Mathematical foundations of Econometrics

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Examples of σ-*algebras*

Question: Consider the following collection of events in Ω. Are they σ-*algebras*?Why?

Answer: All the sets below are σ-*algebras* of Ω:

- $\mathscr{F} = \{\emptyset, \Omega\}$
- $\mathscr{F} = \{\emptyset, \mathrm{A}, \tilde{\mathrm{A}}, \Omega\}$, where A is a subset of Ω, i.e. $\mathrm{A} \subset \Omega$ and $\mathrm{A} \neq \emptyset, \Omega$.
- $\mathscr{F} = \mathscr{P}(\Omega) \overset{def}{=} \{A : A \subset \Omega\}$ is the set of *all subsets of* Ω. It is called the **powerset of** Ω.
- Borel σ-*algebra* $\mathscr{B}$ in $\Omega = \mathbb{R}$, which is essential for the definition of random variables (*it will be discussed later...*)

Exercise: Prove that the second one is a σ-*algebra*.

Why do we need σ-*algebras* ?

σ-*algebras* allow us to model "information". It can be considered as the *information* we can have access to. It contains all the events A for which one can ask: "What is the probabiblity of A happening?"

Health insurance example
Select at random a person and categorize him/her according to their age (young or old) and health status (healthy or sick). The sample space of the experiment is:

$$\Omega = \{YH, YS, OH, OS\}$$

Question: So, baring in mind that you can find the age and the health status of a person (e.g. if you were a doctor in a clinic) which is the σ-*algebra* which best describes the information you can have access to?
Answer:

$\mathscr{F} = \mathbb{P}(\Omega) =$
$\{\emptyset, \Omega, \{YH\}, \{YS\}, \{OH\}, \{OS\}, \{YS, OH, OS\}, \{YH, OH, OS\}, \{YH, YS, OS\},$
$\{YH, YS, OH\}, \{YH, YS\}, \{OH, OS\}, \{YH, OH\}, \{YH, OS\}, \{YS, OH\}, \{YS, OS\}\}$

Why do we need *σ-algebras* ?

...example continued

Assume now that for an insurer only age is public information. Therefore in your experiment, the insurer can only know whether a person is young or old. Then, the information that the insurer can have access to is described by the following *σ-algebra*:

$$\mathscr{F}' = \{\emptyset, \Omega, \{YH, YS\}, \{OH, OS\}\}$$

Question: Can the insurer make inference about a person who is either young and sick or old?
Answer: If he does not have the information about health, he can not distinguish between a healthy and a sick person. More formally, the event $\{YS, OH, OS\}$ is not inside the *σ-algebra* $\mathscr{F}'$.

If the insurer could know the health status of a person, then the information he could acquire would be embodied in the biggest *σ-algebra* $\mathscr{F}$ of all subsets of Ω (*powerset of* Ω)!

Question: In this example, whose information can the σ - *algebra* $\mathscr{F}'' = \{\emptyset, \Omega\}$ describe best?

Why do we need σ-*algebras* instead of *algebras*?

Question: Why don't we just stay with *algebras* instead of defining σ-*algebras* in the probability space?

Many useful results (like Strong Law of Large Numbers) would not apply if a probability space was not structured based on a σ-*algebra*. Simply, many problems could not be solved otherwise.

Consider for instance the following game; a fair coin is tossed constantly. We win one euro every time tails appears and nothing otherwise. Without σ-*algebras*, the probability of the event:

$$\{\text{the average winning converges to } 1/2 \\ \text{as the coin tosses become (infinitely) many}\}$$

can not even be defined.

Properties of *algebras*

Theorem (1)

If an algebra contains a finite number of events, then it is also a σ-algebra. Consequently, an algebra of subsets of a finite set Ω is a σ-algebra.

Theorem (2)

If $\mathscr{F}$ is an algebra, then $\mathrm{A}, \mathrm{B} \in \mathscr{F}$ implies $\mathrm{A} \cap \mathrm{B} \in \mathscr{F}$. Moreover, a collection $\mathscr{F}$ of subsets of a non-empty set Ω is an algebra if it satisfies the first two requirements in the definition of algebra along with:

- *If $\mathrm{A}, \mathrm{B} \in \mathscr{F}$, then $\mathrm{A} \cap \mathrm{B} \in \mathscr{F}$.*

Properties of σ-*algebras*

Theorem (3)

If $\mathscr{F}$ *is a* σ*-algebra, then for any countable sequence of sets* $\mathrm{A}_j \in \mathscr{F}$*, then* $\bigcap_{j=1}^{\infty} \mathrm{A}_j \in \mathscr{F}$*. Moreover, a collection* $\mathscr{F}$ *of subsets of a non-empty set* Ω *is a* σ*-algebra if it satisfies the first two requirements of the definition of* σ*-algebras along with:*

- *If* $\mathrm{A}_j \in \mathscr{F}, j = 1, 2, \ldots$ *, then* $\bigcap_{j=1}^{\infty} \mathrm{A}_j \in \mathscr{F}$.

Example of generated σ-algebras

Definition

The smallest σ-*algebra* containing a given collection $\mathscr{Q}$ of sets is called the σ-*algebra generated by* $\mathscr{Q}$ and is denoted by $\sigma(\mathscr{Q})$.

Let us return to the health insurance example. Suppose that we have the following collection of events $\mathscr{Q} = \{\emptyset, \Omega, \{YH, YS\}\}$.

Question: Which can be a σ-algebra that contains the above collection?
Answer:
$\mathscr{F} = \mathscr{P}(\Omega) =$
$\{\emptyset, \Omega, \{YH\}, \{YS\}, \{OH\}, \{OS\}, \{YH, YS\}, \{YH, OH\}, \{YH, OS\},$
$\{YS, OH\}, \{YS, OS\}, \{OH, OS\}, \{YH, YS, OH\}, \{YH, YS, OS\},$
$\{YS, OH, OS\}, \{OH, OS, YH\}\}$

Question: Is this the σ-algebra generated by the collection $\mathscr{Q}$?
Answer:
No, because it is not the smallest. The smallest is
$\sigma(\mathscr{Q}) = \{\emptyset, \Omega, \{YH, YS\}, \{OH, OS\}\}$

Borel σ-algebras

An important special case of σ-*algebras* is when $\Omega = \mathbb{R}$ and $\mathscr{Q}$ is a collection of all open intervals in $\mathbb{R}$, namely:

$$\mathscr{Q} = \{(\alpha, \beta) : \forall \alpha < \beta, \alpha, \beta \in \mathbb{R}\} \tag{1}$$

Definition

The σ-*algebra* generated by the above collection $\mathscr{Q}$ of all open intervals in $\mathbb{R}$ is called the *Euclidean Borel field* $\mathscr{B}$ or *Borel* σ-*algebra*. Its members are called *Borel sets*.

note: The *Borel* σ-*algebra* can be defined in different ways due to the following theorem.

Theorem

$\mathscr{B} = \sigma(\{[\alpha, \beta] : \forall \alpha \leq \beta, \alpha, \beta \in \mathbb{R}\}) = \sigma(\{(-\infty, \alpha] : \forall \alpha \in \mathbb{R}\})$, *i.e.* σ-*algebras generated by the collection of open intervals, closed intervals and half-open intervals are the same.*