Technology

Ugo Rizzo

Dep. of Economics and Management
University of Ferrara

October 20, 2015

Contents

Technology: preliminary definitions

- Suppose the firm has n possible goods to serve as inputs and/or outputs.$x_A + Y^b$
- If a firm uses y_j^{INPUT} units of a good j as an input and produces y_j^{OUTPUT} of the good as an output, then the **net output** of good j is given by $y_j = y_j^{OUTPUT} - y_j^{INPUT}$
- Net output can be positive or negative
- A **production plan** is simply a list of net outputs of various goods (n). It is a vector **y** in R^n
- The set of all technologically feasible production plans is called the firm's **production possibilities set** and will be denoted by Y.
- In the short-run some inputs can be considered as fixed (capital?, knowledge?)
- The restricted or short-run production possibilities set will be denoted by $Y(\mathbf{z})$; this consists of all feasible net output bundles consistent with the constraint level z.

Input requirement set and Isoquant

- Suppose we are considering a firm that produces only one output. We write the net output bundle as $(y, \text{-}\mathbf{x})$ where $\text{-}\mathbf{x}$ is a vector of inputs that can produce y units of output
- We can then define a special case of a restricted production possibilities set, **the input requirement set**:

$$V(y) = \mathbf{x} \; in \; R^n_+ : (y, \text{-}\mathbf{x}) \; in \; Y$$

- The input requirement set is the set of all input bundles that produce at least y units of output.
- We can also define an **ISOQUANT**:

$$Q(y) = \mathbf{x} \; in \; R^n_+ : \mathbf{x} \; in \; V(y) \; but \; \mathbf{x} \; not \; in \; V(y') \; for \; y' > y$$

- The isoquant gives all input bundles that produce exactly y units of output.

Examples of technology

EXAMPLE: Cobb-Douglas technology

Let a be a parameter such that $0 < a < 1$. Then the Cobb-Douglas technology is defined in the following manner. See Figure 1.1A.

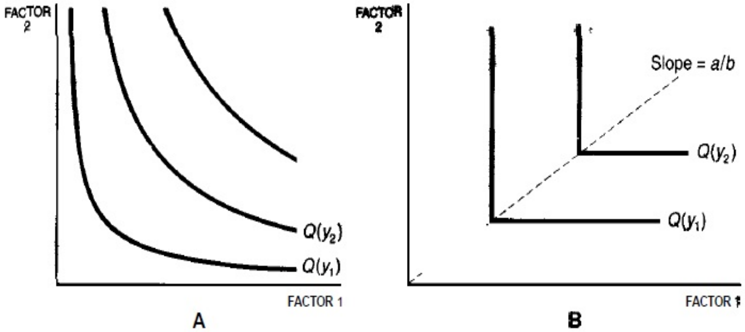
$$Y = \{(y, -x_1, -x_2) \text{ in } R^3 : y \leq x_1^a x_2^{1-a}\}$$
$$V(y) = \{(x_1, x_2) \text{ in } R^2_+ : y \leq x_1^a x_2^{1-a}\}$$
$$Q(y) = \{(\text{XI}, x_2) \text{ in } R^2_+ : y = x_1^a x_2^{1-a}\}$$
$$Y(z) = \{(y, -x_1, -x_2) \text{ in } R^3 : y \leq x_1^a x_2^{1-a}, x_2 = z\}$$
$$T(y, x_1, x_2) = y - x_1^a x_2^{1-a}$$
$$f(x_1, x_2) = x_1^a x_2^{1-a}.$$

EXAMPLE: Leontief technology

Let $a > 0$ and $b > 0$ be parameters. Then the **Leontief** technology is defined in the following manner. See Figure 1.1B.

$$Y = \{(y, -x_1, -x_2) \text{ in } R^3 : y \leq \min(ax_1, bx_2)\}$$
$$V(y) = \{(\text{XI}, x_2) \text{ in } R^2_+ : y \leq \min(ax_1, bx_2)\}$$
$$Q(y) = \{(\text{XI}, x_2) \text{ in } R^2_+ : y = \min(ax_1, bx_2)\}$$
$$T(y, x_1, x_2) = Y - \min(ax_1, bx_2)$$
$$f(x_1, x_2) = \min(ax_1, bx_2).$$

Examples of technology



Cobb-Douglas and Leontief technologies. Panel A depicts the general shape of a Cobb-Douglas technology, and panel B depicts the general shape of a Leontief technology.

Figure 1.1

Activity analysis

- The most straightforward way of describing production sets or input requirement sets is simply to list the feasible production plans.
- Suppose that we can produce an output good using factor inputs 1 and 2.
- Suppose also that there are two different activities or techniques by which this production can take place:
  Technique A: one unit of factor 1 and two units of factor 2 produces one unit of output.
  Technique B: two units of factor 1 and one unit of factor 2 produces one unit of output.
- Then we can represent the production possibilities implied by these two activities by the production set:

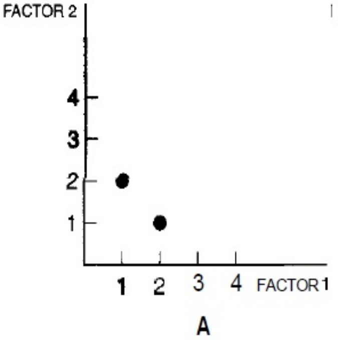
$$Y = [(1, -1, -2), (1, -2, -1)]$$

or the input requirement set:

$$V(1) = [(1, 2), (2, 1)]$$

This input requirement set is depicted in Figure 1.2A

Activity analysis



Activity analysis

- It may be the case that to produce y units of output we could just use y times as much of each input for y = 1,2,. . . . In this case you might think that the set of feasible ways to produce y units of output would be given by

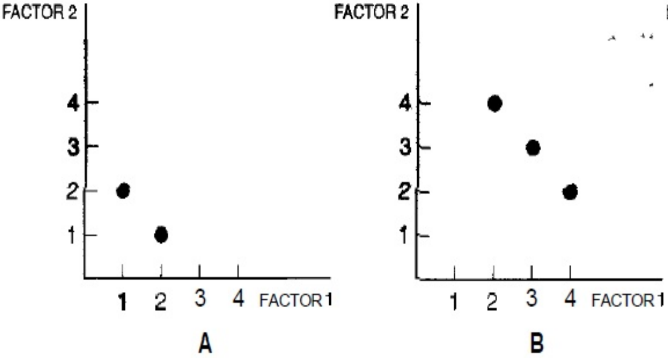
$$V(y) = [(y, 2y), (2y, y)]$$

- However, this set does not include all the relevant possibilities. what if we use a mixture of techniques A and B?
- In this case we have to let yA be the amount of output produced using technique A and yB the amount of output produced using technique B. Then V(y) will be given by the set

$$V(y) = [(y_A + 2y_B, y_B + 2y_A) : y = y_A + y_B]$$

- So, for example, V(2) = [(2,4), (4,2), (3,3)], as depicted in Figure 1.2B

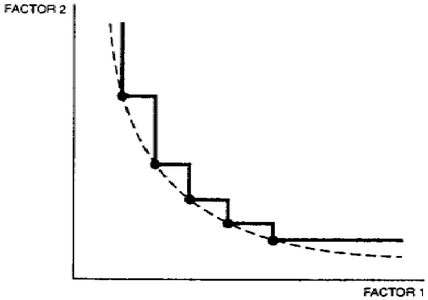
Activity analysis



Parametric representations of technology

- Suppose that we have many possible ways to produce some given level of output.
- Then it might be reasonable to summarize this input set by a "smoothed" input set
- That is, we may want to fit a nice curve through the possible production points. Such a smoothing process should not involve any great problems, if there are indeed many slightly different ways to produce a given level of output.
- If we do make such an approximation to "smooth" the input requirement set, it is natural to look further for a convenient way to represent the technology by a parametric function involving a few unknown parameters.
- These parametric technological representations should not necessarily be thought of as a literal depiction of production possibilities.
- In most applications we only care about having a parametric approximation to a technology over some particular range of input and output levels.
- Parametric representations are very convenient as pedagogic tools...

Parametric representations of technology



Smoothing an isoquant. An input requirement set and a "smooth" approximation to it.

The technical rate of substitution

- Assume that we have some technology summarized by a smooth production function and that we are producing at a particular point $y^* = f(x_1^*, x_2^*)$
- Suppose that we want to increase the amount of input 1 and decrease the amount of input 2 so as to maintain a constant level of output.
- How can we determine this technical rate of substitution between these two factors?
- In the two-dimensional case, the technical rate of substitution is just the slope of the isoquant: how one has to adjust x2 to keep output constant when x1 changes by a small amount, as depicted in Figure 1.6.
- In the n-dimensional case, the technical rate of substitution is the slope of an isoquant surface, measured in a particular direction.

The technical rate of substitution

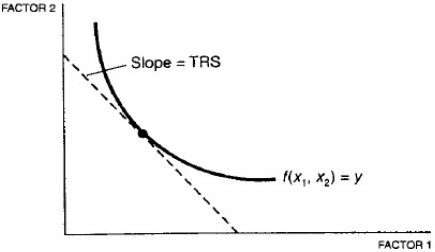


Figure 1.6 **The technical rate of substitution.** The technical rate of substitution measures how one of the inputs must adjust in order to keep output constant when another input changes.

The technical rate of substitution

- By definition $y = f(x_1, x_2)$
- Think of a vector of (small) changes in the input levels which we write as $dx = (dx_1, dx_2)$.
- The associated change in the output is approximated by

$$dy = \frac{\partial f(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial f(x_1, x_2)}{\partial x_2} dx_2$$

- This expression is known as the total differential of the function f(x). Consider a particular change in which only factor 1 and factor 2 change, and the change is such that output remains constant. That is, dx1 and dx2 adjust "along an isoquant":

$$0 = \frac{\partial f(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial f(x_1, x_2)}{\partial x_2} dx_2$$

which can be solved for:

$$\frac{dx_2}{dx_1} = -\frac{\partial f(x1, x2)/\partial x1}{\partial f(x1, x2)/\partial x2}$$

The technical rate of substitution

EXAMPLE: TRS for a Cobb-Douglas technology

Given that $f(x_1, x_2) = x_1^a x_2^{1-a}$, we can take the derivatives to find

$$\frac{\partial f(\mathbf{x})}{\partial x_1} = a x_1^{a-1} x_2^{1-a} = a \left[\frac{x_2}{x_1}\right]^{1-a}$$

$$\frac{\partial f(\mathbf{x})}{\partial x_2} = (1-a) x_1^a x_2^{-a} = (1-a) \left[\frac{x_1}{x_2}\right]^{a}.$$

It follows that

$$\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial f/\partial x_1}{\partial f/\partial x_2} = -\frac{a}{1-a}\frac{x_2}{x_1}.$$

The elasticity of substitution

- The technical rate of substitution measures the slope of an isoquant.
- The elasticity of substitution measures the curvature of an isoquant. Stern (2011, JprodAn) casts doubts....more investigations are required...
- More specifically, the elasticity of substitution measures the percentage change in the factor ratio divided by the percentage change in the TRS, with output being held fixed.
- see BlackorbyAER89.pdf

Returns to scale

- Suppose that we are using some vector of inputs x to produce some output y and we decide to scale all inputs up or down by some amount $t \geqslant 0$. What will happen to the level of output?
- we typically assumed that we could simply replicate what we were doing before and thereby produce t times as much output as before. If this sort of scaling is always possible, we will say that the technology exhibits **constant returns to scale**. More formally:

CONSTANT RETURNS TO SCALE. *A technology exhibits* **constant returns to scale** *if any of the following are satisfied:*

(1) $\mathbf{y}$ *in* Y *implies* $t\mathbf{y}$ *is in* Y, *for all* $t \geq 0$;

(2) $\mathbf{x}$ *in* $V(y)$ *implies* $t\mathbf{x}$ *is in* $V(ty)$ *for all* $t \geq 0$;

(3) $f(t\mathbf{x}) = tf(\mathbf{x})$ *for all* $t \geq 0$; *i.e., the production function* $f(\mathbf{x})$ *is homogeneous of degree 1.*

Returns to scale

INCREASING RETURNS TO SCALE. *A technology exhibits* increasing returns to scale *if* $f(t\mathbf{x}) > tf(\mathbf{x})$ *for all* $t > 1$.

DECREASING RETURNS TO SCALE. *A technology exhibits* decreasing returns to scale *if* $f(t\mathbf{x}) < tf(\mathbf{x})$ *for all* $t > 1$.

- Finally, let us note that the various kinds of returns to scale defined above are **global** in nature. It may well happen that a technology exhibits increasing returns to scale for some values of x and decreasing returns to scale for other values.
- Thus in many circumstances a **local measure** of returns to scale is useful.
- The **elasticity of scale** measures the percent increase in output due to a one percent increase in all inputs-that is, due to an increase in the scale of operations

Returns to scale

Let $y = f(\mathbf{x})$ be the production function. Let t be a positive scalar, and consider the function $y(t) = f(t\mathbf{x})$. If $t = 1$, we have the current scale of operation; if $t > 1$, we are scaling all inputs up by t; and if $t < 1$, we are scaling all inputs down by t.

The elasticity of scale is given by

$$e(\mathbf{x}) = \frac{\frac{dy(t)}{y(t)}}{\frac{dt}{t}},$$

evaluated at $t = 1$. Rearranging this expression, we have

$$e(\mathbf{x}) = \frac{dy(t)}{dt}\frac{t}{y}\bigg|_{t=1} = \frac{df(t\mathbf{x})}{dt}\frac{t}{f(t\mathbf{x})}\bigg|_{t=1}.$$

Note that we must evaluate the expression at $t = 1$ to calculate the elasticity of scale at the point $\mathbf{x}$. We say that the technology exhibits locally increasing, constant, or decreasing returns to scale as $e(\mathbf{x})$ is greater, equal, or less than 1.

Returns to scale

EXAMPLE: Returns to scale and the Cobb-Douglas technology

Suppose that $y = x_1^a x_2^b$. Then $f(tx_1, tx_2) = (tx_1)^a (tx_2)^b = t^{a+b} x_1^a x_2^b = t^{a+b} f(x_1, x_2)$. Hence, $f(tx_1, tx_2) = t f(x_1, x_2)$ if and only if $a + b = 1$. Similarly, $a + b > 1$ implies increasing returns to scale, and $a + b < 1$ implies decreasing returns to scale.

In fact, the elasticity of scale for the Cobb-Douglas technology turns out to be precisely $a + b$. To see this, we apply the definition:

$$\frac{d(tx_1)^a (tx_2)^b}{dt} = \frac{dt^{a+b} x_1^a x_2^b}{dt} = (a+b) t^{a+b-1} x_1^a x_2^b.$$

Evaluating this derivative at $t = 1$ and dividing by $f(x_1, x_2) = x_1^a x_2^b$ gives us the result.

Summing up

- Technical change is said to be Hicks Neutral when:

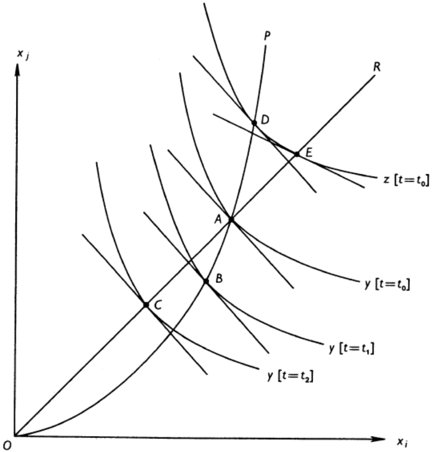


Fig. 1

-

Atkinson e Stiglitz (1969)

- The (recent) literature on technological progress has almost entirely been based on the assumption that its effects can be represented as shifting the production function outwards
- The different points on the curve still represent different processes of production,a nd associatedw ith each of these processest here will be certain technical knowledge specific to that technique
  If one brings about a technological improvement in one of the blue-prints this may have little or no effect on the other blue-prints

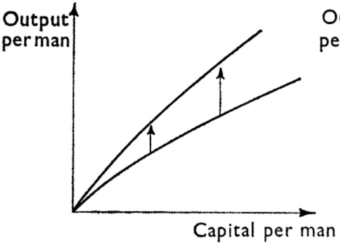


Fig. 1.

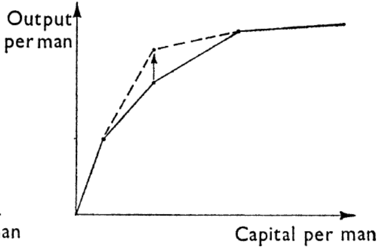


Fig. 2.

-

Necolassical production function

- In the modern synthesis of the subject, production sets are a fundamental concept, production functions are a derived construct, and marginal productivity schedules are an implied attribute of production functions.
- Historically, however, it happened in the opposite order. Marginal productivity came first (Ricardo), then production functions (Wicksteed), then production sets (Koopmans, Arrow and Debreu). In the "finished" structure of modern theory, the concepts that developed later are logically antecedent to those that appeared earlier. The development of the more recent arrivals has been strongly influenced by their logical role in an already extant theoretical structure; they did not have much chance to develop a life of their own

Necolassical production function, no TC - Properties

- (strict) monotonicity
- (quasi) concavity
- Essentiality
- Nonempty
- Finite, nonnegative, real valued
- Continuous and twice differentiable

Disembodied technical change

- Technical change represents a shift in the production function over time (not embodied in any particular input or group of inputs)
- A stable relationship between output, input and time (t) is presumed to exist: $y = f(x, t)$
- Technical change is measured by how output changes as time elapses with the input bundle held costant
- Strong restriction therefore irrealistic: change in technology may easily require new inputs
- Production function maintains the same basic form as time elaspes
- Example: Cobb-Douglas

Embodied technical change

- Require differentiating the production function itself as well as the incput bundle over time: $y_t = f_t(X_t, t)$ where $f_t(X_t, t)$ and $f_T(X_T, T)$ need not be the same functional forms and the components of the input X_T and X_t may be different
- Unfortunately, embodied technical change consistent with this representation is very difficult analytically.
- As a consequences not often adopted as a model

Augmented technical change

- The idea behind factor-augmenting technical change is simple: Input quality varies with time so that, for example, hiring one unit of labor in year 1 does not necessarily yield the same effective labor units as in year 2. The passage of time makes a difference in how that input affects production.
- Notice, however, that this is not the same thing as what we have called embodied technical change because a stable relationship between output, input, and time still exist. Although the effectiveness of inputs varies over time, their essential character does not.

Estimation diagram

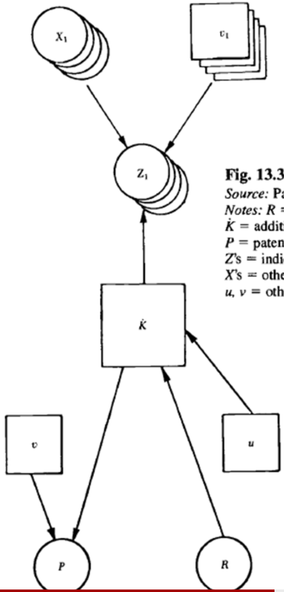


Fig. 13.3 Knowledge production function: a simplified path analysis diagram

Source: Pakes and Griliches (1984), figure 3.1.

Notes: R = research expenditures.

$\dot{K}$ = additions to economically valuable knowledge.

P = patents, a quantitative indicator of the number of inventions.

Z's = indicators of expected or realized benefits from invention.

X's = other observed variables influencing the Z's.

u, v = other unobserved influences, assumed random and mutually uncorrelated.