Matrices and Linear Algebra

Quantitative methods for Economics and Business

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Part I

Matrices and Linear Algebra

Definition

A matrix is an $m \times n$ array of $m \cdot n$ scalars from $\mathbb{R}$. The individual values in the matrix are called **entries**.

Examples:

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The *size* of the array is written as $m \times n$, where

$$m \quad \text{is the number of rows}$$

$$n \quad \text{is the number of columns}$$

Notation:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$a_{ij} := \text{the entry on the row } i \text{ and on the column } j$$

Specific cases

A matrix with dimension $1 \times n$ is called **row vector**:

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

A matrix with dimension $m \times 1$ is called **column vector** or simply **vector**:

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

A **unit vector** is a vector of ones:

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Special matrices

If $m = n$, the matrix is called **square**. In this case we have:

- A matrix A is said to be **diagonal** if

$$a_{ij} = 0 \quad \text{for } i \neq j$$

- A diagonal matrix A may be denoted by $diag(d_1, d_2, \ldots, d_n)$ where

$$a_{ii} = d_i \qquad \text{and} \qquad a_{ij} = 0 \quad \text{for } i \neq j.$$

  The diagonal matrix $diag(1, 1, \ldots, 1)$ is called the **identity matrix** and is usually denoted by

$$I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

  The diagonal matrix $O = diag(0, \ldots, 0)$ is called the **zero matrix**.

- A square matrix L is said to be **lower triangular** if

$$l_{ij} = 0 \quad \text{for } i < j$$

  that is

$$L = \begin{pmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix}$$

- A square matrix U is said to be **upper triangular** if

$$u_{ij} = 0 \quad \text{for } i > j$$

  that is

$$U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{pmatrix}$$

Definition

Let A be an $m \times n$ matrix.
Define the **transpose** of A, denoted by A^T, to be the $n \times m$ matrix (i.e. n rows and m columns) with entries

$$(A^T)_{ij} = a_{ji}.$$

In other words:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

Examples:

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \end{pmatrix} \qquad A^T = \begin{pmatrix} 2 & 1 \\ -1 & 4 \\ 3 & -2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

We have that

$$(A^T)^T = A$$

A square matrix A (i.e. an $n \times n$ matrix) is called **symmetric** if

$$a_{ij} = a_{ji}$$

that is

$$A^T = A$$

Example: The matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & -4 \\ 2 & -4 & 0 \end{pmatrix}$$

is symmetric.

Equality, Addition, Scalar multiplication

Two matrices A and B are **equal** if and only if they have the same size and

$$a_{ij} = b_{ij} \quad \text{for all } i, j.$$

If A and B are matrices of the same size then the **sum** of A and B is defined by $C = A + B$, where

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all } i, j.$$

If A is any matrix and $\alpha \in \mathbb{R}$ then the **scalar multiplication** $B = \alpha A$ is defined by

$$b_{ij} = \alpha a_{ij} \quad \text{for all } i, j.$$

Examples

$$\begin{pmatrix} 2 & 1 \\ -3 & 4 \\ 7 & 0 \end{pmatrix} + \begin{pmatrix} 6 & -1.5 \\ 0 & 1 \\ 1 & \pi \end{pmatrix} = \begin{pmatrix} 8 & -0.5 \\ -3 & 5 \\ 8 & \pi \end{pmatrix}$$

$$-3 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ -9 & -12 \end{pmatrix}$$

Matrix addition "inherits" many properties from $\mathbb{R}$.

Theorem: If A, B, C are $m \times n$ matrices and $\alpha, \beta \in \mathbb{R}$, then

- $A + B = B + A$ (commutivity)
- $A + (B + C) = (A + B) + C$ (associativity)
- $\alpha(A + B) = \alpha A + \alpha B$ (distributivity of a scalar)
- if $B = O$ (a matrix of all zeros) then $A + B = A + O = A$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $\alpha(\beta A) = \alpha\beta A$
- $0A = O$
- $\alpha O = O$
- $(A + B)^T = A^T + B^T$

Inner or scalar product

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two vectors.
The **scalar** or **inner product** of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

It is a scalar (i.e. a number)!!!

Remark: Alternative notation for the scalar product is $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.
Scalar product is defined only for vectors of the same length!!!

Example: let $\mathbf{x} = (1, 0, 3, -1)$ and $\mathbf{y} = (0, 2, -1, 2)$. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = 1 \cdot 0 + 0 \cdot 2 + 3 \cdot (-1) + (-1) \cdot 2 = -5$$

Matrix product

Assume that A is $m \times n$ and B is $n \times p$.
Denote by

$$\mathbf{r}_i(A) \text{ the } i\text{−th row of } A$$
$$\mathbf{c}_j(B) \text{ the } j\text{−th column of } B$$

The **product** $D = AB$ is the $m \times p$ matrix defined by

$$d_{ij} = \langle \mathbf{r}_i(A), \mathbf{c}_j(B) \rangle$$

that means

$$d_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}, \quad \text{for } 1 \leq i \leq m,\ 1 \leq j \leq p$$

Remark: In order to perform matrix multiplication, we need that the number of columns of A is equal to the number of rows of B!

Example:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 1 \end{pmatrix}$$

Set $C = AB$, then

$$\begin{aligned}
d_{11} &= \langle \mathbf{r}_1(A), \mathbf{c}_1(B) \rangle = 1 \cdot 2 + 0 \cdot 3 + 1 \cdot (-1) = 1 \\
d_{12} &= \langle \mathbf{r}_1(A), \mathbf{c}_2(B) \rangle = 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 = 2 \\
d_{21} &= \langle \mathbf{r}_2(A), \mathbf{c}_1(B) \rangle = 3 \cdot 2 + 2 \cdot 3 + 1 \cdot (-1) = 11 \\
d_{22} &= \langle \mathbf{r}_2(A), \mathbf{c}_2(B) \rangle = 3 \cdot 1 + 2 \cdot 0 + 1 \cdot 1 = 4
\end{aligned}$$

We obtain

$$D = AB = \begin{pmatrix} 1 & 2 \\ 11 & 4 \end{pmatrix}$$

Properties of matrix product

If AB exists, does it happen that BA exists and $AB = BA$?
The answer is usually no.

- First AB and BA exist if and only if A is $m \times n$ and B is $n \times m$. Even if this is so, the sizes of AB and BA are different (AB is $m \times m$ and BA is $n \times n$) unless $m = n$.
  Example:

$$A = (1, 2) \quad B = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad AB = (1) \qquad BA = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}$$

- However even if m = n we may have $AB \neq BA$.
  Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & 3 \\ -3 & 7 \end{pmatrix} \qquad BA = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$$

If $AB = O$, does it happen that $A = O$ or $B = O$?
The answer is usually no.

Example: Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

It happens that

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but

$$A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$B \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Theorem (Matrix Multiplication Rules).

Assume A, B, and C are matrices for which all products below make sense. Then

- $A(BC) = (AB)C$ (associativity)
- $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
- $AI = A$ and $IA = A$
- $\alpha(AB) = (\alpha A)B$
- $AO = O$ and $OB = O$
- $(AB)^T = B^T A^T$

The simple case of 2×2 matrices

Consider a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The **determinant** of A is defined as

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2$$

The case of 3×3 matrices

Consider a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The **determinant** of A is defined as

$$\det(A) = a_{11} \cdot (-1)^{1+1} \det(A_{11}) + a_{12} \cdot (-1)^{1+2} \det(A_{12}) + a_{13} \cdot (-1)^{1+3} \det(A_{13})$$

where

$$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \quad A_{12} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \quad A_{13} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Then

$$\det(A) = a_{11} \cdot \det(A_{11}) - a_{12} \cdot \det(A_{12}) + a_{13} \cdot \det(A_{13})$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

The same rule may be carried out by selecting any row or column of A:

for each row index i ($i = 1, 2, 3$), we have

$$\det(A) = a_{i1}\cdot(-1)^{i+1}\det(A_{i1}) + a_{i2}\cdot(-1)^{i+2}\det(A_{i2}) + a_{i3}\cdot(-1)^{i+3}\det(A_{i3})$$

for each column index j ($j = 1, 2, 3$), we have

$$\det(A) = a_{1j}\cdot(-1)^{1+j}\det(A_{1j}) + a_{2j}\cdot(-1)^{2+j}\det(A_{2j}) + a_{3j}\cdot(-1)^{3+j}\det(A_{3j})$$

where A_{ij} is the submatrix obtained from A after deleting the $i-$th row and the $j-$th column of A itself.

For instance, consider

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \\ 1 & -2 & 1 \end{pmatrix}$$

Then we have

$$\det(A) = 1 \cdot \det \begin{pmatrix} -1 & 4 \\ -2 & 1 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix}$$

It follows that

$$\det(A) = (-1 + 8) - 2(3 - 4) + (-6 + 1) = 4$$

Any row or column can be selected in order to evaluate the determinant: try to work on the third column and verify the result!

The general case of $n \times n$ matrices

Consider a square $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The **determinant** of A is defined as

$$\det(A) = a_{i1} \cdot (-1)^{i+1} \det(A_{i1}) + a_{i2} \cdot (-1)^{i+2} \det(A_{i2}) + \cdots + a_{in} \cdot (-1)^{i+n} \det(A_{in})$$

for each row index i ($i = 1, 2, \dots, n$), or equivalently as

$$\det(A) = a_{1j} \cdot (-1)^{1+j} \det(A_{1j}) + a_{2j} \cdot (-1)^{2+j} \det(A_{2j}) + \cdots + a_{nj} \cdot (-1)^{n+j} \det(A_{nj})$$

for each column index j ($j = 1, 2, \dots, n$), where

A_{ij} is the submatrix obtained from A after deleting the $i-$th row and the $j-$th column of A itself.

Some properties of the determinant

Consider an $n \times n$ matrix A.

- If two rows of A are interchanged to obtain B, then

$$\det(B) = -\det(A)$$

- If any row of A is multiplied by a scalar α, the resulting matrix B has determinant

$$\det(B) = \alpha \det(A)$$

- If any two rows of A are equal, then

$$\det(A) = 0$$

- If A has two rows equal up to a multiplicative constant, then

$$\det(A) = 0$$

- If B is obtained by summing to a row or a column of A a linear combination of the other rows or columns of A, respectively, then

$$\det(B) = \det(A)$$

- $\det(A^T) = \det(A)$
- $\det(AB) = \det(A) \cdot \det(B)$
- $\det(\alpha A) = \alpha^n \det(A)$
- If $A = diag(d_1, d_2, \ldots, d_n)$, then $\det(A) = d_1 \cdot d_2 \ldots d_n$. It follows that

$$\det(I) = 1 \cdot 1 \ldots 1 = 1$$

Definition

Let A be an $n \times n$ square matrix.
A is **invertible** if there exists a matrix B that satisfies the following relationship

$$AB = BA = I$$

Under the assumption that B exists, it is unique!!! We denote B by

$$A^{-1} \qquad \textbf{inverse} \text{ of } A.$$

Then

$$AA^{-1} = A^{-1}A = I$$

It is possible to prove that:

$$A \text{ is invertible (i.e. } A^{-1} \text{ exists)} \quad \Leftrightarrow \quad \det(A) \neq 0$$

in that case, A is **non singular**.

Simple case: 2×2 matrices

Consider $n = 2$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Assume that $\det(A) \neq 0$. Then A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} \qquad \det(A) = 10$$

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Some properties of the inverse

Consider two $n \times n$ matrices A, B and a scalar $\alpha \in \mathbb{R}$.

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(\alpha A)^{-1} = \alpha^{-1}A^{-1}$
- $\det(A^{-1}) = 1/\det(A)$ (why?)

An application: solution of linear systems

Data: $n \times n$ matrix A, $n \times 1$ column vector $\mathbf{b}$
Unknown:: $n \times 1$ column vector $\mathbf{x}$
Problem: Find $\mathbf{x}$ such that

$$A\mathbf{x} = \mathbf{b}$$

that is equivalent to solve the following system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Assume that A is non-singular (i.e. $\det(A) \neq 0$), then the previous system has got a unique solution, which is obtained as

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

thus

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example: The following system

$$\begin{cases} 3x_1 + 2x_2 = 1 \\ 4x_1 + 3x_2 = -2 \end{cases}$$

is equivalent to

$$A\mathbf{x} = \mathbf{b}$$

with

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

We have $\det(A) = 1 \neq 0$, then A is invertible and

$$A^{-1} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}$$

The solution of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ -10 \end{pmatrix}$$

Definition

If a vector $\mathbf{v} \neq \mathbf{0}$ satisfies the equation

$$A\mathbf{v} = \lambda \mathbf{v},$$

for some scalar λ, then λ is said to be an **eigenvalue** of the matrix A, and $\mathbf{v}$ is said to be an **eigenvector** of A corresponding to the eigenvalue λ.

Example: If

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

and

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then

$$A\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 5\mathbf{v}$$

So $\lambda = 5$ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector corresponding to this eigenvalue.

The definition of eigenvector requires that $\mathbf{v} \neq \mathbf{0}$.
The reason for this is that if $\mathbf{v} = \mathbf{0}$ were allowed, then any number λ would be an eigenvalue since the statement $A\mathbf{0} = \lambda\mathbf{0}$ holds for any λ.
On the other hand, we can have $\lambda = 0$ and $\mathbf{v} \neq \mathbf{0}$.

Example: If

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$$

and

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

then

$$A\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\mathbf{v}$$

So $\lambda = 0$ is an eigenvalue of A and $\mathbf{v} \neq \mathbf{0}$ is an eigenvector corresponding to this eigenvalue.

How do we find the eigenvalues and eigenvectors of a matrix?

Suppose $\mathbf{v} \neq \mathbf{0}$ is an eigenvector of A. Then for some $\lambda \in \mathbb{R}$, we have $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

or, equivalently,

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

This happens when

$$\det(A - \lambda I) = 0$$

Then:

$$\lambda \text{ is an eigenvalue of } A \quad \Leftrightarrow \quad \det(A - \lambda I) = 0$$

$p(\lambda) = \det(A - \lambda I)$ is a polynomial function of λ.
$p(\lambda) = 0$ is called the **characteristic equation** of the matrix A

An example

Consider

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Then

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix}$$

and

$$p(\lambda) = \det(A - \lambda I) = (1-\lambda)^2 - 9 = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$$

From $p(\lambda) = 0$ we obtain that $\lambda_1 = 4$ and $\lambda_2 = -2$ are the eigenvalues of A.

As an exercise, we want to find an eigenvector corresponding to λ_1. Thus we have to solve the linear system $(A - 4I)\mathbf{v} = \mathbf{0}$, i.e.

$$\begin{pmatrix} 1-4 & 3 \\ 3 & 1-4 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It is equivalent to

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This leads to the two equations $-3v_1 + 3v_2 = 0$ and $3v_1 - 3v_2 = 0$. Notice that the first equation is a multiple of the second one, so there is really only one equation to solve

$$3v_1 - 3v_2 = 0$$

The general solution to the homogeneous system is given by

$$v_1 = v_2 = c$$

therefore, all vectors $\mathbf{v}$ such that

$$\mathbf{v} = \begin{pmatrix} c \\ c \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{(where } c \neq 0 \text{ is arbitrary)}$$

are eigenvectors of A corresponding to $\lambda_1 = 4$.

Conclusion:

In general, eigenvectors are not unique! If $\mathbf{v}$ is an eigenvector for A corresponding to a given eigenvalue λ, then so is $c\mathbf{v}$, for any number $c \neq 0$.

As an exercise, find the eigenvectors of A corresponding to the other eigenvalue $\lambda_2 = -2$ (in the previous example).