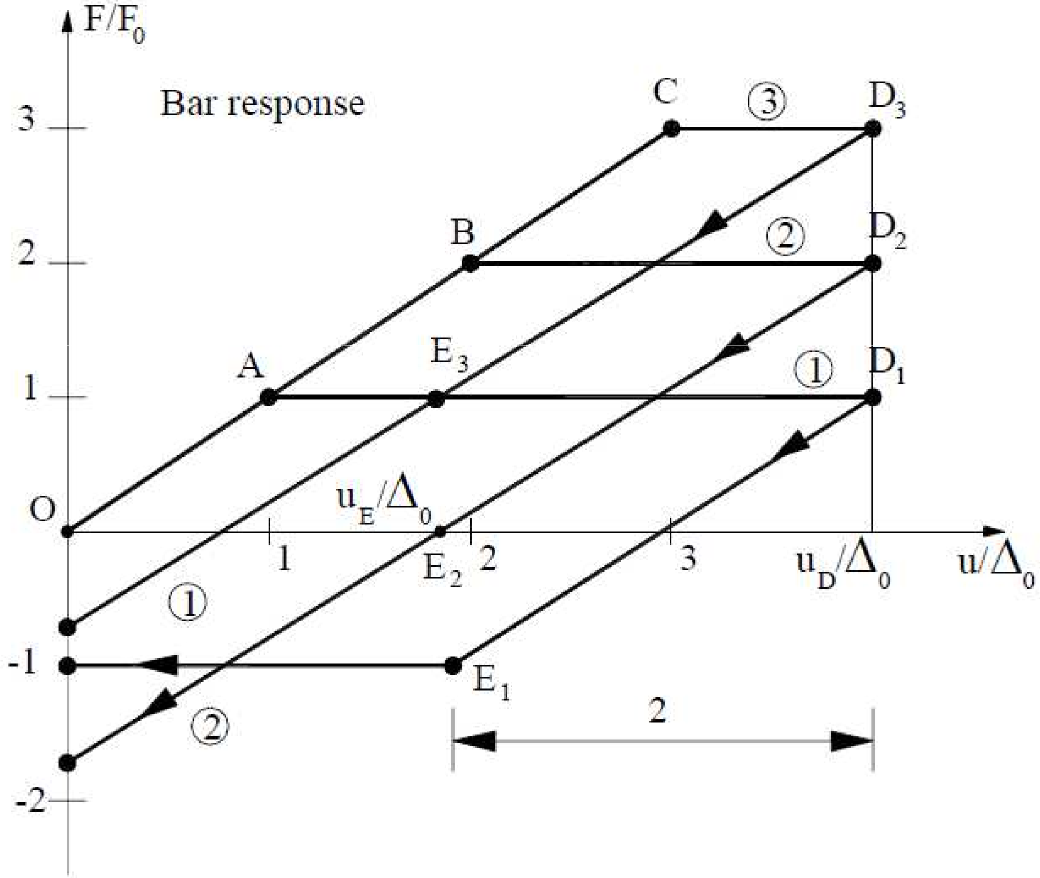


Fig.1.24



$$u_E = u_D - 2\Delta_0$$

Fig.1.28

Example: Statically indeterminate truss - three EPP parallel bars

Step 5-Unloading Bar Response $u<u_D\,,\ \dot{u}<0$

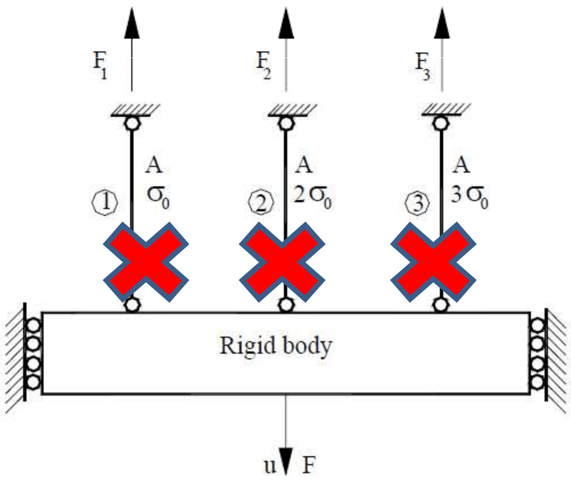


Fig.1.24

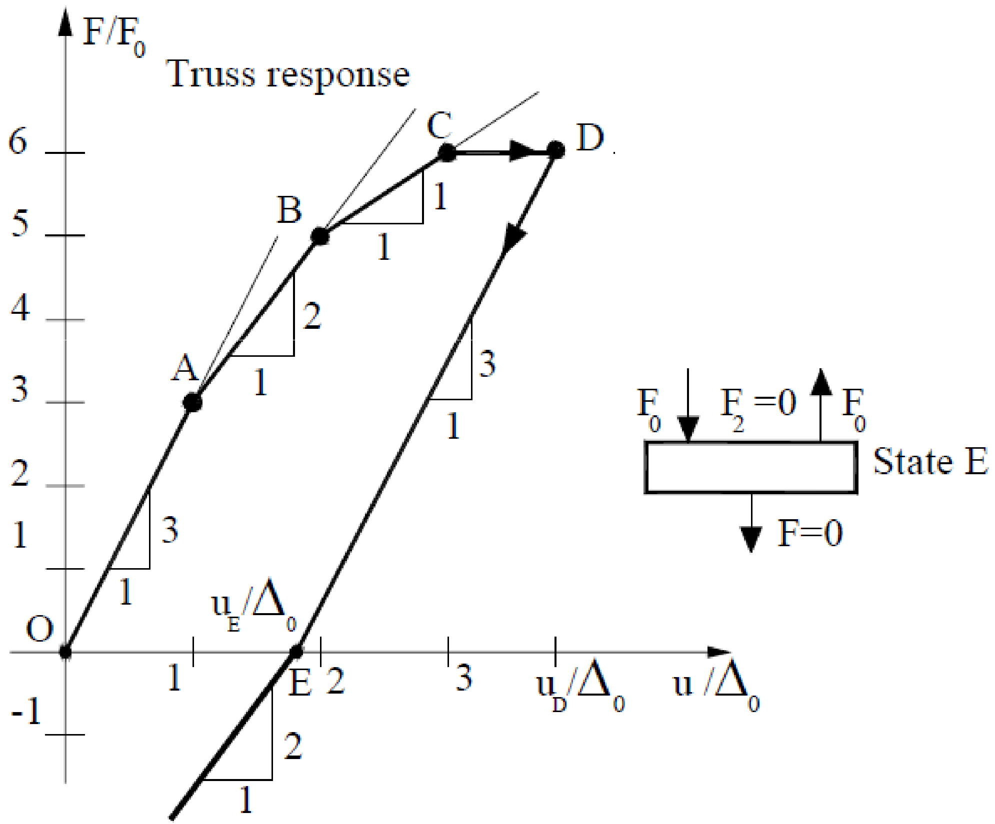


Fig.1.29

Example: Statically indeterminate truss - three EPP parallel bars

Ciclyc response

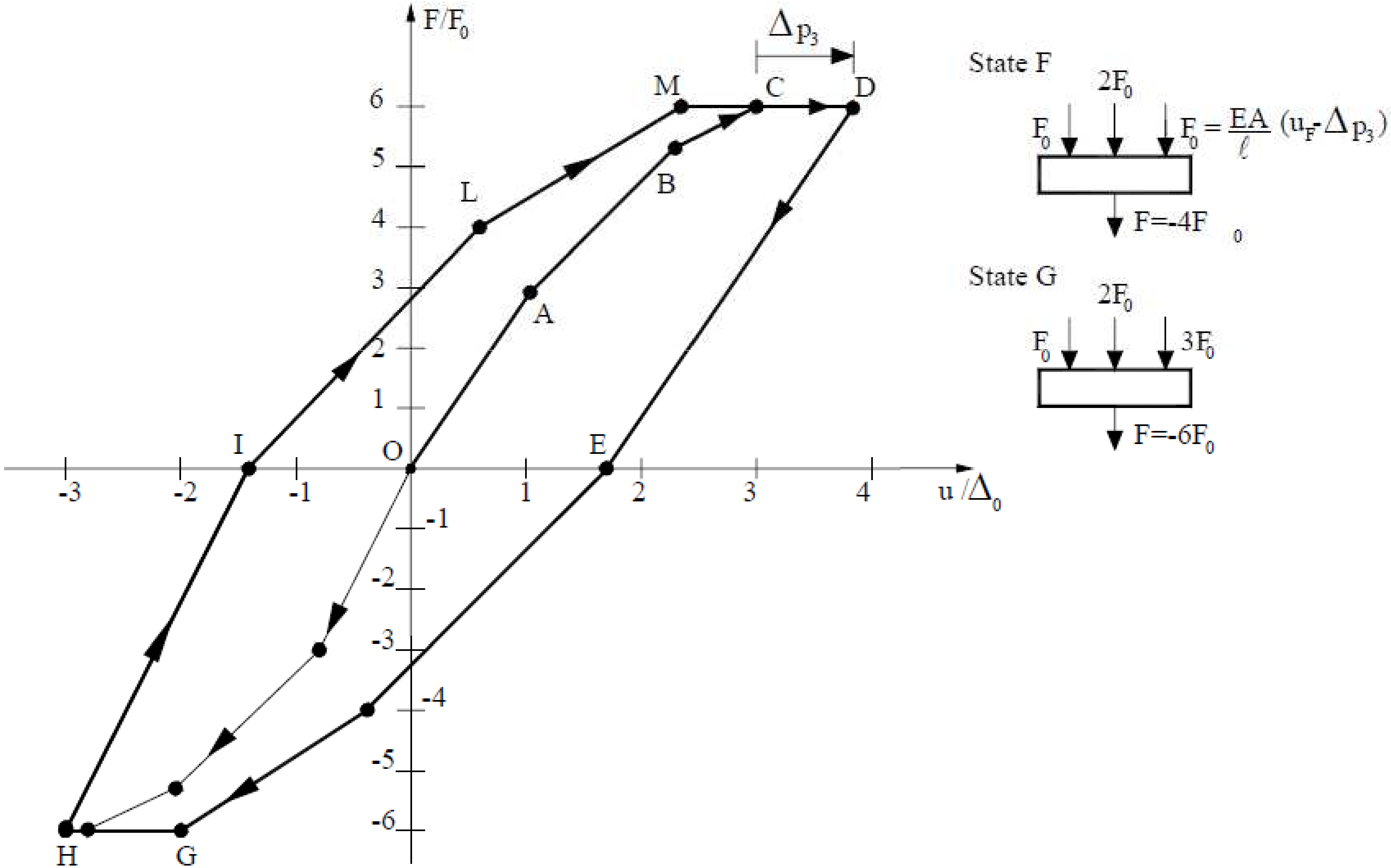


Fig.1.30

Example: Statically indeterminate truss - three EPP parallel bars

Influence of initial distortion on collapse

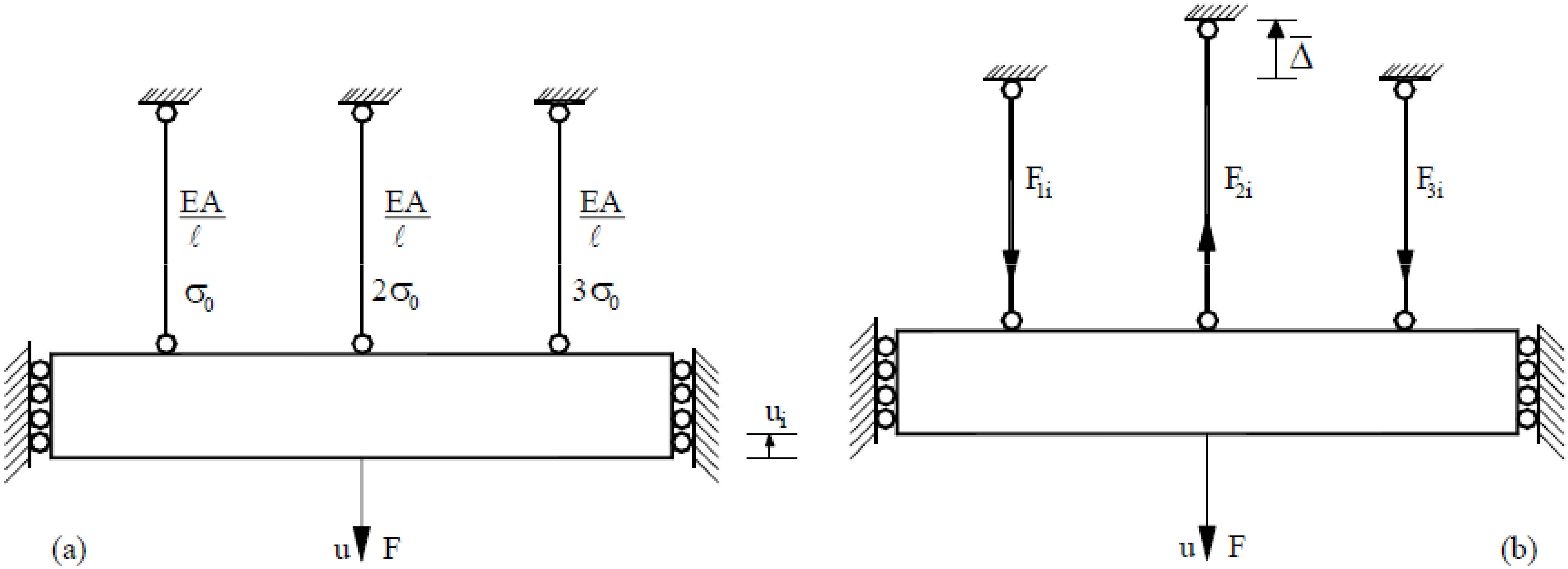
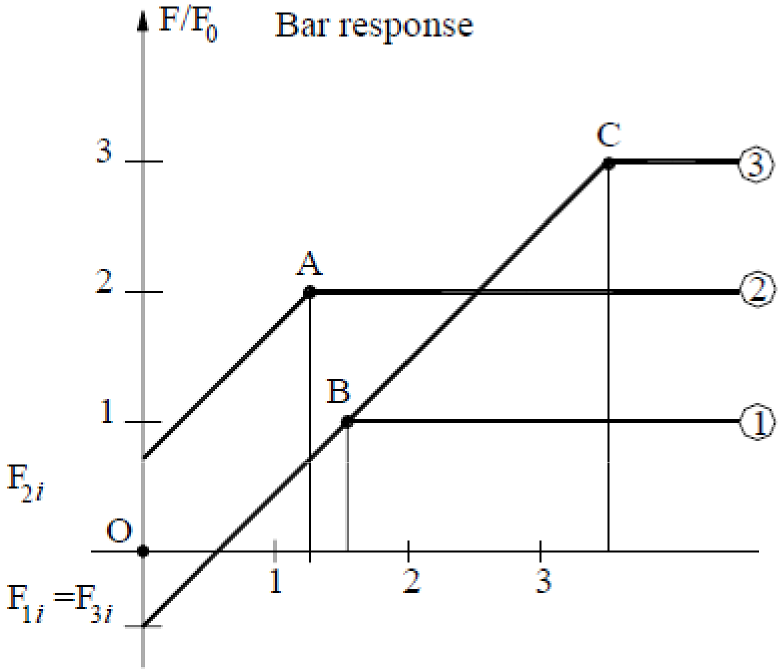


Fig.1.31

(a): Natural configuration (b): Initial/Reference configuration

- $\bar{\Delta}$ displacement imposed – bar ② $\Rightarrow$ u_i displacement of the rigid block;
- At the reference configuration $F = F_{2i} - F_{1i} - F_{3i} = 0$; $F_{3i} = F_{1i} \Rightarrow F_{2i} = -2F_{1i}$

Example: Statically indeterminate truss - three EPP parallel bars



- The response diagram is different
- The collapse load the same.

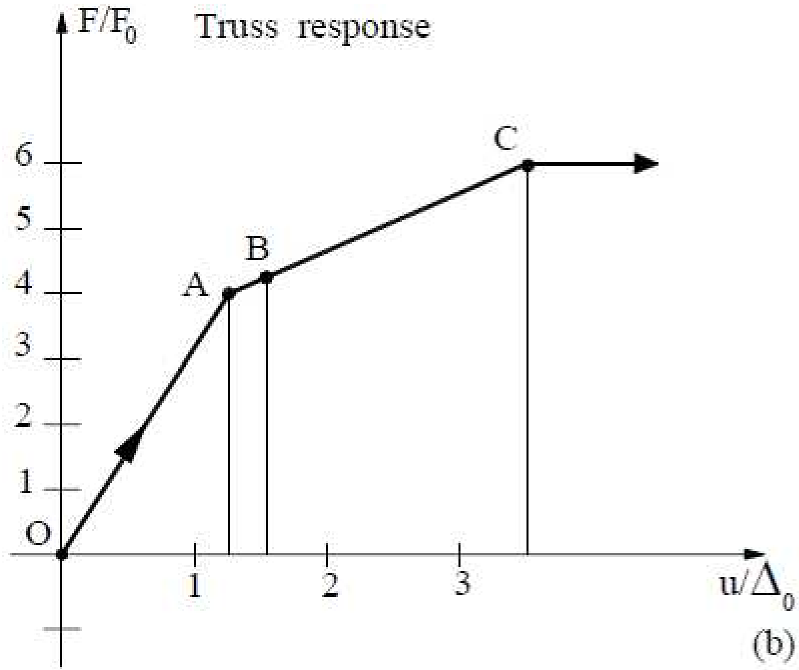


Fig.1.32