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Multivariate problems and matrix algebra

Multivariate problems

Multivariate statistical analysis deals with data containing observations on two or more characteristics (variables) each measured on a set of objects (statistical units)

Example 1: examination marks, about 5 courses (Mechanics, Vectors, Algebra, Analysis, Statistics), achieved by 88 students

Example 2: weights of cork deposites (centigrams) for 28 trees in the four directions (N, E, S, W)

Example 3: flower measurements (sepal length, sepal width, petal length, petal width) on 50 flowers belonging to a certein species of iris

Multivariate problems

n statistical units

$\Rightarrow$ *nk* pieces of information

k variables

Available information $\rightarrow$ Dataset $\rightarrow$ $n \times k$ matrix

Example: data matrix with 5 students where X_1=age in years at entry to university, X_2=marks out of 100 in an examination at the end of the first year and X_3=sex.

	Variables		
units	X_1	X_2	X_3
1	18.45	70	1
2	18.41	65	0
3	18.39	71	0
4	18.70	72	0
5	18.34	94	1

Multivariate problems

Some multivariate problems:

Example 1: study how the mark in the examination of «Statistics» (dependent variable) *is affected by* or *can be predicted as function of* the marks in other examinations or other variables such as age, sex, etc. (explanatory variables) → **regression problem**

Example 2: study how to combine the information on the performance of the students on the 5 examinations to determine the global performance of each student with just one, or two or less than 5 values → **factor analysis, principal component analysis, composite indicator**

Example 3: study how to group students with similar performances by considering the whole set of examinations → **cluster analysis**

Multivariate problems

The general $n \times k$ matrix which represents a dataset with *n* statistical units and *k* variables can be written as follows:

Variables

Units	X_1	…	X_v	…	X_k
1	x_{11}	…	x_{1v}	…	x_{1k}
…	…	…	…	…	…
u	x_{u1}	…	x_{uv}	…	x_{uk}
…	…	…	…	…	…
n	x_{n1}	…	x_{nv}	…	x_{nk}

This matrix can be denoted **X** or (x_{uv})

$$\mathbf{x}_u = \begin{pmatrix} x_{u1} \\ \cdots \\ x_{uv} \\ \cdots \\ x_{uk} \end{pmatrix} \qquad \mathbf{x}_{(v)} = \begin{pmatrix} x_{1v} \\ \cdots \\ x_{uv} \\ \cdots \\ x_{nv} \end{pmatrix}$$

Matrix algebra

A m×n matrix A is a table with m rows and n columns:

$$\mathbf{A} = \begin{pmatrix} 3 & 8 & 3 & 5 \\ 9 & 1 & 1 & 8 \\ 4 & 6 & 4 & 2 \end{pmatrix} \qquad a_{23} = 1$$

In this case the matrix has 3 rows and 4 columns. If *m=n* then it is called **square matrix**

$$a_{ij}$$

i denotes the row

j denotes the column

Matrix algebra

A matrix with dimension $1 \times n$ is called **row vector**:

$$\underset{1\times5}{\mathbf{a}} = \begin{pmatrix} 6 & 3 & 1 & 7 & 2 \end{pmatrix}$$

A matrix with dimension $m \times 1$ is called **column vector** or simply vector:

$$\underset{5\times1}{\mathbf{c}} = \begin{pmatrix} 6 \\ 3 \\ 1 \\ 7 \\ 2 \end{pmatrix}$$

A **unit vector** is a vector of ones:

$$\underset{5\times1}{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Matrix algebra

Given the matrices ***A*** and ***B***, their sum is defined as ***C***=***A*** + ***B***, where $c_{ij} = a_{ij} + b_{ij}$

Example:

$$\mathbf{A} = \begin{pmatrix} 3 & 8 & 3 & 5 \\ 9 & 1 & 1 & 8 \\ 4 & 6 & 4 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & 3 & 5 & 9 \\ 9 & 1 & 4 & 3 \\ 6 & 6 & 9 & 6 \end{pmatrix} \Rightarrow \mathbf{A} + \mathbf{B} = \mathbf{C} = \begin{pmatrix} 6 & 11 & 8 & 14 \\ 18 & 2 & 5 & 11 \\ 10 & 12 & 13 & 8 \end{pmatrix}$$

$$a_{23} = 1 \quad b_{23} = 4 \qquad c_{23} = 5$$

Matrix algebra

The product of a $m \times n$ matrix ***A*** and a scalar (single value) λ is called **scalar multiplication** and it consists in a matrix with the same dimension of ***A***, obtained by multiplying each element of ***A*** by λ

C=λ***A*** $\Leftrightarrow$ $c_{ij} = \lambda a_{ij}$

Example:

$$\lambda = 2 \qquad \mathbf{A} = \begin{pmatrix} 3 & 8 & 3 & 5 \\ 9 & 1 & 1 & 8 \\ 4 & 6 & 4 & 2 \end{pmatrix}$$

$$2\mathbf{A} = \begin{pmatrix} 6 & 16 & 6 & 10 \\ 18 & 2 & 2 & 16 \\ 8 & 12 & 8 & 4 \end{pmatrix}$$

Matrix algebra

The **inner product** of two vectors **a** and **b** is possible if the vectors have the same number of elements and it is equal to $\boldsymbol{a}'\boldsymbol{b} = \sum_i a_i b_i$

The **product between two matrices** A and B is possible if the number of columns of A is equal to the number of rows of B.

Given the $m \times n$ matrix A and the $n \times h$ matrix B, the product C=AB is a $m \times h$ matrix. The element in row i and column j is equal to the inner product between row i of A and column j of B.

C=AB $\Leftrightarrow c_{ij} = \mathbf{a}_i'\mathbf{b}_{(j)}$

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \quad \Rightarrow \quad \mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{pmatrix} 7 & 14 \\ -1 & -2 \\ 6 & 12 \end{pmatrix}$$

Matrix algebra

…where the elements of ***C*** are equal to:

$$\mathbf{C} = \begin{pmatrix} c_{11} = 1 \cdot 1 + 2 \cdot 3 = 7 & c_{12} = 1 \cdot 2 + 2 \cdot 6 = 14 \\ c_{21} = -1 \cdot 1 + 0 \cdot 3 = -1 & c_{22} = -1 \cdot 2 + 0 \cdot 6 = -2 \\ c_{31} = 3 \cdot 1 + 1 \cdot 3 = 6 & c_{32} = 3 \cdot 2 + 1 \cdot 6 = 12 \end{pmatrix}$$

Matrix algebra

Note that:

$$\underset{(m\times n)}{\mathbf{A}} \cdot \underset{(n\times h)}{\mathbf{B}} = \underset{(m\times h)}{\mathbf{C}}$$

Thus the product between a row vector and a column vector is a scalar; the product between a column vector and a row vector is a matrix:

$$\underset{1\times n}{\mathbf{a}} \cdot \underset{n\times 1}{\mathbf{b}} = \underset{1\times 1}{c} \qquad \underset{n\times 1}{\mathbf{b}} \cdot \underset{1\times n}{\mathbf{a}} = \underset{n\times n}{\mathbf{C}}$$

Matrix algebra

Examples:

$$\underset{1\times 2}{\mathbf{a}} = \begin{pmatrix} 2 & 4 \end{pmatrix} \qquad \underset{2\times 1}{\mathbf{b}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \Rightarrow \quad \mathbf{a}\times\mathbf{b} = 2\cdot 5 + 4\cdot 2 = 18$$

$$\underset{2\times 1}{\mathbf{b}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \underset{1\times 2}{\mathbf{a}} = \begin{pmatrix} 2 & 4 \end{pmatrix} \quad \Rightarrow \quad \mathbf{b}\times\mathbf{a} = \begin{pmatrix} 10 & 20 \\ 4 & 8 \end{pmatrix}$$

Matrix algebra

The **transpose** of the matrix $\boldsymbol{A}=(a_{ij})$ is the matrix $\boldsymbol{A}'=(a_{ji})$ whose rows correspond to the columns of $\boldsymbol{A}$:

Example:

$$\mathbf{A} = \begin{pmatrix} 3 & 6 & 4 \\ 2 & 8 & 9 \\ 2 & 5 & 1 \end{pmatrix} \qquad \mathbf{A}' = \begin{pmatrix} 3 & 2 & 2 \\ 6 & 8 & 5 \\ 4 & 9 & 1 \end{pmatrix}$$

The square matrix $\boldsymbol{A}=(a_{ij})$ is **symmetric** if $a_{ij}= a_{ji}$ or equivalently if $\boldsymbol{A}' = \boldsymbol{A}$.

Example:

$$\mathbf{A} = \begin{pmatrix} 3 & 6 & 4 \\ 6 & 8 & 9 \\ 4 & 9 & 1 \end{pmatrix}$$

Matrix algebra

A **null matrix** is a matrix with all elements equal to 0.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & . \\ 0 & 0 & . & 0 \end{pmatrix}$$

A **diagonal matrix** is a square matrix whose elements not in the main diagonal are all equal to 0.

$$diag(a_1, \cdots, a_n) = \begin{pmatrix} a_1 & 0 & . & 0 \\ 0 & a_2 & . & . \\ . & . & . & . \\ 0 & 0 & . & a_n \end{pmatrix}$$

Matrix algebra

The transpose satisfies the following properties:

1. (***A***')'=***A***
2. (***A***+***B)***'=**A**'+**B**'
3. (***AB***)'=***B***'***A***'

A **diagonal matrix** is a square matrix whose elements not in the main diagonal are all equal to 0.

Matrix algebra

The **trace** of $\boldsymbol{A}$=(a_{ij}) is the sum of the elements in the main diagonal of $\boldsymbol{A}$:

$$tr(\boldsymbol{A})=\Sigma_i a_{ii}$$

Example:

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 1 \\ 7 & 5 & 8 \\ 9 & 7 & 5 \end{pmatrix} \qquad tr(\mathbf{A}) = 2+5+5=12$$

Main diagonal

Matrix algebra

The trace satisfies the following properties for

$\boldsymbol{A}$ $(m \times m)$, $\boldsymbol{B}$ $(m \times m)$, $\boldsymbol{C}$ $(m \times n)$, $\boldsymbol{D}$ $(n \times m)$ and a scalar λ:

1. $\mathrm{tr}(\lambda) = \lambda$
2. $\mathrm{tr}(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{A}')$
3. $\mathrm{tr}(\boldsymbol{A}+\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}) + \mathrm{tr}(\boldsymbol{B})$
4. $\mathrm{tr}(\boldsymbol{CD}) = \mathrm{tr}(\boldsymbol{DC}) = \Sigma_{i,j}\, c_{ij}\, d_{ji}$
5. $\mathrm{tr}(\boldsymbol{CC}') = \mathrm{tr}(\boldsymbol{C}'\boldsymbol{C}) = \Sigma_{i,j}\, c_{ij}^2$

Matrix algebra

Given the *2×2* matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

The **determinant** of ***A*** is

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \quad a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Matrix algebra

Given the $m \times m$ matrix

$$\mathbf{A}$$

The **determinant** of ***A*** is

$$\det(\mathbf{A}) = |\mathbf{A}| = \sum_{j=1}^{m} a_{ij} A_{ij} = \sum_{i=1}^{m} a_{ij} A_{ij}$$ for any i,j

where the **cofactor** A_{ij} is the product of $(-1)^{i+j}$ and the determinant of the matrix obtained after deleting ith row and jth column of ***A*** (minor)

Case $m=3$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \det(\mathbf{A}) = |\mathbf{A}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Matrix algebra

Computation of the determinant of a 3rd order matrix (Sarrus rule):

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \left(a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}\right)$$

Matrix algebra

Example:

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 6 & 2 \\ 5 & 7 & 4 \end{pmatrix}$$

$$\det(A) = 3\begin{vmatrix} 6 & 2 \\ 7 & 4 \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} + 1\begin{vmatrix} 4 & 6 \\ 5 & 7 \end{vmatrix} = 3(24-14) - 4(16-10) + 1(28-30) = 4$$

or alternatively:

$$\det(\mathbf{A}) = \begin{pmatrix} 3 & 4 & 1 & 3 & 4 \\ 4 & 6 & 2 & 4 & 6 \\ 5 & 7 & 4 & 5 & 7 \end{pmatrix} =$$

$$= 3 \cdot 6 \cdot 4 + 4 \cdot 2 \cdot 5 + 1 \cdot 4 \cdot 7 - (5 \cdot 6 \cdot 1 + 7 \cdot 2 \cdot 3 + 4 \cdot 4 \cdot 4) =$$

$$= 140 - 136 = 4$$

Matrix algebra

Properties of the determinant

1. If $\boldsymbol{A}=diag(a_1,\dots,a_n)$ then $det(\boldsymbol{A})=a_1 \cdot a_2 \cdot \dots \cdot a_n = \Pi_i\, a_i$
2. $det(\lambda \boldsymbol{A}) = |\lambda \boldsymbol{A}| = \lambda^n\, |\boldsymbol{A}|$
3. $det(\boldsymbol{AB}) = |\boldsymbol{AB}| = |\boldsymbol{A}| \cdot |\boldsymbol{B}|$
4. If ***A*** has two equal rows or two equal columns then $det(\boldsymbol{A})=0$
5. If ***A*** has a row of zeros or a column of zeros then $det(\boldsymbol{A})=0$
6. If ***B*** is the matrix obtained exchanging the position of two rows or two columns of ***A*** then $det(\boldsymbol{B}) = -det(\boldsymbol{A})$
7. $det(\boldsymbol{A})=det(\boldsymbol{A}')$
8. If ***B*** is the matrix obtained by summing to a row or a column of ***A*** a linear combination of the other rows or columns of ***A*** respectively then $det(\boldsymbol{B})=det(\boldsymbol{A})$
9. A square matrix ***A*** is **non-singular** if $det(\boldsymbol{A}) \neq 0$; otherwise ***A*** is singular

Matrix algebra

Example:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix}$$

$$\det(\mathbf{A}) = 2 \cdot 6 - 3 \cdot 1 = 9 \qquad \det(\mathbf{B}) = 2 \cdot 3 - 5 \cdot 7 = -29$$

$$\det(\mathbf{A}) \cdot \det(\mathbf{B}) = 9 \cdot (-29) = -261$$

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 + 3 \cdot 7 & 2 \cdot 5 + 3 \cdot 3 \\ 1 \cdot 2 + 6 \cdot 7 & 1 \cdot 5 + 6 \cdot 3 \end{pmatrix} = \begin{pmatrix} 25 & 19 \\ 44 & 23 \end{pmatrix}$$

$$\det(\mathbf{AB}) = 25 \cdot 23 - 19 \cdot 44 = -261$$

Matrix algebra

The **inverse** of the square matrix ***A*** is the unique matrix ***A***$^{-1}$ satisfying:

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$

diag(1)= Identity matrix

$$\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$$

The inverse ***A***$^{-1}$ exists if and only if ***A*** is non singular, that is, if and only if det(***A***)≠0.

Matrix algebra

The **identity matrix** is a diagonal matrix where all the elements in the main diagonal are equal to 1.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & . & 0 & 0 \\ 0 & 1 & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & 1 & 0 \\ 0 & 0 & . & 0 & 1 \end{pmatrix}$$

Properties of $\mathbf{I}$

$$\mathbf{A} \times \mathbf{I} = \mathbf{I} \times \mathbf{A} = \mathbf{A}$$

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$

$$\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}$$

Matrix algebra

Properties of the inverse:

1. $(\lambda \boldsymbol{A})^{-1} = \lambda^{-1}\boldsymbol{A}^{-1}$
2. $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
3. The unique solution of $\boldsymbol{Ax}=\boldsymbol{b}$ is $\boldsymbol{x}=\boldsymbol{A}^{-1}\boldsymbol{b}$

4. $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

Example 1:

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix} \Rightarrow (\mathbf{AB})^{-1} = \begin{pmatrix} 25 & 19 \\ 44 & 23 \end{pmatrix}^{-1} = \frac{1}{(-261)} \begin{pmatrix} 23 & -19 \\ -44 & 25 \end{pmatrix} = \begin{pmatrix} -0.088 & 0.073 \\ 0.169 & -0.096 \end{pmatrix}$$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix}^{-1} \Rightarrow \begin{pmatrix} -0.103 & 0.172 \\ 0.241 & -0.069 \end{pmatrix} \cdot \begin{pmatrix} 0.667 & -0.333 \\ -0.111 & 0.222 \end{pmatrix} = \begin{pmatrix} -0.088 & 0.073 \\ 0.169 & -0.096 \end{pmatrix}$$

Matrix algebra

Example 2:

Let us consider the following system of equations

$$\begin{cases} 2x_1 + 3x_2 = 13 \\ x_1 + 2x_2 = 8 \end{cases} \Longrightarrow \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix} \Longrightarrow \boldsymbol{Ax} = \boldsymbol{b}$$

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 8 \end{pmatrix} = \frac{1}{2 \cdot 2 - 3 \cdot 1} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 13 \\ 8 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \cdot 13 - 3 \cdot 8 \\ -1 \cdot 13 + 2 \cdot 8 \end{pmatrix} = \begin{pmatrix} 26 - 24 \\ -13 + 16 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Matrix algebra

A square matrix $\boldsymbol{A}$ is **orthogonal** if $\boldsymbol{AA'} = \boldsymbol{I}$

The following properties hold:

1. $\boldsymbol{A'} = \boldsymbol{A}^{-1}$
2. $\boldsymbol{A'A} = \boldsymbol{I}$
3. $|\boldsymbol{A}| = \pm 1$
4. $\boldsymbol{a}_i'\boldsymbol{a}_j=0,\ i\neq j;\ \boldsymbol{a}_i'\boldsymbol{a}_i=0,\ \forall i;\ \boldsymbol{a}_{(i)}'\boldsymbol{a}_{(j)}=0,\ i\neq j;\ \boldsymbol{a}_{(i)}'\boldsymbol{a}_{(i)}=0,\ \forall i;$

Example:

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boldsymbol{A}' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \boldsymbol{A}^{-1} \text{ because } \boldsymbol{AA}' = \boldsymbol{I}$$

Matrix algebra

Vectors $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k$ are called **linearly dependent** if there exist numbers $\lambda_1,\ldots,\lambda_K$ not all zero such that $\lambda_1\boldsymbol{x}_1+\ldots+\lambda_K\boldsymbol{x}_k=\mathbf{0}$.
Otherwise the *k* vectors are linearly independent.

Let $\boldsymbol{W}$ be a subspace of $\boldsymbol{R}^n$. Then a basis of $\boldsymbol{W}$ is a maximal linearly independent set of vectors.

Every basis of $\boldsymbol{W}$ contains the same (finite) number of elements. This number is the dimension of $\boldsymbol{W}$.

If $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k$ is a basis for $\boldsymbol{W}$ then every element $\boldsymbol{x}$ in $\boldsymbol{W}$ can be expressed as a linear combination of $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k$.

Example:
The dimension of $\mathbf{W}=\boldsymbol{R}^3$ is 3. A basis for $\boldsymbol{R}^3$ is $\boldsymbol{x}_1=(1,0,0)'$, $\boldsymbol{x}_2=(0,1,0)'$ and $\boldsymbol{x}_3=(0,0,1)'$. As a matter of fact $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are linearly independent and every vector $\boldsymbol{a}=(a_1,a_2,a_3)'$ can be expressed as linear combination of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$: $\boldsymbol{a}= a_1\boldsymbol{x}_1 +a_2\boldsymbol{x}_2+a_3\boldsymbol{x}_3$

Matrix algebra

The **rank** of a $n \times k$ matrix ***A*** is defined as the maximum number of linearly independent columns (rows) in ***A***.

The following properties hold for the rank of ***A***, denoted with *r(**A**)*:

1. *r(**A**)* is the largest order of those (square) submatrices of ***A*** with non null determinants.
2. 0≤ *r(**A**) ≤min(n,k)*
3. *r(**A**)=r(**A'**)*
4. *r(**A'A**)= r(**AA'**)= r(**A**)*
5. If *n=k* then *r(**A**)=k* if and only if A is non-singular

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 \\ -2 & 2 & 5 \\ 1 & -1 & 9 \end{pmatrix} \quad \det(\mathbf{A}) = 0 \quad \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} = 0 \quad \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = -11 \neq 0$$

Thus the r(***A***)=2

Matrix algebra

If ***A*** is a square matrix of order *n*, in some problems we are interested in finding a vector x and a scalar λ which satisfy the following property:

eigenvalue

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

eigenvector

A trivial solution is ***x***=***0***, any $\lambda \in \boldsymbol{R}$

Matrix algebra

The *n* ***eigenvalues*** of ***A*** $\lambda_1,\ldots,\lambda_n$ are the *n* solutions of the characteristic equation

$$\left|\mathbf{A}-\lambda\mathbf{I}\right| = \mathbf{0}$$

Properties of the eigenvalues of ***A***:

1. $|\boldsymbol{A}|=\Pi_i \lambda_i$
2. $tr(\boldsymbol{A})= \Sigma_i \lambda_i$
3. $r(\boldsymbol{A})$ equals the number of non-zero eigenvalues
4. The set of all eigenvectors for an eigenvalue λ_i is called the eigenspace of ***A*** for λ_i
5. Any symmetric $n \times n$ matrix ***A*** can be written as $\boldsymbol{A}=\Gamma\Lambda\Gamma'=\Sigma_i\lambda_i\gamma_{(i)}\gamma_{(i)}'$ where Λ is a diagonal matrix of eigenvalues of ***A*** and Γ is an orthogonal matrix whose columns are eigenvectors with $\gamma_{((i)}'\gamma_{(i)}=1$

Matrix algebra

Example:

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

The characteristic equation is:

$$\begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

By computing the determinant we have:

$$(2-\lambda)\cdot\begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} -2 & 3 \\ 3 & -1-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} -2 & 3 \\ 1-\lambda & 1 \end{vmatrix} =$$

$$= (2-\lambda)\left[-1+\lambda^2-3\right]-2-2\lambda+9-2-3+3\lambda = (\lambda+2)(\lambda-1)(3-\lambda) = 0$$

The solutions represent the 3 eigenvalues of ***A***:

$$\lambda_1 = 1 \qquad \lambda_2 = -2 \qquad \lambda_3 = 3$$

Matrix algebra

The eigenvalue with maximum absolute value $\lambda_3=3$ is called dominant

There is an infinite number of eigenvectors ***x*** which satisfy $(\boldsymbol{A}-3\boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}$

$$\begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A possible solution is ***x***= (1,1,1)' , thus a standardized eigenvector is $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)'$